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Infinite games on finitely coloured graphs with applications to automata on infinite trees

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Abstract

We examine a class of infinite two-person games on finitely coloured graphs. The main aim is to construct finite memory winning strategies for both players. This problem is motivated by applications to finite automata on infinite trees. A special attention is given to the exact amount of memory needed by the players for their winning strategies. Based on a previous work of Gurevich and Harrington and on subsequent improvements of McNaughton we propose a unique framework that allows to reestablish and to improve various results concerning memoryless strategies due to Emerson and Jutla, Mostowski, Klarlund. © 1998—Elsevier Science B.V. All rights reserved

Keywords: Infinite games; Infinite trees; Rabin complementation lemma

Contents

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1. Introduction

The main subject of this work concerns infinite games played on oriented finitely coloured graphs. The graphs used as game arenas can be either finite or infinite, in fact the major application of these games – the complementation lemma for automata on infinite trees – involves games on infinite graphs. However, the set C of colours that label graph vertices is always supposed to be finite.

Two players, player 0 and player 1, play on a graph G by moving alternatively a token along the edges of G. Let $\mathscr{F}_0 \subseteq \mathscr{P}(C)$ be a collection of sets of colours. After an infinite number of moves player 0 is declared the winner if the set X consisting of colours visited infinitely often by the token during the play belongs to $\mathscr{F}_0$, otherwise $X \in \mathscr{P}(C) \setminus \mathscr{F}_0 = \mathscr{F}_1$ and player 1 wins. The partition $(\mathscr{F}_0, \mathscr{F}_1)$ of $\mathscr{P}(C)$ onto two disjoint collections is called the winning condition.

The problem consists in determining for both players their respective sets of winning positions and to construct for them finite memory winning strategies. The solution presented in this paper gets its inspiration from two sources: Gurevich and Harrington's [11] celebrated short proof of Rabin's complementation lemma for automata on infinite trees [25] and McNaughton's splendid application of their method to infinite games on finite graphs [19].

Since the publication in 1969 of Rabin's proof of the decidability of monadic second-order theory of trees [25], the problem of finding a simplified proof for the most difficult part of his demonstration – the complementation lemma for automata on infinite trees constitutes an ongoing challenge attracting much attention.

The idea to use games to prove this result appears in Büchi [3] and was applied successfully by Gurevich and Harrington [11]. The last paper settles also positively a problem posed by Rabin if the complementation lemma can be demonstrated without ordinal numbers (Rabin's proof uses ordinals up to ω_1 – the first uncountable ordinal). However, there is a price to pay for the elimination of ordinals, the proof in [11] is non-constructive, it is shown that always one of the two players has a winning finite memory strategy without actually exhibiting the winning player. On the other hand, the great merit of [11] is to introduce a precise definition of the winning player's memory. The Gurevich and Harrington paper was followed by numerous other attempts to clarify and simplify the proof of the complementation lemma.

The leading idea of Muchnik's proof [22] is the same as in [11]: the induction on the number of states allows to present a more complicated game as a suitable composition of simpler games. Both papers differ mainly in the method used to obtain such a decomposition. Muchnik's demonstration is more detailed and therefore simpler to follow than that of Gurevich and Harrington. However it presents also one serious handicap. Muchnik considers games with Muller winning condition, where the winning player needs some (finite) memory. Although from his construction it is clear that the winning strategy uses a finite amount of memory the construction of this memory is left completely to the reader. For this reason Muchnik's paper is more difficult to exploit than [11] if we are concerned with the size of winning player's memory.

The paper [30] of Yakhnis and Yakhnis is a direct follow-up to [11]. It presents a constructive version of [11], which allows to exhibit explicitly the winning player and to construct for him a finite memory winning strategy. Their argument was subsequently simplified by Zeitman [31]. An interesting novelty introduced in Zeitman's paper consists in considering games on graphs rather than on infinite trees. Her method is also presented in the recent monograph by Börger et al. [2].

Muller and Schupp [23] use alternating tree automata. Their proof is non-constructive and takes two steps. To show the existence of a winning strategy for one of the players they invoke the determinacy of Borel games [18, 13] (in fact a simpler result about determinacy of $G_{\delta\sigma}$ games due to Davis [5] is sufficient). In the next step they show how to transform a perfect (unbounded) memory strategy into an equivalent finite memory strategy for a player using a winning condition in Streett form.

Emerson and Jutla [9] prove the complementation lemma in the framework of the μ-calculus. They show also that for games with so-called parity winning condition the winning player has always a memoryless winning strategy. The same result was obtained independently by Mostowski [21]. We postpone the discussion of these papers to Section 3 where parity games will be examined. Let us note only that the μ-calculus is a popular research area and new proofs of the complementation lemma using the μ-calculus techniques appear [1, 12].

Finally we can end this list of papers devoted to the complementation lemma with a recent article by Klarlund [14]. He improves on the results of [9, 21] by showing that the player using Rabin winning condition has always a memoryless winning strategy. We shall discuss his paper at length in Section 5.

Although the papers cited above use often quite different techniques and vary considerably in the degree of difficulty, still they seem to be accessible only for a mature reader. What is also remarkable is the fact that, with the exception of [25, 1], they all employ game theoretic terminology.

Nevertheless three lessons can be learned from the attempts to present a legible game theoretic proof of Rabin lemma:

(1) Separating completely the results concerning finite memory determinacy of games from applications we gain in clarity and transparency. This point can be illustrated by problems the reader encounters in Gurevich and Harrington proof (Emerson and Jutla, expressing what seems to be a largely shared sentiment, comment on [11]: "While the presentation is brief, the argument is still extremely difficult.") In our opinion these difficulties result mainly from the fact that in [11] the determinacy of games is

too much intertwined with applications. The other reason to separate the determinacy from applications is that there are also other applications than the complementation lemma, the finite memory game determinacy can be used to show that each nonempty recognizable set of trees contains a regular tree [26]. Thus separating determinacy from applications we avoid also repetitions. (An attentive reader may observe such repetitions in Muchnik's paper [22], where determinacy is not separated from applications.)

(2) It is easier and more natural to handle and understand games played on general (eventually infinite) graphs rather than on infinite trees, even if both types of games are equivalent. And after all, graphs and not just infinite trees constitute a natural framework for some applications. For example Gurevich and Harrington play the games on infinite trees equipped with an equivalence relation identifying vertices that are roots of isomorphic subtrees. While in this way they obtain an interesting result concerning the existence of highly regular accepting runs, this equivalence relation is completely irrelevant to the underlying game theoretic problem. It would be more transparent to prove the determinacy of games on graphs and next to apply this result to graphs obtained as quotients of infinite trees by the equivalence relation.

(3) In general, a winning condition can be represented either abstractly as a set of plays (infinite paths) obtained as a boolean combination of G_δ sets, or in a more concrete way using the set of states (or colours in the terminology we have adopted in this paper) visited infinitely often. Although both formulations are equivalent the last one is preferable. This becomes evident if we compare [30, 31] with [11].

In 1993 McNaughton published a paper devoted to infinite games on finite graphs. Based on techniques of Gurevich and Harrington, he gives an algorithm allowing to calculate the set of winning positions for each player and their respective finite memory winning strategies. This is accompanied by a detailed complexity analysis of proposed procedures. The transparency and readability of McNaughton paper contrasts sharply with all game theoretic proofs of Rabin lemma. However, finite memory determinacy of games on *finite* graphs does not seem to be sufficient to obtain the complementation lemma, therefore our main goal is to show that Gurevich–Harrington–McNaughton methods can be applied with a similar clarity to games on infinite graphs.

Our second aim is to get more insight into the role the memory plays in the construction of winning strategies. If we use for both players full LAR memory, as it was done in [11], then their winning strategies need $n!$ states, where n is the number of colours (in applications n is the number of states in the tree automaton). However, Emerson and Jutla [9], Mostowski [21] and Klarlund [14] show that sometimes players can have memoryless winning strategies. Their results were obtained by different methods. In our presentation we show how the ideas of Gurevich and Harrington together with the important contribution of McNaughton enable to demonstrate these facts in a unified way in a single proof.

However, presenting a unifying framework for various well-known results may not justify completely a new paper on Rabin's complementation lemma. And, after all, undoubtedly, each specialist in the domain has already chosen his or her favourite proof in the abundant literature cited above.

This leads us to our second aim. Taking into account the importance of the subject to automata theory and to logics and also the interest that such a nontrivial result presents by itself we believe that offering a proof unifying known facts in a way, as we hope, accessible to a nonspecialist is also a worthy enterprise. Having such a nonexpert reader in mind we tried to make the paper as self-contained as possible. In fact this approach allowed also to uncover new facts concerning memoryless strategies that may present some interest to specialists.

In Section 2 we introduce formally the notions of games and strategies. Important auxiliary concepts of traps, attractors and attracting strategies are defined there as well. In Section 3 we show that for games using parity winning conditions both players have memoryless winning strategies. This result as well as its proof are only specialized cases of the material presented in the next section; in particular, all subsequent sections are independent of Section 3. Nevertheless, a separate section on parity games seems to be opportune since, while the determinacy proof is in this case much simpler, memoryless determinacy of parity games implies finite memory determinacy of more general games [20, 28, 29], in particular it implies Rabin's complementation lemma.

Section 4 contains the main result of the paper – determinacy of games by finite memory strategies. We use essentially LAR memories introduced by Gurevich and Harrington but with one important modification. For each player we define a set of useful colours and in his LAR memory we record only the colours that are useful for him, the other colours are ignored. We should note that although the proof of determinacy presented in Section 4 is based on old ideas of [11, 19] the particular structure of the winning set uncovered in our proof seems to be new and it turned out to be pertinent to the problem of the exact memory size required by the winning player for a fixed winning condition, see Dziembowski et al. [7] (we shall comment on this paper later on). Thus our proof may present some interest for specialists as it can serve as an introduction to [7].

Section 4 ends with two short subsections; the first of them analyses the role the memory plays in winning strategies, the second one shows briefly how to construct a finite memory winning strategy with a memory that is not an LAR memory.

The question of when winning strategies obtained in Section 4 reduce to memoryless strategies is discussed in Section 5. As it turns out the set of useful colours of a given player is empty – in this case his winning strategy becomes memoryless – iff his winning condition can be expressed in Rabin form. On the other hand, both players have empty sets of useful colours iff the winning condition is equivalent to the chain condition, i.e. we recover the results of [14, 9, 21] as particular cases of our main result. A direct adaptation of an example due to McNaughton shows that these conditions assuring the existence of winning memoryless strategies are not only sufficient but also necessary for games played on partially coloured graphs.

However, the games that arise in applications are played on totally coloured graphs. For such games the players may need less memory for their winning strategies. Unfortunately the detailed construction of such strategies becomes rather cumbersome in this case. For this reason in Section 6 we will only examine in detail the special case of *memoryless* strategies for games on totally coloured graphs. As it turns out the necessary and sufficient condition assuring the existence of a memoryless winning strategy either for one or for both players is strictly weaker then the corresponding conditions for partially coloured graphs. This seems to be new and it demonstrates that the results of Emerson and Jutla [9], Mostowski [21], Klarlund [14] concerning memoryless strategies are not optimal for the games that occur in applications. We should note however that playing on totally colored arenas we can save at most a linear amount of memory, see remarks at the end of Section 6.

Finally Section 7 is devoted to applications to automata on infinite trees. It was added only for the sake of completeness since it is widely known how finite memory determinacy of games implies the Rabin complementation lemma as well as the decidability of the emptiness problem and the existence of a regular tree in each nonempty recognizable set of trees.

Let us end with a short discussion of recent results concerning the memory size that is necessary for the winning player.

Some partial results for games on special classes of finite graphs appear in Lescow [17].

As we have already mentioned, from Klarlund [14] and from McNaughton's example we know that a player needs no memory in all games if and only if his winning condition is expressible in Rabin form. A natural question is how much memory the player needs if the winning condition is not in this form, more precisely how much memory he needs for all possible games with a fixed winning condition? This difficult problem was completely settled in a recent paper by Dziembowski et al. [7]. Contrary to LAR memory the memory used in [7] is not updated by a finite automaton. Their achievement is more remarkable when we realize that previously even the attempts to present an alternative to LAR memory were scarce – in fact we are aware of only one such attempt due to Yakhnis and Yakhnis [30]. However, the memory data structures constructed in [30] have the same advantages and the same drawbacks as LAR memory, these memories are updated by finite automata which makes them easy to describe however for the same reason they cannot be optimal for all winning conditions.

In all the other papers cited previously, either memoryless strategies are constructed for special classes of winning conditions [9, 14, 21], or only some classes of graphs and/or winning conditions are considered [17], or the memory construction is left to the reader [22], or LAR memory [31] or its variant due to Büchi [4] are used [23].

2. Preliminaries

The set of words (finite sequences) over X is noted by X^* and $\varepsilon \in X^*$ is the empty word. For any word $x = x_1x_2x_3 \ldots x_k \in X^*$, $(\forall i,\ x_i \in X)$, $|x| = k$ is the length of x. We shall also meet infinite words $x = x_1x_2x_3 \ldots$ of length ω, where ω is the smallest infinite ordinal. The cardinality of a set X is noted by card(X). By $\mathscr{P}(X) = \{Y \mid Y \subseteq X\}$ we note the collection of all subsets of X, while $\mathscr{P}_0(X)$ is the collection of all nonempty subsets of X. Finally, $X \not\subseteq Y$ means that X is not a subset of Y, i.e. that $X \setminus Y \neq \emptyset$.

An arena is a tuple $G = (V_0, V_1, E, \varphi, C)$, where

(1) V_0 and V_1 are nonempty and disjoint sets of vertices,

(2) $E \subseteq V_0 \times V_1 \cup V_1 \times V_0$ is the set of edges such that, for each $v \in V_0 \cup V_1$, the set $vE = \{v' \in V_0 \cup V_1 \mid (v, v') \in E\}$, called the set of *successors* of v, is finite[2] nonempty,

(3) C is a finite nonempty set of colours,

(4) φ is a colouring mapping, it is a partial mapping from $V_0 \cup V_1$ into C.

The vertices belonging to V_0 (V_1) are called 0-vertices (1-vertices respectively). The union $V_0 \cup V_1$ will be denoted by V. A vertex $v \in V$ belonging to the domain of φ is said to be coloured by the colour $\varphi(v) \in C$, the vertices that are not in the domain of φ are *uncoloured*.

We do not assume anything about the cardinality of V, this set can be finite or infinite of any cardinality, only the set of colours is always supposed to be finite.

Two players, player 0 and player 1, play on G by moving a token between vertices. If the token is in a 0-vertex $v \in V_0$ then player 0 chooses a successor v' of v and moves it there, if the token is in a 1-vertex then it is player's 1 turn to move the token to some successor vertex. In this way, by subsequent moves executed alternatively by players 0 and 1, the token visits vertices of G.

Since each vertex has at least one successor the subsequent move is always possible and after ω moves we obtain an infinite path

$$p = v_0 v_1 v_2 \ldots, \quad \text{where} \quad \forall i, (v_i, v_{i+1}) \in E$$

consisting of vertices visited by the token that started its walk at a vertex v_0. In the sequel all such infinite paths in G are called *plays* and we add the last condition to the definition of arenas:

(5) for each play $p = v_0 v_1 v_2 \ldots$ in G there are infinitely many i such that v_i is coloured (this is equivalent with the requirement that there is no play going exclusively through uncoloured vertices).

We shall use also the notion of partial plays which are finite nonempty sequences $v_0, \ldots, v_k$, $k \geqslant 0$, of vertices such that $\forall i, 0 \leqslant i < k, (v_i, v_{i+1}) \in E$.

To declare one of the players the winner of a play p, we should specify winning criteria. *Muller condition* is given by two complementary collections of nonempty subsets of C

$$\mathscr{F}_0 \subseteq \mathscr{P}_0(C) \quad \text{and} \quad \mathscr{F}_1 = \mathscr{P}_0(C) \setminus \mathscr{F}_0$$

[2] Although in arenas that appear in applications vertices have finitely many successors all the results concerning game determinacy formulated in this paper hold also for arenas that do not satisfy this condition. In fact the finiteness of vE is never used in proofs with the exception of Section 2.3 where some minor adjustments are necessary.

that are called winning conditions for players 0 and 1 respectively. Let

$$\inf(p)=\{c\in C \mid \text{for infinitely many } i, \quad c=\varphi(v_i)\}$$

be the set of colours visited infinitely often in a play $p=v_0v_1v_2\ldots$ (we say that the token visits a colour $c\in C$ if it visits a vertex coloured by c). Then player 0 wins p if $\inf(p)\in\mathscr{F}_0$, otherwise $\inf(p)\in\mathscr{F}_1$ and player 1 wins. (Note that by (5), $\inf(p)$ is always nonempty.) Winning conditions concern only full infinite plays, there is no winner for a partial play since such a play is simply considered as not yet finished.

The couple $\mathscr{G}=(G,(\mathscr{F}_0,\mathscr{F}_1))$ consisting of an arena and a winning condition is a game (on the arena G).

In the sequel $\sigma\in\{0,1\}$ will always stand for one of the two players, his adversary will be noted by $1-\sigma$.

2.1. *Strategies*

Informally, a strategy for a player $\sigma\in\{0,1\}$ is a method that σ applies to choose a successor vertex whenever the token visits a vertex $v\in V_\sigma$. A strategy is winning for σ if it allows σ to win all resulting plays against any possible moves of his adversary. There are several types of strategies possible. In general, the subsequent move of player σ may depend not only on the current token position but also on the previous token positions. If all the previous token positions are taken into account we have a strategy with perfect information considered in descriptive set theory [13]. Formally, such a strategy for player σ is a mapping assigning to each partial play $v_1\ldots v_n$ such that $v_n\in V_\sigma$ a subset of v_nE. As it is known by the result of Martin [18, 13], for the class of Borel games, which is much larger than the class of games we consider here, for each initial token position one of the players has a winning strategy with perfect information. However, for these strategies the player should dispose of unbounded memory to store the complete sequence of previous token positions, which makes them useless for our purposes. What we need is a property that Gurevich and Harrington [11] call a forgetful determinacy. It asserts the existence of winning finite memory strategies.

A *finite memory strategy* for player σ is a mapping $f_\sigma : V_\sigma\times M\to\mathscr{P}(V_{1-\sigma})$, where M is a finite memory with a size depending on the winning condition $(\mathscr{F}_0,\mathscr{F}_1)$ but independent of the arena G. Each time the token changes the position player σ updates his memory as a function of the new token position and the previous memory state. More precisely, besides the strategy f_σ player σ is equipped with a mapping δ_σ : $M\times V\to M$, which for the previous memory state $m\in M$ and for a new token position $v\in V$ gives the new memory state $\delta_\sigma(m,v)$ of σ (note that σ updates his memory at each token movement, independently of the identity of the player moving it). If σ plays according to the strategy f_σ and the token visits a vertex $v\in V_\sigma$ and $m\in M$ is the current memory state of σ then player σ moves the token to any vertex $w\in f_\sigma(v,m)\subseteq vE$. A special important case of finite memory strategies are memoryless strategies where the subsequent move depends only on the current token position and no information about previous token positions is needed.

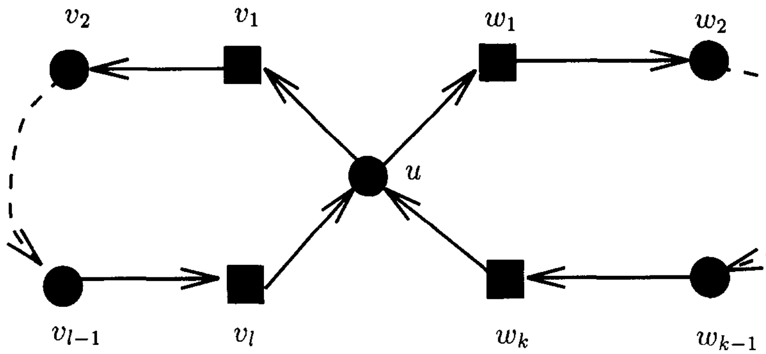

Fig. 1.

A *memoryless strategy* for a player σ is a mapping $f_\sigma : V_\sigma \rightarrow \mathscr{P}(V_{1-\sigma})$ such that $\forall v \in V_\sigma, f_\sigma(v) \subseteq vE$.

The concept of σ playing according to f_σ is captured by the notion of plays consistent with f_σ.

A play $p = v_1 v_2 v_3 \ldots$ is *consistent* with f_σ if

$$\forall v_i, \quad \text{if} \quad v_i \in V_\sigma \quad \text{then} \quad v_{i+1} \in f_\sigma(v_i)$$

The definition of consistency applies in the obvious way to partial plays.

Unfortunately, as we show below, for Muller winning conditions a player can have a winning strategy but no winning memoryless strategy.

Let $\mathscr{F} \subseteq \mathscr{P}_0(C)$. Following McNaughton [19], we call a *split* in $\mathscr{F}$ any pair of sets $X_1, X_2 \in \mathscr{F}$ such that $X_1 \cup X_2 \notin \mathscr{F}$. The following observation is essentially due to McNaughton:

Lemma 1. *If in a Muller condition $(\mathscr{F}_0, \mathscr{F}_1)$ the set $\mathscr{F}_{1-\sigma}$ has a split then there exists a game $(G, (\mathscr{F}_0, \mathscr{F}_1))$ such that player σ has a winning strategy but no memoryless winning strategy.*

Proof. Suppose that $X_1, X_2 \in \mathscr{F}_{1-\sigma}$ but $X_1 \cup X_2 \in \mathscr{F}_\sigma$.

Take the graph of Fig. 1, where circles represent σ-vertices and squares $1-\sigma$-vertices. Colour vertices in such a way that $X_1 = \{\varphi(v_i) \mid 1 \leqslant i \leqslant l\}$, $X_2 = \{\varphi(w_i) \mid 1 \leqslant i \leqslant k\}$, i.e. X_1 is the set of colours labelling the vertices v_i and X_2 colours the vertices w_i. The vertex u is left uncoloured. Player σ has an obvious winning strategy: whenever the token visits the vertex u he should move it alternatively to v_1 and w_1. With this strategy the token visits infinitely often all colours of $X_1 \cup X_2 \in \mathscr{F}_\sigma$. However to implement such a strategy σ needs some memory (one bit is sufficient) to record the parity of the visit in u.

On the other hand, σ has no memoryless winning strategy: It is obvious that memoryless strategies f such that $f(u)$ equals either $\{v_1\}$ or $\{w_1\}$ are not winning for σ. Otherwise, if $f(u) = \{v_1, w_1\}$ then player σ has no information which of the two successors of u to choose, in particular if he chooses one of them finitely many times then the resulting play is consistent with f but winning for $1-\sigma$, i.e. such a strategy is not winning for σ. $\square$

It is clear that in general the existence of the winning strategy for a given player depends on the initial token position. It turns out however that instead of looking for a winning strategy for a fixed initial token position it is more convenient to construct for each player σ the set W^σ of all his winning positions and a strategy w^σ that assures his victory for all plays starting anywhere in W^σ (this idea of constructing a winning strategy that is independent of the initial position appears in McNaughton [19]). The set W^σ has a special form that we describe below.

2.2. Subarenas and traps

Let $U \subseteq V$ be any set of vertices of an arena $G=(V_0, V_1, E, \varphi, C)$. The partially coloured subgraph of G induced by U will be denoted by $G[U]$,

$$G[U]=(V_0 \cap U, V_1 \cap U, E \cap (U \times U), \varphi_{|U}, C)$$

where $\varphi_{|U}$ is the restriction of φ to U.

$G[U]$ is a *subarena* of G if it is an arena, i.e. if each vertex of U has at least one successor in U. It may happen that vertices of U are coloured by elements of some proper subset B of C. In this case we can (and sometimes will) assume that $G[U]$ is an arena coloured by B rather than by C, i.e. we set $G[U]=(V_0 \cap U, V_1 \cap U, E \cap (U \times U), \varphi_{|U}, B)$.

Let $\sigma \in \{0,1\}$. A *σ-trap* (or a trap for σ) in an arena $G=(V_0, V_1, E, \varphi, C)$ is any nonempty set U of vertices of G such that

$$\forall v \in U \cap V_\sigma, \quad vE \subseteq U \quad \text{and} \quad \forall v \in U \cap V_{1-\sigma}, \quad vE \cap U \neq \emptyset$$

If the token is in a σ-trap U then player $1-\sigma$ can play a strategy consisting in choosing always successors inside of U. Since each $(1-\sigma)$-vertex in U has always at least one successor in U player $1-\sigma$ can always take a move consistent with this strategy. On the other hand, since all successors of σ-vertices in U are also in U player σ has no possibility to force the token outside of U.

Let us note finally that if U is a σ-trap in G then $G[U]$ is a subarena of G, the inverse, however, is not true in general, there are subarenas that are not traps.

Example 2. Let us consider the arena G of Fig. 2 (colours are omitted, circles denote 0-vertices, squares 1-vertices). Then the set $\{v_1, v_8\}$ is a 1-trap, while the sets $\{v_1, v_2, v_3, v_4, v_8\}$ and $\{v_4, v_5\}$ are 0-traps. The set $\{v_4, v_5, v_6, v_7\}$ induces a subarena in G but is neither a 0-trap nor a 1-trap in G.

The reader can verify readily the following fact describing the structure of nested σ-traps.

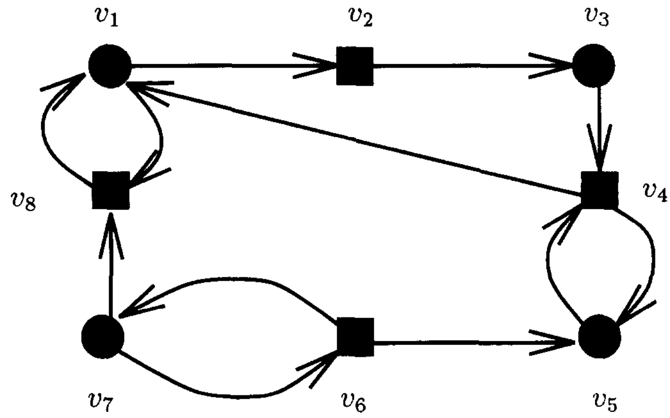

Fig. 2.

Lemma 3. *Let $G=(V_0,V_1,E,\varphi,C)$ be an arena and let $X\subseteq V$ be a σ-trap in G. For each nonempty subset Y of X, Y is a σ-trap in G iff Y is a σ-trap in the subarena $G[X]$ induced by X.*

Note that the equivalence of Lemma 3 does not hold for nested traps of different types: if X is a σ-trap in G and Y is a $1-\sigma$-trap in $G[X]$ then in general Y is not a trap of any kind (neither σ nor $1-\sigma$) in G.

2.3. Attractor sets and attractor strategies

In this subsection we describe an important auxiliary strategy. In fact it is nothing else but a well-known strategy used in open games in descriptive set theory [10].

Let X be any nonempty set of vertices of an arena G. We are looking for the greatest set $\mathrm{Attr}^\sigma(G,X)\subseteq V$ of vertices such that player σ has a strategy allowing him to attract the token from any vertex of $\mathrm{Attr}^\sigma(G,X)$ to X in a finite (possibly 0) number of steps.

Consider the following inductively defined sequence of sets:

$$X_0=X$$

and

$$X_{i+1}=X_i\cup\{v\in V_\sigma \mid vE\cap X_i\neq\emptyset\}\cup\{v\in V_{1-\sigma}\mid vE\subseteq X_i\} \tag{1}$$

and set

$$\mathrm{Attr}^\sigma(G,X)=\bigcup_{i\geqslant 0}X_i$$

With each vertex $v \in \mathrm{Attr}^\sigma(G,X)$ we associate the rank of v: $\mathrm{rank}(v) = \min\{i \mid v \in X_i\}$. Now it suffices to note that if $\mathrm{rank}(v) = i+1$, i.e. if $v \in X_{i+1} \setminus X_i$, then

- either $v \in V_\sigma$ and v has at least one successor in X_i, i.e. v has at least one successor of rank $\leqslant i$,
- or $v \in V_{1-\sigma}$ and all successors of v are in X_i, i.e. they have all ranks $\leqslant i$.

Thus the obvious (memoryless) strategy for player σ to attract the token to X consists in choosing at each step vertices with a rank smaller than that of the current vertex:

$$\text{for} \quad v \in V_\sigma, \quad \mathrm{attr}^\sigma(G,X)(v) = \{w \in vE \mid \mathrm{rank}(w) < \mathrm{rank}(v)\}$$

In any play p starting from a vertex of $\mathrm{Attr}^\sigma(G,X)$ and consistent with the strategy $\mathrm{attr}^\sigma(G,X)$ the ranks of visited vertices form a strictly decreasing sequence and therefore after a finite number of steps the token hits the set X of vertices of rank 0.

To show that $\mathrm{Attr}^\sigma(G,X)$ is the greatest set such that player σ has a strategy to attract the token to X it is sufficient to verify that $V \setminus \mathrm{Attr}^\sigma(G,X)$ is a σ-trap, which would imply that player $1-\sigma$ has a strategy to keep the token in $V \setminus \mathrm{Attr}^\sigma(G,X)$ forever.

If a σ-vertex v has a successor $w \in \mathrm{Attr}^\sigma(G,X)$ then $w \in X_i$ for some i implying $v \in X_{i+1}$, i.e. v lies in $\mathrm{Attr}^\sigma(G,X)$ itself. On the other hand, if a $1-\sigma$-vertex v has all successors in $\mathrm{Attr}^\sigma(G,X)$ then taking n to be the maximum of the ranks of these successors we obtain $v \in X_{n+1}$, i.e. $v \in \mathrm{Attr}^\sigma(G,X)$ as well. (Note that this maximum is correctly defined only if v has finitely many successors and therefore the argument above is not valid if vertices are allowed to have infinitely many successors.)

In this way we have proved that

Lemma 4. *The set $V \setminus \mathrm{Attr}^\sigma(G,X)$ is a σ-trap in G.*

Let us note also the following simple fact.

Lemma 5. *Let $X \subseteq V$ be a σ-trap in an arena G. Then the $1-\sigma$-attractor set $\mathrm{Attr}^{1-\sigma}(G,X)$ is also a σ-trap in G.*

Proof. Instead of verifying directly if the conditions defining a σ-trap hold for $\mathrm{Attr}^{1-\sigma}(G,X)$ we can note simply that player $1-\sigma$ has a strategy to keep the token forever in $\mathrm{Attr}^{1-\sigma}(G,X)$. This strategy consists in attracting first the token into X and, once in X, in choosing always successors in X. $\square$

Remarks. As it turns out, except for the proof of Lemma 4, the assumption that vertices have finitely many successors is never used in this paper. To deal with arenas that are allowed to have infinitely many successors it suffices to modify the computation of $\mathrm{Attr}^\sigma(G,X)$ in the following way. We define by transfinite induction an increasing sequence X_ξ of subsets of V. The set $X_{\xi+1}$ for a nonlimit ordinal $\xi+1$ is obtained by formula (1) where i should be replaced by ξ. For a limit ordinal ξ we set $X_\xi = \bigcup_{\eta<\xi} X_\eta$. Then $\mathrm{Attr}^\sigma(G,X) = X_\zeta$, where ζ is the smallest ordinal such that

$X_\zeta = X_{\zeta+1}$. The definition of the rank (ranks are now ordinals) and of the attracting strategy remain unchanged. That this strategy attracts the token to X in a finite number of steps results from the fact that each strictly decreasing sequence of ordinals is finite. Lemma 4 remains true with this more general attractor definition since a σ-vertex with a successor in X_ζ belongs to $X_{\zeta+1}$ and similarly an $1-\sigma$-vertex with all successors in X_ζ belongs to $X_{\zeta+1}$. Thus the assertion of Lemma 4 follows from $\mathrm{Attr}^\sigma(G,X) = X_\zeta = X_{\zeta+1}$.

3. Parity games

As a warming exercise preceding the more serious case of games with memory that will be considered later on we examine here a restricted class of games that are called parity games. In these games $C = C_n = \{0,\dots,n\}$, i.e. the set of colours consists of integers between 0 and some fixed non negative integer n. For any play $p = v_0v_1v_2\dots$ on G by

$$\sup(p) = \max\{i \in C_n \mid i = \varphi(v_k) \quad \text{for infinitely many } k\}$$

we denote the maximal colour visited infinitely often. Player 0 wins p if $\sup(p)$ is even, otherwise player 1 wins, i.e. $\sigma = \sup(p) \bmod 2$ is the winning player.

Obviously, parity games constitute just a very special class of games with Muller condition. However, since as it is well known each automaton with Muller acceptance condition can be transformed to an equivalent automaton with parity condition [20], memoryless determinacy of parity games is sufficient to prove complementation lemma for tree automata, cf. [9].

Let $X \subseteq V$ be a $(1-\sigma)$-trap in G. A memoryless strategy f for player σ is said to be winning on X if

- $\forall v \in X \cap V_\sigma$, $\emptyset \neq f(v) \subseteq X$ and
- each play $p = v_0v_1v_2\dots$ starting from any vertex v_0 of X and consistent with f is winning for player σ.

Thus each winning memoryless strategy f for a given player σ is always associated with a trap X for his adversary $1-\sigma$. The condition $f(v) \subseteq X$, for $v \in X \cap V_\sigma$, indicates that if the token is in X then player σ playing according to f will keep it inside X, by $f(v) \neq \emptyset$ such a move is always possible. Since X is a $(1-\sigma)$-trap player $1-\sigma$ has no strategy to force the token outside of X.

Theorem 6. *Let $G = (V_0, V_1, E, \varphi, C_n)$, where $C_n = \{0,\dots,n\}$, be an arena for the parity game. Then the set V of vertices can be partitioned onto two sets W^0 and W^1, called winning sets for player 0 and 1 respectively, and such that, for $\sigma \in \{0,1\}$, W^σ is a $1-\sigma$ trap in G and player σ has a winning memoryless strategy w^σ on W^σ.*

We will give two proofs of this theorem. Both are carried by induction on n.

First proof of Theorem 6. If $n=0$ then each play p visits infinitely often the only existing colour 0 and player 0 wins all possible plays using the trivial strategy that moves the token to any successor vertex.

Suppose that $n \geqslant 1$. Let

$$\sigma = n \bmod 2 \tag{2}$$

be the player that wins if the token visits infinitely often the greatest colour n. We construct by transfinite induction sequences $W_\xi^{1-\sigma}$ and $w_\xi^{1-\sigma}$ such that

(I) each $W_\xi^{1-\sigma} \subseteq V$ is a σ-trap in G and $w_\xi^{1-\sigma}$ is a memoryless winning strategy of player $1-\sigma$ on $W_\xi^{1-\sigma}$,

(II) if $\eta < \xi$ then $W_\eta^{1-\sigma} \subset W_\xi^{1-\sigma}$, i.e. $W_\xi^{1-\sigma}$ is strictly increasing, and $w_\xi^{1-\sigma}$ is an extension of $w_\eta^{1-\sigma}$ from $W_\eta^{1-\sigma}$ to $W_\xi^{1-\sigma}$.

Initially $W_0^{1-\sigma} = \emptyset$. For a limit ordinal ξ we set $W_\xi^{1-\sigma} = \bigcup_{\eta<\xi} W_\eta^{1-\sigma}$ and similarly $w_\xi^{1-\sigma}$ is the union of the strategies $w_\eta^{1-\sigma}$ for $\eta < \xi$ ($w_\xi^{1-\sigma}$ is well-defined since if $\eta_1, \eta_2 < \xi$ then one of the strategies $w_{\eta_1}^{1-\sigma}$ and $w_{\eta_2}^{1-\sigma}$ extends the other). It is easy to see that, as a union of σ-traps, $W_\xi^{1-\sigma}$ is also a σ-trap. The strategy $w_\xi^{1-\sigma}$ is winning on $W_\xi^{1-\sigma}$ since any play p starting in $W_\xi^{1-\sigma}$ and consistent with this strategy is also consistent with a strategy $w_\eta^{1-\sigma}$ for some $\eta < \xi$.

The definition of $W_{\xi+1}^{1-\sigma}$ for a non limit ordinal $\xi+1$ takes more steps. Let $X_\xi = \mathrm{Attr}^{1-\sigma}(G, W_\xi^{1-\sigma})$ Since $W_\xi^{1-\sigma}$ is a σ-trap, X_ξ is also a σ-trap (cf. Lemma 5) and player $1-\sigma$ has an obvious winning memoryless strategy x_ξ on X_ξ, he attracts the token in a finite number of steps to $W_\xi^{1-\sigma}$ and next plays always according to his winning strategy $w_\xi^{1-\sigma}$ on $W_\xi^{1-\sigma}$.

Let $Y_\xi = V \setminus X_\xi$. As a complement of a $1-\sigma$-attractor, by Lemma 4, Y_ξ is a $1-\sigma$-trap in G, in particular $G[Y_\xi]$ is a subarena of G. Now we forget for a moment the vertices of X_ξ and we play on $G[Y_\xi]$. Let us take the set $N_\xi = \{v \in Y_\xi \mid n = \varphi(v)\}$ of all vertices of $G[Y_\xi]$ coloured by the maximal colour n and consider the complement of the attractor $\mathrm{Attr}^\sigma(G[Y_\xi], N_\xi)$:

$$Z_\xi = Y_\xi \setminus \mathrm{Attr}^\sigma(G[Y_\xi], N_\xi) \tag{3}$$

Note two facts: (1) Z_ξ is a σ-trap in the arena $G[Y_\xi]$ (as a complement of a σ-attractor in this arena) and (2) vertices of Z_ξ are exclusively coloured by elements of $\{0, \ldots, n-1\}$.

Therefore we can solve the parity game on $G[Z_\xi]$ by applying the inductive hypothesis and we find a partition of Z_ξ onto the winning sets Z_ξ^0 and Z_ξ^1 for players 0 and 1 and two corresponding winning memoryless strategies z_ξ^0 and z_ξ^1.

We can define finally the set $W_{\xi+1}^{1-\sigma}$: $W_{\xi+1}^{1-\sigma} = X_\xi \cup Z_\xi^{1-\sigma}$.

The definition of the strategy $w_{\xi+1}^{1-\sigma}$ on $W_{\xi+1}^{1-\sigma}$ is obvious: if the token is in X_ξ then player $1-\sigma$ plays according to x_ξ, otherwise, if the token is in $Z_\xi^{1-\sigma}$ then he uses his strategy $z_\xi^{1-\sigma}$ that was found solving the parity game on $G[Z_\xi]$.

Let us verify that σ-vertices in $W^{1-\sigma}_{\xi+1}$ have successors only inside of $W^{1-\sigma}_{\xi+1}$. On one hand, σ-vertices in X_ξ have successors only in X_ξ since X_ξ is a σ-trap in G. On the other hand, σ-vertices of $Z^{1-\sigma}_\xi$ cannot have successors neither in $\mathrm{Attr}^\sigma(G[Y_\xi], N_\xi)$ (the complement of a σ-attractor is a σ-trap) nor in Z^σ_ξ (since $Z^{1-\sigma}_\xi$ itself is a σ-trap in $G[Z_\xi]$). It is also quite obvious that each $1-\sigma$-vertex of $W^{1-\sigma}_{\xi+1}$ has at least one successor in $W^{1-\sigma}_{\xi+1}$. Thus we can conclude that $W^{1-\sigma}_{\xi+1}$ is a σ-trap.

Let $p = v_0 v_1 v_2 \ldots$ be a play starting from a vertex $v_0 \in W^{1-\sigma}_{\xi+1}$ and consistent with the strategy $w^{1-\sigma}_{\xi+1}$. There are two possibilities:

(1) If the token hits at some moment the set X_ξ then from this moment on it will stay in X_ξ forever and player $1-\sigma$ playing according to his winning strategy x_ξ on X_ξ wins the play (recall that once the token in X_ξ, the adversary σ has no possibility to move the token outside).

(2) If the token stays forever in $Z^{1-\sigma}_\xi$ never hitting X_ξ then all the play is consistent in fact with the strategy $z^{1-\sigma}_\xi$ and $1-\sigma$ wins as well.

Let ζ be the smallest ordinal such that

$$W^{1-\sigma}_\zeta = W^{1-\sigma}_{\zeta+1} \tag{4}$$

We claim that $W^{1-\sigma} = W^{1-\sigma}_\zeta$ is the winning set for player $1-\sigma$ in the whole parity game on G (thus we stop the construction when the presented method fails to extend the winning set of player $1-\sigma$). Obviously $w^{1-\sigma}_\zeta$ becomes the winning strategy $w^{1-\sigma}$ for player $1-\sigma$.

It remains to construct a winning memoryless strategy for player σ on $W^\sigma = V \setminus W^{1-\sigma}$.

First note however that $W^{1-\sigma}_\zeta \subseteq X_\zeta = \mathrm{Attr}^{1-\sigma}(G, W^{1-\sigma}_\zeta) \subseteq W^{1-\sigma}_{\zeta+1} = W^{1-\sigma}_\zeta$ implying $W^{1-\sigma} = \mathrm{Attr}^{1-\sigma}(G, W^{1-\sigma})$, thus W^σ is an $1-\sigma$-trap as required (as a complement of $(1-\sigma)$-attractor).

Let

$$N = \{v \in W^\sigma \mid n \in \varphi(v)\} \quad \text{and} \quad Z = W^\sigma \setminus \mathrm{Attr}^\sigma(G[W^\sigma], N) \tag{5}$$

Again the arena $G[Z]$ is coloured by $\{0, \ldots, n-1\}$. This time however, when we solve using the induction hypothesis the parity game on $G[Z]$, we find the winning set of player σ to be empty (otherwise it would be added to $W^{1-\sigma}_\zeta$ contradicting (4)). Thus σ has a memoryless strategy z^σ that allows him to win everywhere on $G[Z]$.

The strategy w^σ is defined in the following way: for $v \in W^\sigma \cap V_\sigma$ set

$$w^\sigma(v) = \begin{cases} z^\sigma(v) & \text{if } v \in Z \\ \mathrm{attr}^\sigma(G[W^\sigma], N)(v) & \text{if } v \in \mathrm{Attr}^\sigma(G[W^\sigma], N) \setminus N \\ vE \cap W^\sigma & \text{if } v \in N \end{cases} \tag{6}$$

Thus player σ plays in the following way on W^σ. If the token visits a vertex $v \in N$ coloured by the maximal colour n then he moves it to any successor vertex inside of his winning set W^σ (there is always at least one such successor vertex since W^σ is a $1-\sigma$-trap). If the token visits $\mathrm{Attr}^\sigma(G[W^\sigma], N) \setminus N$ then σ attracts it in a finite number of steps to N, i.e. the token will visit the maximal colour after a finite number of moves. If the token is in Z then σ plays his winning strategy z^σ on Z.

Let p be any play consistent with w^σ and starting at some vertex of W^σ. Then either the token visits infinitely often the maximal colour n (i.e. the set N) and σ wins by (2) or, from some moment on, the token stays forever inside of Z and in this case some infinite suffix of p is consistent with z^σ and player σ wins as well. □

Second proof of Theorem 6. It is possible to give a bit shorter non-constructive proof of Theorem 6. Again we proceed by induction on n and we sketch quickly the inductive step. We assume again that (2) holds. Let $\mathfrak{W}^{1-\sigma} = \{W_q^{1-\sigma}\}_{q\in Q}$ be the family of all σ-traps in G such that player $1-\sigma$ has a winning memoryless strategy $w_q^{1-\sigma}$ on $W_q^{1-\sigma}$.

Let $W^{1-\sigma} = \bigcup_{q\in Q} W_q^{1-\sigma}$. We show that $W^{1-\sigma} \in \mathfrak{W}^{1-\sigma}$, i.e. $W^{1-\sigma}$ is the greatest element of $\mathfrak{W}^{1-\sigma}$.

First note that $W^{1-\sigma}$ is a σ-trap as the union of σ-traps. A memoryless strategy $w^{1-\sigma}$ on $W^{1-\sigma}$ is constructed in the following way. Fix a well-ordering relation $<$ on Q. Then for $v \in W^{1-\sigma} \cap V_{1-\sigma}$, we set $w^{1-\sigma}(v) = w_q^{1-\sigma}(v)$, where q is the minimal element of Q (w.r.t. $<$) such that $v \in W_q^{1-\sigma}$.

Let $p = v_0v_1v_2\ldots$ be a play consistent with $w^{1-\sigma}$ and let, for all $i, q_i = \min\{q \in Q \mid v_i \in W_q^{1-\sigma}\}$. Obviously $v_i \in W_{q_i}^{1-\sigma}$. What is more interesting is that the successor vertex v_{i+1} belongs to $W_{q_i}^{1-\sigma}$ as well (either v_i is an σ-vertex and then all its successors, in particular v_{i+1}, belong to the σ-trap $W_{q_i}^{1-\sigma}$ or v_i is an $1-\sigma$-vertex and then $v_{i+1} = w^{1-\sigma}(v_i) = w_{q_i}^{1-\sigma}(v_i) \in W_{q_i}^{1-\sigma}$). However $v_{i+1} \in W_{q_i}^{1-\sigma}$ implies that $q_{i+1} \leqslant q_i$. Since an infinite non-increasing sequence of elements of a well-ordered set is ultimately constant we conclude that some suffix of p is consistent with one of the strategies $w_q^{1-\sigma}$ and $1-\sigma$ wins p.

The winning strategy $w^{1-\sigma}$ on $W^{1-\sigma}$ can be extended to a winning strategy on $\mathrm{Attr}^{1-\sigma}(G, W^{1-\sigma})$ (by attracting the token to $W^{1-\sigma}$ and next playing $w^{1-\sigma}$), i.e. $\mathrm{Attr}^{1-\sigma}(G, W^{1-\sigma}) \in \mathfrak{W}^{1-\sigma}$. But the maximality of $W^{1-\sigma}$ implies that in fact we should have the equality $W^{1-\sigma} = \mathrm{Attr}^{1-\sigma}(G, W^{1-\sigma})$ and we can see that, as a complement of a $1-\sigma$-attractor, W^σ is a $1-\sigma$-trap.

The winning strategy w^σ for player σ on W^σ is constructed exactly as in the first proof, i.e. we take N and Z as in (5), solve inductively the parity game on $G[Z]$ (again σ wins everywhere on $G[Z]$ otherwise we could extend $W^{1-\sigma}$) and compose the strategies as in (6). □

In the proof above to show that $1-\sigma$ has the maximal winning set we have used the fact that each set can be well-ordered. Alternatively, one can deduce it easily from the Zorn lemma [6].

The fact that both players have memoryless winning strategies in parity games was proved for the first time independently by Emerson and Jutla [9] and Mostowski [21] (McNaughton [19] gives a simple proof for games on finite graphs). And this result is sufficient for applications to automata on infinite trees. In fact it seems that at present the simplest and most elementary way to obtain the Rabin complementation lemma for a particular class of tree automata consists in three steps: (1) first prove the equivalence of this class with parity automata (let us note here that it is well-known that Muller/Rabin/Streett/parity accepting conditions are all equivalent, probably for the first time the equivalence of parity and Muller automata was noted explicitly by A. Mostowski [20] and this equivalence implies the other equivalences above), (2) in the second step prove by induction the memoryless determinacy for parity games (all inductive proofs of memoryless determinacy published up to now, cf. [21, 29] are in fact quite similar to the proofs presented in this section) and finally, (3) apply the determinacy to show the complementation lemma for parity automata.

There are also non-inductive proofs of memoryless determinacy of parity games. However, they seem to be a bit more difficult than their inductive counterparts. Typically such proofs are given in the framework of the μ-calculus, the proof due to Emerson and Jutla [9] belongs to this class. Their proof goes through two stages. First they give a μ-calculus formula F expressing the set W of winning positions of a player – in this formula the number of alternations of the least and the greatest fixpoint is proportional to the length of the chain. Now it is possible to deduce that the complement of W is indeed the set of winning positions for the adversary from the fact that the negation of F has the same form as F after exchanging the roles of both players. The important feature of the proofs using the μ-calculus is that it is possible to calculate the winning set of both players independently. This contrasts with the inductive proofs where the first player for which the winning set is calculated is predetermined by the winning condition (in the proof given in this section we are obliged to begin with player $1-\sigma$, for $\sigma = n \bmod 2$, to apply the induction on n). In the proofs using the μ-calculus the role of both players is perfectly symmetrical; the winning set of one player is determined independently from the winning set of the other. In the second step of their proof Emerson and Jutla label vertices of the winning set W by sequences of ordinals, the lexicographic order on these sequences is used by the winning player to choose the successor vertex (the ordinal sequences labelling vertices are of fixed length depending on n, thus they are well ordered by the lexicographic order). Although not difficult to follow, the approach of [9] seems to be less elementary than an approach via an inductive proof. First, some fluency in the μ-calculus is necessary to understand the μ-formula expressing the winning set and next, the construction of their winning strategy – which they separate from the construction of the winning set – is also more complicated.

We should admit however, that the μ-calculus approach gives more than just memoryless determinacy since the fact that winning sets are expressible as μ-calculus formulas is of independent interest. And it is not at all clear if this result can be obtained directly from inductive proofs (of course we can always deduce it in a circuitous way but this is not what we are looking for here). The "non-constructive" inductive proofs like [21, 29] or the second proof from the present section show only the existence of winning sets and winning strategies and therefore are impossible to translate into μ-formulas. In the first proof in this section the winning set is "constructed" by induction, however in (3) set difference is used, which is not monotonic with respect to the second argument, and this makes a translation into a μ-formula problematic.

4. Determinacy of games by means of finite memory strategies

As noted in Lemma 1 in general the players need some memory for their winning strategies. The memory that we shall use is LAR memory introduced by Gurevich and Harrington [11]. However, in contrast to [11] where all colours were recorded in LAR memory, in the approach presented below the set of recorded colours depends on the winning condition.

4.1. Determinacy of games by LAR*-strategies*

The Later Appearance Record (LAR) for a set B of colours, $B \subseteq C$, is simply a finite deterministic transition system $\mathrm{LAR}_B = (C, M_B, \delta_B)$ with the input alphabet C and where the set M_B of states and the transition mapping $\delta_B : M_B \times C^* \to M_B$ are defined in the following way:

M_B consists of all words $x \in B^*$ such that each colour $c \in B$ appears exactly once in x, in particular if $B = \emptyset$ then M_B contains just one state – the empty word. Obviously, the number of elements of M_B equals to the number of permutations on B. For $m \in M_B$ and $c \in C$,

$$\delta_B(m,c) = \begin{cases} x_1x_2c & \text{if } c \in B \quad \text{and} \quad m = x_1cx_2 \text{ for some } x_1, x_2 \in B^* \\ m & \text{if } c \in C \backslash B \end{cases}$$

with the usual extension to all words of $C^* : \delta_B(m,\varepsilon) = m$ for the empty word ε and $\delta_B(m,uc) = \delta_B(\delta_B(m,u),c)$ for $u \in C^*$ and $c \in C$.

To describe conveniently how player's memory is updated we assume in this section that the mapping φ colouring vertices is extended to a total mapping into $C \cup \{\varepsilon\}$ and we set $\varphi(v) = \varepsilon$ for uncoloured vertices, i.e. we label them with the empty word.

Suppose that the memory used by player σ is a LAR_B memory M_B for some $B \subseteq C$, his current memory state equals $m \in M_B$ and that the token visits a vertex v. If the token is moved to a successor vertex w, $(v,w) \in E$, either by player σ or by his adversary, then player σ updates his memory to the new state $\delta_B(m, \varphi(w))$. In particular, the memory remains unchanged if the token visits an uncoloured vertex or a vertex coloured by $c \in C \backslash B$.

A nonempty (finite or infinite) sequence $h = (v_0, m_0), (v_1, m_1), (v_2, m_2), \ldots$ of consecutive token positions $v_i \in V$ and LAR_B memory states $m_i \in M_B$ constitutes a LAR_B *history* if

- $p = v_0v_1v_2\ldots$ is a play (or a partial play) in G and
- $\forall i, m_i \in M_B$ and $m_{i+1} = \delta_B(m_i, \varphi(v_{i+1}))$.

An LAR_B strategy for a player σ is a mapping

$$f : V_\sigma \times M_B \to \mathscr{P}(V_{1-\sigma})$$

such that $\forall v \in V_\sigma$ and $\forall m \in M_B$, $f(v,m) \subseteq vE$.

The intuitive notion of player σ playing according to f (his adversary can make any valid moves) is now captured by the concept of histories consistent with the strategy f:

A LAR_B history $h=(v_0,m_0),(v_1,m_1),(v_2,m_2),\ldots$ is said to be *consistent* with an LAR_B strategy f for player σ if

$$\forall i, \quad \text{if } v_i \in V_\sigma \quad \text{then } v_{i+1} \in f(v_i,m_i)$$

Let $U \subseteq V$ be a $(1-\sigma)$-trap. A LAR_B strategy f for player σ is *winning on* U (for σ) if it satisfies the following two conditions:

(**W** I) $\forall v \in U \cap V_\sigma, \forall m \in M_B,\ \emptyset \neq f(v,m) \subseteq U$ and

(**W** II) for any vertex $v_0 \in U$ and any memory $m_0 \in M_B$ and any history $h=(v_0,m_0)$, $(v_1,m_1),(v_2,m_2),\ldots$ starting at (v_0,m_0) and consistent with f, the corresponding play $p=v_0v_1v_2\ldots$ is winning for player σ.

Thus if the token is in U then (1) player σ playing according to f has always a move consistent with f ($f(v,m)\neq\emptyset$), (2) taking such a move he never sends the token outside of U ($f(v,m)\subseteq U$), and (3) his adversary cannot sent the token outside of U (U is a $(1-\sigma)$-trap).

Note also that there is no distinguished initial token position inside of U, similarly there is no specific initial memory state. Using his winning strategy player σ should win with the token starting from any vertex of U and with any memory state. This property not only dispenses us once for all from specifying the initial conditions but, what is more important, it allows to compose strategies conveniently; once the token enters U player σ can apply his winning strategy on U without bothering about where exactly the token entered U and what was his memory state at that moment.

Given a fixed Muller condition $(\mathscr{F}_0,\mathscr{F}_1)$ over C we distinguish for each player σ a set $U_\sigma \subseteq C$ of his useful colours. Our next aim is to construct a winning LAR_{U_σ} strategy for σ. Note that in general the sets of useful colours for players 0 and 1 are different. Thus the situation of both players is not exactly symmetrical, one of them may use less auxiliary memory than the other. An especially interesting case arises when the set of useful colours of player σ is empty since then his winning strategy reduces to a memoryless strategy.

Let $\mathscr{F} \subseteq \mathscr{P}_0(C)$ be a collection of nonempty subsets of C. Then $\mathfrak{Usf}(\mathscr{F})$ is a subset of C defined in the following way:

$$\mathfrak{Usf}(\mathscr{F}) = \bigcup_{\substack{X_1,X_2\notin\mathscr{F}\\ X_1\cup X_2\in\mathscr{F}}} X_1 \triangle X_2 \tag{7}$$

where $X_1 \triangle X_2 = (X_1 \backslash X_2) \cup (X_2 \backslash X_1)$ stands for the symmetric difference of sets X_1 and X_2.

Let $\overline{\mathscr{F}} = \mathscr{P}_0(C)\setminus\mathscr{F}$. Let us recall that pairs $X_1, X_2 \in \overline{\mathscr{F}}$ such that $X_1 \cup X_2 \in \mathscr{F}$ are called splits in $\overline{\mathscr{F}}$ (see Section 2.1). Thus $\mathfrak{Usf}(\mathscr{F})$ is the union of $X_1 \triangle X_2$ over all splits X_1, X_2 in $\overline{\mathscr{F}}$, in particular $\mathfrak{Usf}(\mathscr{F})$ is empty iff $\overline{\mathscr{F}}$ does not contain splits.

For any player $\sigma \in \{0,1\}$, the set $\mathfrak{Usf}(\mathscr{F}_\sigma)$ is said to be the sets of his *useful colours* with respect to the condition $(\mathscr{F}_0, \mathscr{F}_1)$. Now we are ready to formulate the main result.

Theorem 7. *Let $\mathscr{G} = (G, (\mathscr{F}_0, \mathscr{F}_1))$ be a game on an arena G. Then there exists a partition of the set V of vertices of G onto two sets W^0 and W^1 that are called winning sets for player 0 and 1, respectively, and such that for each player $\sigma \in \{0,1\}$*

- *W^σ is a $(1-\sigma)$-trap in G and*
- *player σ has a winning LAR_{U_σ} strategy w^σ on W^σ, where $U_\sigma = \mathfrak{Usf}(\mathscr{F}_\sigma)$ is the set of his useful colours w.r.t. $(\mathscr{F}_0, \mathscr{F}_1)$.*

If a game $\mathscr{G}$ satisfies the conditions stated in Theorem 7 then we say that $\mathscr{G}$ is solvable by LAR strategies. The rest of the section is devoted to the proof of Theorem 7.

Remark. McNaughton [19] considered a class of games on finite graphs with a distinguished set $W \subseteq V$ of vertices, with the winning condition given by a partition of $\mathscr{P}(W)$ onto two sets $\mathscr{F}_0$ and $\mathscr{F}_1$ and where player σ wins a play p iff the set $\{w \in W \mid w$ is visited infinitely often in $p\}$ belongs to $\mathscr{F}_\sigma$. Direct translation of the results of his paper to our framework would produce winning strategies with the memory size depending on the number of coloured vertices. In the present paper we are looking for strategies where the memory size depends rather on the number of colours, moreover we need a method adapted also for infinite graphs. This implies two modifications: to get a strategy with the memory depending on the number of colours we should replace the induction on the number of distinguished vertices used in [19] by the induction on the number of colours and to cope with infinite graphs we can use a transfinite induction. Both modifications are in fact rather straightforward. What is more subtle and new in our paper with respect to [19, 11] is that we record only some visited colours.

4.2. From subgame strategies to strategies on full games

Notation. For $B \subseteq C$, π_B will denote the erasing morphism $\pi_B : C^* \to C^*$, $\pi_B(c) = c$ if $c \in B$ and $\pi_B(c) = \varepsilon$ for $c \in C\setminus B$. Note that if $A \subseteq B \subseteq C$ and $m \in M_B$ then $\pi_A(m)$ belongs to LAR memory M_A.

In the process of constructing LAR strategies the following situation arises frequently. Suppose that we have a LAR_A strategy $f : V_\sigma \times M_A \to \mathscr{P}(V_{1-\sigma})$ for a player σ defined on some subarena of G and we need to convert it into a LAR_B strategy g with the set B of recorded colours greater than A, $A \subseteq B \subseteq C$. An obvious way to do it is by setting for $v \in V_\sigma$ and $m \in M_B$, $g(v,m) = f(v, \pi_A(m))$, i.e. we take the current memory $m \in M_B$ and after erasing superfluous colours not belonging to A we apply the strategy f. The following lemma exhibits a direct correspondence between LAR_A histories consistent with f and LAR_B histories consistent with g, in particular it implies that the plays resulting from these histories are exactly the same.

Lemma 8. *Under the conditions stated above, let* $h=(v_1,m_1),(v_2,m_2),(v_3,m_3),\ldots$ *be a (possibly partial)* LAR_B *history consistent with* g. *Then* $h'=(v_1,\pi_A(m_1)),(v_2,\pi_A(m_2)),(v_3,\pi_A(m_3)),\ldots$ *is a* LAR_A *history consistent with the strategy* f.

Proof. Easy case analysis shows that, for all $m\in M_B$ and $c\in C$, $\pi_A(\delta_B(m,c))=\delta_A(\pi_A(m),c)$, which yields directly our assertion. $\square$

4.3. Solving the games

The proof of Theorem 7 is carried by induction on the number of colours.

If C contains just one colour then for one of the players, say σ, we have $\mathscr{F}_\sigma=\{C\}$ while for the other player $\mathscr{F}_{1-\sigma}=\emptyset$. Since in this case all possible plays are winning for σ, he can use the trivial memoryless strategy $w^\sigma(v)=vE$ for all $v\in V_\sigma$.

Let $\mathrm{card}(C)=n>1$. By the induction hypothesis the theorem holds for all games with less than n colours.

In the sequel we assume that σ is this of the two players for which

$$C\in\mathscr{F}_\sigma \tag{8}$$

The case $\mathscr{F}_\sigma=\mathscr{P}_0(C)$ is trivial since again all possible plays are winning for σ and he can play according to the trivial memoryless strategy $w^\sigma(v)=vE$ for $v\in V_\sigma$. Thus in the sequel we suppose that $\mathscr{F}_\sigma\neq\mathscr{P}_0(C)$, i.e. $\mathscr{F}_{1-\sigma}$ is nonempty.

Let $\max(\mathscr{F}_{1-\sigma})$ be the family of all maximal elements of $\mathscr{F}_{1-\sigma}$ with respect to the inclusion relation. From this moment on we assume that $\max(\mathscr{F}_{1-\sigma})$ contains a fixed number of $k\geqslant 1$ elements:

$$\max(\mathscr{F}_{1-\sigma})=\{C_0,\ldots,C_{k-1}\} \tag{9}$$

(Obviously, (8) implies that each C_i is a proper subset of C. Note also that any two different elements of $\max(\mathscr{F}_{1-\sigma})$ constitute a split in $\mathscr{F}_{1-\sigma}$.)

4.3.1. Winning set and winning strategy for player $1-\sigma$

Our first step is to build the winning set $W^{1-\sigma}$ and the winning $\mathrm{LAR}_{U_{1-\sigma}}$ strategy $w^{1-\sigma}$ for the player $1-\sigma$, where $U_{1-\sigma}=\mathfrak{Usf}(\mathscr{F}_{1-\sigma})$ is the set of his useful colours. To this end we construct by transfinite induction a sequence $W_\xi^{1-\sigma}$ of subsets of V and simultaneously a sequence $w_\xi^{1-\sigma}$ of strategies satisfying the following conditions:

(S1) each $W_\xi^{1-\sigma}\subseteq V$ is a σ-trap in G and $w_\xi^{1-\sigma}$ is a winning $\mathrm{LAR}_{U_{1-\sigma}}$ strategy for player $1-\sigma$ on $W_\xi^{1-\sigma}$,

(S2) the sequence $W_\xi^{1-\sigma}$ is strictly increasing ($W_\eta^{1-\sigma}\subset W_\xi^{1-\sigma}$ for $\eta<\xi$) and if $\eta<\xi$ then strategies $w_\eta^{1-\sigma}$ and $w_\xi^{1-\sigma}$ coincide on $W_\eta^{1-\sigma}$ for any possible memory values.

Let us note that the fact that $W_\xi^{1-\sigma}$ is a strictly increasing sequence of subsets of a fixed set V determines its maximal length, it is bounded by the minimal ordinal of cardinality greater then card(V). In particular for countable arenas the length of the sequence $W_\xi^{1-\sigma}$ is bounded by ω_1, the first uncountable ordinal.

Initially $W_0^{1-\sigma}=\emptyset$.

If ξ is a limit ordinal then we set simply $W_\xi^{1-\sigma}=\bigcup_{\eta<\xi}W_\eta^{1-\sigma}$. As a union of σ-traps, $W_\xi^{1-\sigma}$ is also a σ-trap in G. The strategy $w_\xi^{1-\sigma}$ is just the union of the strategies $w_\eta^{1-\sigma}$: for a $1-\sigma$-vertex $v\in W_\xi^{1-\sigma}$ and $m\in M_{U_{1-\sigma}}$, $w_\xi^{1-\sigma}(v,m)=w_\eta^{1-\sigma}(v,m)$, where η is any ordinal less than ξ and such that $v\in W_\eta^{1-\sigma}$ (by (S2) this definition of $w_\xi^{1-\sigma}$ is unambiguous).

To see that $w_\xi^{1-\sigma}$ is winning for $1-\sigma$ it suffices to note that if h is a $\mathrm{LAR}_{U_{1-\sigma}}$ history consistent with $w_\xi^{1-\sigma}$ and starting at a vertex $v\in W_\xi^{1-\sigma}$ then $v\in W_\eta^{1-\sigma}$ for some $\eta<\xi$ and h is consistent with the strategy $w_\eta^{1-\sigma}$.

The construction of $W_{\xi+1}^{1-\sigma}$ and $w_{\xi+1}^{1-\sigma}$ for a nonlimit ordinal $\xi+1$ is more involved. Let

$$X_\xi=\mathrm{Attr}^{1-\sigma}(G,W_\xi^{1-\sigma}) \tag{10}$$

Since $W_\xi^{1-\sigma}$ is a σ-trap, by Lemma 5, X_ξ is a σ-trap. The strategy x_ξ of player $1-\sigma$ on X_ξ is obvious: he attracts the token in a finite number of steps to $W_\xi^{1-\sigma}$ and next plays according to his winning strategy $w_\xi^{1-\sigma}$. Formally, for $v\in V_{1-\sigma}$ and $m\in M_{U_{1-\sigma}}$,

$$x_\xi(v,m)=\begin{cases} w_\xi^{1-\sigma}(v,m) & \text{if } v\in W_\xi^{1-\sigma}\\ \mathrm{attr}^{1-\sigma}(G,W_\xi^{1-\sigma})(v) & \text{if } v\in X_\xi\setminus W_\xi^{1-\sigma}\end{cases} \tag{11}$$

Let $Y_\xi=V\setminus X_\xi$.

As a complement of a $1-\sigma$-attractor Y_ξ is a $1-\sigma$-trap in G, in particular $G[Y_\xi]$ is a subarena of G.

For all $0\leqslant t<k$, where $k=\mathrm{card}(\max(\mathscr{F}_{1-\sigma}))$, cf. (9), let

$$N_{t,\xi}=\{v\in Y_\xi\mid \varphi(v)\in C\setminus C_t\}$$

be the set of vertices of the arena $G[Y_\xi]$ that are coloured by colours *not* in C_t. Let

$$Z_{t,\xi}=Y_\xi\setminus\mathrm{Attr}^\sigma(G[Y_\xi],N_{t,\xi})$$

i.e. $Z_{t,\xi}$ is the complement, with respect to Y_ξ, of the set $\mathrm{Attr}^\sigma(G[Y_\xi],N_{t,\xi})$ consisting of vertices where player σ has a strategy in the subarena $G[Y_\xi]$ to force the token to visit a colour of $C\setminus C_t$.

Note that

(A) each vertex of $Z_{t,\xi}$ is either coloured by a colour of C_t or uncoloured and

(B) as a complement of a σ-attractor, $Z_{t,\xi}$ is a σ-trap in $G[Y_\xi]$, in particular $G[Z_{t,\xi}]$ is a subarena of $G[Y_\xi]$ and of G.

From (A) it follows that the subarena $G[Z_{t,\xi}]$ can be considered as coloured by C_t rather than C. Accordingly, when playing on $G[Z_{t,\xi}]$, we can also modify the winning conditions of both players by taking

$$\mathscr{F}_0^t=\mathscr{F}_0\cap\mathscr{P}_0(C_t) \quad\text{and}\quad \mathscr{F}_1^t=\mathscr{F}_1\cap\mathscr{P}_0(C_t) \tag{12}$$

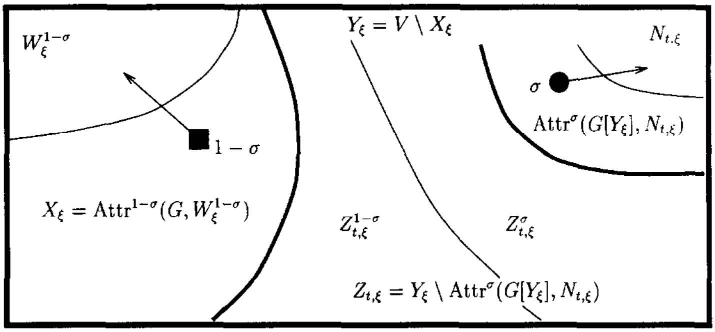

Fig. 3. The set $W^{1-\sigma}_{\xi+1}$ is the union of X_ξ and $Z^{1-\sigma}_{t,\xi}$, where $Z^{1-\sigma}_{t,\xi}$ is nonempty, except in the last inductive step. The strategy $w^{1-\sigma}_{\xi+1}$ for player $1-\sigma$ on $W^{1-\sigma}_{\xi+1}$ is constructed in the natural way: if the token is in $Z^{1-\sigma}_{t,\xi}$ then $1-\sigma$ plays according to the strategy $z^{1-\sigma}_{t,\xi}$, otherwise if the token is in X_ξ then $1-\sigma$ attracts it in a finite number of steps to $W^{1-\sigma}_{\xi}$ and next plays according to $w^{1-\sigma}_{\xi}$.

as the winning conditions for player 0 and 1 respectively. (We should note that $(\mathscr{F}^t_0, \mathscr{F}^t_1)$ defined above is a partition of $\mathscr{P}_0(C_t)$. Moreover, the winning conditions $(\mathscr{F}^t_0, \mathscr{F}^t_1)$ and $(\mathscr{F}_0, \mathscr{F}_1)$ designate the same winner for all plays $p = v_0 v_1 v_2 \dots$ such that $\forall i,\ v_i \in Z_{t,\xi}$.)

Let us examine the resulting game

$$(G[Z_{t,\xi}], (\mathscr{F}^t_0, \mathscr{F}^t_1)) \tag{13}$$

Since this game is played on the arena coloured by a proper subset C_t of C we can apply the induction hypothesis to find the partition of $Z_{t,\xi}$ onto two sets $Z^0_{t,\xi}$ and $Z^1_{t,\xi}$ winning for players 0 and 1, respectively, and also their winning strategies $z^0_{t,\xi}$ and $z^1_{t,\xi}$ on these sets.

Now two cases arise.

(C1) If there exists t, $0 \leqslant t < k$, such that the winning set $Z^{1-\sigma}_{t,\xi}$ of player $1-\sigma$ in the subgame (13) is nonempty then we choose any such t (to fix attention we can always take the least $0 \leqslant t < k$ such that $Z^{1-\sigma}_{t,\xi} \neq \emptyset$) and we set (cf. Fig. 3)

$$W^{1-\sigma}_{\xi+1} = X_\xi \cup Z^{1-\sigma}_{t,\xi}$$

(C2) If for all t, $0 \leqslant t < k$, $Z^{1-\sigma}_{t,\xi} = \emptyset$ then player σ has a winning strategy on the whole set $Z_{t,\xi}$ ($Z_{t,\xi} = Z^{\sigma}_{t,\xi}$) and we set

$$W^{1-\sigma} = W^{1-\sigma}_{\xi+1} = X_\xi \tag{14}$$

i.e. we terminate the transfinite induction and we claim the set $W^{1-\sigma}$ above is the winning set for player $1-\sigma$ in the whole game $(G, (\mathscr{F}_0, \mathscr{F}_1))$.

Before proving that the strategy $w^{1-\sigma}_{\xi+1}$ described informally on Fig. 3 is winning for player $1-\sigma$ we should verify if $W^{1-\sigma}_{\xi+1}$ is really a σ-trap in G.

We have already noted that $Z_{t,\xi}$ is a σ-trap in $G[Y_\xi]$. On the other hand, the set $Z^{1-\sigma}_{t,\xi}$ – being the winning set for player $1-\sigma$ – is a σ-trap in the arena $G[Z_{t,\xi}]$. Thus, by Lemma 3, $Z^{1-\sigma}_{t,\xi}$ is a σ-trap in $G[Y_\xi]$. Therefore each $(1-\sigma)$-vertex in $Z^{1-\sigma}_{t,\xi}$ has at least one successor in $Z^{1-\sigma}_{t,\xi}$ while σ-vertices of $Z^{1-\sigma}_{t,\xi}$ cannot have successors in $Y_\xi \setminus Z^{1-\sigma}_{t,\xi}$. This, and the fact that X_ξ is a σ-trap in G, imply that $W^{1-\sigma}_{\xi+1}$ is a σ-trap in G.

Let us consider the sets

$$U^t_\sigma = \mathfrak{Usf}(\mathscr{F}^t_\sigma) \quad \text{and} \quad U^t_{1-\sigma} = \mathfrak{Usf}(\mathscr{F}^t_{1-\sigma})$$

of useful colours for players σ and $1-\sigma$, respectively, in the game (13). When we solve this game then the winning strategy $z^{1-\sigma}_{t,\xi}$ for the player $1-\sigma$ on his winning set $Z^{1-\sigma}_{t,\xi}$ is a $\mathrm{LAR}_{U^t_{1-\sigma}}$ strategy. However, we can note the inclusions:

$$\begin{aligned} U^t_\sigma &= \mathfrak{Usf}(\mathscr{F}^t_\sigma) \subseteq \mathfrak{Usf}(\mathscr{F}_\sigma) = U_\sigma \\ U^t_{1-\sigma} &= \mathfrak{Usf}(\mathscr{F}^t_{1-\sigma}) \subseteq \mathfrak{Usf}(\mathscr{F}_{1-\sigma}) = U_{1-\sigma} \end{aligned} \tag{15}$$

(If $X_1, X_2 \in \mathscr{F}^t_{1-\sigma} \subseteq \mathscr{F}_{1-\sigma}$ is a split in $\mathscr{F}^t_{1-\sigma}$ then $X_1 \cup X_2 \in \mathscr{F}^t_\sigma \subseteq \mathscr{F}_\sigma$, thus X_1, X_2 is also a split in $\mathscr{F}_{1-\sigma}$, whence the first inclusion. The second one follows by symmetry.)

In particular, the last inclusion implies that, for $m \in M_{U_{1-\sigma}}$, $\pi_{U^t_{1-\sigma}}(m) \in M_{U^t_{1-\sigma}}$. It shows that the formal definition of $w^{1-\sigma}_{\xi+1}$ given below is sound. For $v \in W^{1-\sigma}_{\xi+1} \cap V_{1-\sigma}$ and $m \in \mathrm{M}_{U_{1-\sigma}}$,

$$w^{1-\sigma}_{\xi+1}(v,m) = \begin{cases} z^{1-\sigma}_{t,\xi}(v, \pi_{U^t_{1-\sigma}}(m)) & \text{if } v \in Z^{1-\sigma}_{t,\xi} \\ x_\xi(v,m) & \text{if } v \in X_\xi \end{cases} \tag{16}$$

It remains to show that $w^{1-\sigma}_{\xi+1}$ is winning on $W^{1-\sigma}_{\xi+1}$ for player $1-\sigma$. The first condition (WI) required for the winning strategy – $\emptyset \neq w^{1-\sigma}_{\xi+1}(v,m) \subseteq W^{1-\sigma}_{\xi+1}$ – is obviously satisfied. Let $h = (v_0, m_0), (v_1, m_1), (v_2, m_2), \ldots$ be any history consistent with $w^{1-\sigma}_{\xi+1}$ and starting from a vertex $v_0 \in W^{1-\sigma}_{\xi+1}$. Two cases arise:

Case 1: There exists an i such that $v_i \in X_\xi$.

Then from the moment i onward the history is consistent with x_ξ and therefore $1-\sigma$ wins. (Before entering X_ξ the token may have visited $Z^{1-\sigma}_{t,\xi}$. Although $Z^{1-\sigma}_{t,\xi}$ is a σ-trap in $G[Y_\xi]$, $Z^{1-\sigma}_{t,\xi}$ is not necessarily a σ-trap in the whole arena G – there may exist σ vertices in $Z^{1-\sigma}_{t,\xi}$ with successors in X_ξ and player σ can use this possibility to move the token from $Z^{1-\sigma}_{t,\xi}$ to X_ξ. However, he has no possibility to move the token back from X_ξ to $Z^{1-\sigma}_{t,\xi}$.)

Case 2: For all vertices v_i of the history h, $v_i \in Z^{1-\sigma}_{t,\xi}$.

Then the inclusion $U^t_{1-\sigma} \subseteq U_{1-\sigma}$ and Lemma 8 imply that $(v_0, \pi_{U^t_{1-\sigma}}(m_0))$, $(v_1, \pi_{U^t_{1-\sigma}}(m_1)), (v_2, \pi_{U^t_{1-\sigma}}(m_2)), \ldots$ is consistent with the strategy $z^{1-\sigma}_{t,\xi}$. Since $z^{1-\sigma}_{t,\xi}$ is winning on $Z^{1-\sigma}_{t,\xi}$ in the play (13) we get $\inf(v_0 v_1 v_2 \ldots) \in \mathscr{F}^t_{1-\sigma} \subseteq \mathscr{F}_{1-\sigma}$ and $1-\sigma$ wins again.

This completes the construction of $W^{1-\sigma}$ and $w^{1-\sigma}$.

Remarks. Let us examine the structure of $W^{1-\sigma}$ from the global perspective. We can see that the set $W^{1-\sigma}$ contains a sequence of disjoint subarenas $G[Z^{1-\sigma}_{t,\xi}]$ satisfying the following three conditions:

(P1) subarenas $G[Z^{1-\sigma}_{t,\xi}]$ are coloured by proper subsets C_t of the set C and player $1-\sigma$ has a winning substrategy $z^{1-\sigma}_{t,\xi}$ on each $G[Z^{1-\sigma}_{t,\xi}]$, (P2) if his adversary σ moves the token outside of some $Z^{1-\sigma}_{t,\xi}$, where $\xi > 0$, then player $1-\sigma$, using an attracting strategy, can attract the token to a subarena $G[Z^{1-\sigma}_{t,\eta}]$, with $\eta < \xi$, (P3) for $\xi = 0$ the first set $Z^{1-\sigma}_{t,0}$ constructed at the step 1 is a σ-trap in G thus player σ cannot move the token outside of $Z^{1-\sigma}_{t,0}$.

Since there is no strictly decreasing infinite sequence of ordinals, by (P2) we can see that player σ can move the token outside of $\bigcup_{\xi} Z^{1-\sigma}_{t,\xi}$ only finitely many times. Therefore in each play consistent with the strategy $w^{1-\sigma}$ eventually the token will enter some of the sets $Z^{1-\sigma}_{t,\xi}$ where it will remain forever. From this moment on the history will be consistent with $z^{1-\sigma}_{t,\xi}$.

Note also that when the token is in $W^{1-\sigma} \setminus \bigcup_{\xi} Z^{1-\sigma}_{t,\xi}$ then player $1-\sigma$ does not use and does not need any memory since he applies a memoryless attracting strategy until the token hits some $G[Z^{1-\sigma}_{t,\xi}]$. (Playing on $G[Z^{1-\sigma}_{t,\xi}]$ he may need some memory, the size of this memory depending on the condition (12).)

4.3.2. Winning strategy for player σ

It remains to construct a winning strategy w^{σ} for player σ on $W^{\sigma} = V \setminus W^{1-\sigma}$. First note that from (14) and (10) it follows that, as required, W^{σ} is a $1-\sigma$-trap (as a complement of an $1-\sigma$-attractor).

As in the preceding subsection, for all $0 \leqslant t < k$, we set

$$N_t = \{v \in W^{\sigma} \mid \varphi(v) \in (C \setminus C_t)\} \quad \text{and} \quad Z_t = W^{\sigma} \setminus \mathrm{Attr}^{\sigma}(G[W^{\sigma}], N_t)$$

Recall that each set Z_t – being a complement of a σ-attractor – is a σ-trap in the arena $G[W^{\sigma}]$ and all vertices of $G[Z_t]$ are coloured by elements of C_t. Let us recall also that in the preceding section the inductive definition of $W^{1-\sigma}$ terminated precisely when we have detected that in all subgames

$$(G[Z_t], (\mathscr{F}_0^t, \mathscr{F}_1^t)), \quad 0 \leqslant t < k \tag{17}$$

where $(\mathscr{F}_0^t, \mathscr{F}_1^t)$ are given by (12), the winning set for player $1-\sigma$ is empty. Therefore it is the player σ who has a winning strategy everywhere on $G[Z_t]$. This strategy – let us call it z_t^{σ} – was obtained by the inductive hypothesis (arena $G[Z_t]$ being coloured by a proper subset C_t of C the induction on the number of colours applies).

The idea is to build the strategy w^{σ} for player σ by composing the strategies z_t^{σ} and the attracting strategies in such a way that

- either from some moment on the token remains forever in some of the sets Z_t enabling σ to win by applying the strategy z_t^{σ} or

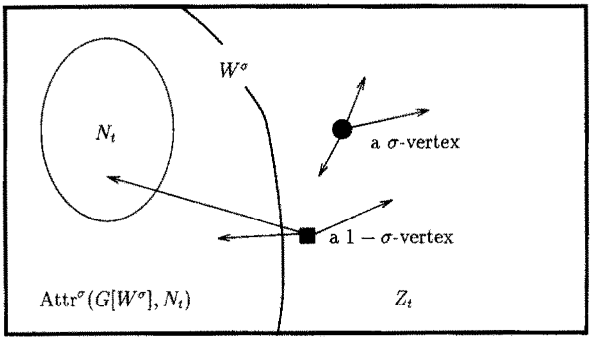

Fig. 4. For each t, $0 \leqslant t < k$, the set W^σ is partitioned on two sets: $\mathrm{Attr}^\sigma(G[W^\sigma], N_t)$ and Z_t. If the token is in $\mathrm{Attr}^\sigma(G[W^\sigma], N_t)$ then player σ can attract it to a vertex coloured by a colour *not* in C_t. If the token is in Z_t then σ can play using his winning strategy z_t^σ. However, $(1-\sigma)$ vertices of Z_t may have successors in $\mathrm{Attr}^\sigma(G[W^\sigma], N_t)$ enabling player $(1-\sigma)$ to move the token from Z_t to $\mathrm{Attr}^\sigma(G[W^\sigma], N_t)$ and preventing σ from applying any fixed z_t^σ again and again.

– the token is moved again and again outside of each Z_t by the adversary player $1-\sigma$ (Fig. 4). In this case player σ uses the attracting strategies $\mathrm{attr}^\sigma(G[W^\sigma], N_t)$ to attract the token infinitely often to each of the sets N_t. If all the sets N_t are visited infinitely often then for each t there is a colour $c \in C \setminus C_t$ visited infinitely often. This implies that the set $\inf(p)$ of colours visited infinitely often cannot be a subset of any $C_t \in \max(\mathscr{F}_{1-\sigma})$ and therefore $\inf(p) \notin \mathscr{F}_{1-\sigma}$ and player σ wins again.

Player σ uses his memory in order to determine which of the possible substrategies should be applied at a given moment.

Goal associated with LAR memory. With each LAR_{U_σ} memory $m \in M_{U_\sigma}$ of player σ we associate an integer $\mathbf{goal}(m)$, $0 \leqslant \mathbf{goal}(m) < k$, called the goal of m, that is calculated in the following way.

For each t, $0 \leqslant t < k$, and for each word $z \in C^*$ let $\mathbf{S}_t(z)$ be the length of the longest suffix of z consisting of letters of C_t:

$$\mathbf{S}_t(z) = \max\{|z_2| \mid z_2 \in C_t^*,\ z_1 \in C^* \text{ and } z = z_1 z_2\}$$

Now, for $m \in \mathrm{LAR}_{U_\sigma}$, $\mathbf{goal}(m)$ is defined to be the t, $0 \leqslant t < k$, for which the value of $\mathbf{S}_t(m)$ is maximal, if there are several t giving the same maximal value of $\mathbf{S}_t(m)$ then we take the smallest of them.

If $k > 1$ then in the given token position several different substrategies may be applicable (for example if $v \in Z_i \cap Z_j$, where $i \neq j$ then we do not know in v if we should apply z_i^σ or z_j^σ or maybe yet another substrategy). The role of $\mathbf{goal}(m)$ is to eliminate this ambiguity. (Note that using LAR memory player σ has no possibility to remember even which substrategy he used in the preceding step).

The formal definition of w^σ is the following. Let v be a σ-vertex of W^σ, $m \in M_{U_\sigma}$, and suppose that $t = \mathbf{goal}(m)$. Then

$$w^\sigma(v,m) = \begin{cases} z_t^\sigma(v, \pi_{U_\sigma^t}(m)) & \text{if } v \in Z_t \\ \mathrm{attr}^\sigma(G[W^\sigma], N_t)(v) & \text{if } v \in \mathrm{Attr}^\sigma(G[W^\sigma], N_t) \setminus N_t \\ vE \cap W^\sigma & \text{if } v \in N_t \end{cases} \tag{18}$$

Let us recall that the strategies z_t^σ winning for σ in the games (17) are $\mathrm{LAR}_{U_\sigma^t}$ strategies, where $U_\sigma^t = \mathfrak{Usf}(\mathscr{F}_\sigma^t)$ is the set of useful colours for player σ w.r.t. $(\mathscr{F}_0^t, \mathscr{F}_1^t)$. But, by (15), $U_\sigma^t \subseteq U_\sigma$, therefore $\pi_{U_\sigma^t}(m) \in M_{U_\sigma^t}$ for $m \in M_{U_\sigma}$. This shows that w^σ is well defined. Moreover, one can note easily that $\emptyset \neq w^\sigma(v,m) \subseteq W^\sigma$ for $v \in W^\sigma \cap V_\sigma$, i.e. w^σ satisfies (W I) – the first of the two conditions required in the definition of a winning strategy.

It remains to show that any LAR_{U_σ} history $h = (v_0, m_0), (v_1, m_1), (v_2, m_2), \ldots$ starting from a vertex $v_0 \in W^\sigma$ and consistent with w^σ is winning for player σ. Let $p = v_0 v_1 v_2 \ldots$ be the corresponding play. We examine two complementary cases.

Case 1: For all $0 \leqslant i < k$, $\inf(p)$ is not a subset of C_i.

From the definition of the sets C_i (cf. (9)) it follows that each element of $\mathscr{F}_{1-\sigma}$ is a subset of some $C_i \in \max(\mathscr{F}_{1-\sigma})$. Therefore $\inf(p)$ cannot belong to $\mathscr{F}_{1-\sigma}$, i.e. $\inf(p) \in \mathscr{F}_\sigma$ and σ wins p.

Case 2: There exists i, $0 \leqslant i < k$, such that $\inf(p) \subseteq C_i$.

Let us take a factorization of p of the form $p = (v_0 \ldots v_l)(v_{l+1} \ldots v_{n-1}) v_n v_{n+1} \ldots$ where

- v_l is the last vertex in p coloured by a colour *not* in $\inf(p)$ and
- $\forall c \in \inf(p), \exists i,\ l < i < n$ such that $c = \varphi(v_i)$, i.e. visiting the vertices between v_{l+1} and v_{n-1} the token visits all colours of $\inf(p)$, possibly with repetitions, and maybe also some uncoloured vertices. (Clearly such a factorization always exists.)

Thus at the moment n the last visit of any colour of $\inf(p)$ is more recent than the last visit of any colour of $C \setminus \inf(p)$. The definition of LAR_{U_σ} transition system implies that the memory m_n of player σ at the moment n can be factorized as uz, where u is some permutation of the useful colours that are visited finitely often or never in p while z is a permutation of the useful colours visited infinitely often in p (in other words u is a permutation of the set $U_\sigma \setminus \inf(p)$ and z is a permutation of $U_\sigma \cap \inf(p)$). Since from the moment n onwards the token visits only the colours of $\inf(p)$, the definition of LAR_{U_σ} transition function implies that for all subsequent memory states only the colours of z are permuted, i.e. the memory m_j has the form

$$\forall j \geqslant n, \quad m_j = u z_j \tag{19}$$

where z_j is a permutation of $U_\sigma \cap \inf(p)$ and u is a fixed (independent of j) permutation of $U_\sigma \setminus \inf(p)$.

To determine $\mathbf{goal}(m_j)$ for all $j \geqslant n$ we need to find $\mathbf{S}_l(m_j)$ for all $0 \leqslant l < k$.

Let us partition the set $\{0,\dots,k-1\}$ onto two sets:

$$I=\{i \mid 0 \leqslant i < k \text{ and } \inf(p) \nsubseteq C_i \in \max(\mathscr{F}_{1-\sigma})\}$$

and

$$L=\{l \mid 0 \leqslant l < k \text{ and } \inf(p) \subseteq C_l \in \max(\mathscr{F}_{1-\sigma})\}$$

(note that the condition defining Case 2 says that $L \neq \emptyset$). We shall show that

$$\text{for all } i \in I \text{ and all } l \in L, \quad \mathbf{S}_i(m_j) < \mathbf{S}_l(m_j) \quad \text{for } j \geqslant n \tag{20}$$

and

$$\text{for any fixed } l \in L, \text{ the value } \mathbf{S}_l(m_j) \text{ is constant for all } j \quad (j \geqslant n) \tag{21}$$

Let $i \in I$ and $l \in L$. Then C_i, C_l, being different elements of $\max(\mathscr{F}_{1-\sigma})$, constitute a split in $\mathscr{F}_{1-\sigma}$. Therefore $C_l \backslash C_i \subseteq C_l \triangle C_i \subseteq \mathfrak{Usf}(\mathscr{F}_\sigma) = U_\sigma$. Definitions of I and L imply that at least one colour c visited infinitely often belongs to $C_l \backslash C_i \subseteq U_\sigma$. In the factorization (19) this colour c belongs to the suffix z_j of colours of U_σ that are visited infinitely often, therefore the longest suffix of m_j consisting of letters of C_i does not contain the letter c of z_j, i.e.

$$\mathbf{S}_i(m_j) < |z_j| = \operatorname{card}(U_\sigma \cap \inf(p)) \tag{22}$$

On the other hand, for $l \in L$ all colours of z_j in the factorization (19) belong to C_l, thus

$$\mathbf{S}_l(m_j) = \mathbf{S}_l(u) + |z_j| = \mathbf{S}_l(u) + \operatorname{card}(U_\sigma \cap \inf(p)) \geqslant \operatorname{card}(U_\sigma \cap \inf(p)) \tag{23}$$

(where $\mathbf{S}_l(u)$ is the length of the longest suffix composed of letters of C_l in the word u in the factorization (19)).

In particular, we can see that (23) implies directly the assertion (21) while comparing (22) and (23) we get (20).

The definition of **goal** and assertions (20), (21) yield directly that from the moment n onwards the goal constantly equals l for some fixed $l \in L$,

$$\exists l \in L,\ \forall j \geqslant n, \quad \mathbf{goal}(m_j) = l$$

Since $\varphi(v_i) \in \inf(p)$ for all coloured vertices v_i for $i \geqslant n$ and, on the other hand, as $l \in L$, $\inf(p) \subseteq C_l$, we can see that from the moment n onwards the token visits only colours of C_l (and possibly some uncoloured vertices). We shall show that this implies that $v_i \in Z_l$ for all $i \geqslant n$.

Indeed, suppose the contrary, i.e. that $v_i \in W^\sigma \backslash Z_l = \operatorname{Attr}^\sigma(G[W^\sigma], N_l)$ for some $i \geqslant n$. Then, since the goal constantly equals l player σ playing according to w^σ would apply the attracting strategy $\operatorname{attr}^\sigma(G[W^\sigma], N_l)$ until the token enters N_l. However, this means that the token will visit a colour of $C \backslash C_l$ contradicting our previous assertion that only colours of C_l are visited in vertices v_i for $i \geqslant n$.

Summarizing, we have proved that

$$\exists l,\ \forall j \geqslant n,\quad v_j \in Z_l \quad \text{and} \quad \mathbf{goal}(m_j) = l$$

which means that from the moment n onwards the token stays in Z_l and, since the goal remains equal l, player σ applies always his winning strategy z_l^σ on Z_l (formally, using Lemma 8, we get that $(v_n, \pi_{U_\sigma^l}(m_n)), (v_{n+1}, \pi_{U_\sigma^l}(m_{n+1})), (v_{n+2}, \pi_{U_\sigma^l}(m_{n+2})), \ldots$ is consistent with z_l^σ). Thus $\inf(p) = \inf(v_n v_{n+1} v_{n+2} \ldots) \in \mathscr{F}_\sigma^l \subseteq \mathscr{F}_\sigma$ and σ wins. This ends the proof of Theorem 7. □

4.4. *Memory in winning strategies – split trees*

In this subsection we shall discuss shortly the role that the memory plays in LAR-strategies. We can trace it more clearly in the case of the player σ from the proof of Theorem 7. His memory $m \in \mathfrak{Usf}(\mathscr{F}_\sigma)$ is used only to calculate $\mathbf{goal}(m)$ in order to choose one of the substrategies.

In particular, if $\max(\mathscr{F}_{1-\sigma}) = \{C_0\}$ consists of only one element then the choice of the substrategy depends only on the current token position and the memory is not used at all at the topmost level to make the right choice, σ either plays his winning strategy on Z_0 or attracts the token to N_0 so that the token visits a colour of $C \setminus C_0$ (of course some memory maybe needed to play on the lower level on Z_0). Note that in this case $\mathfrak{Usf}(\mathscr{F}_\sigma) \cap (C \setminus C_0) = \emptyset$, i.e. there is no useful colour in $C \setminus C_0$ for player σ.

If $\max(\mathscr{F}_{1-\sigma}) = \{C_0, \ldots, C_{k-1}\}$ contains several elements then each pair of different elements of $\max(\mathscr{F}_{1-\sigma})$ is a split in $\mathscr{F}_{1-\sigma}$. Thus the set $P = \bigcup_{i \neq j} C_i \triangle C_j$ is included in $\mathfrak{Usf}(\mathscr{F}_\sigma)$ and in fact only the colours from P are relevant for player σ to the choice of the goal at the topmost level of induction. This becomes obvious if we note that $\mathfrak{Usf}(\mathscr{F}_\sigma) \setminus P \subseteq \bigcap_{0 \leqslant i < k} C_i$, i.e. all the other colours useful for σ belong to all the sets $C_i \in \max(\mathscr{F}_{1-\sigma})$ and therefore do not help to discriminate between possible goals at the topmost level (they may be useful on lower induction levels, in substrategies).

There is a simple way to visualize what happens at all induction levels.

A *split tree* associated with a condition $(\mathscr{F}_0, \mathscr{F}_1)$ is a finite tree T with vertices labelled by couples (α, B), where $\alpha \in \{0, 1\}$ and $B \in \mathscr{F}_\alpha$. This tree is constructed inductively in the following way. The root of T is labelled by (σ, C), where C is the set of all colours and σ is the player for which $C \in \mathscr{F}_\sigma$. Suppose that a vertex x of T is labelled by (α, B), $\alpha \in \{0, 1\}$.

- If $\{B_0, \ldots, B_{l-1}\}$ are the maximal subsets of B belonging to $\mathscr{F}_{1-\alpha}$ then x has l sons labelled $(1-\alpha, B_0), \ldots, (1-\alpha, B_{l-1})$.
- Otherwise, if all subsets of B belong to $\mathscr{F}_\alpha$ then the vertex x is a leaf of T.

In the sequel when referring to a vertex of a split tree T labelled by a couple (α, B) we shall call it frequently an α-vertex labelled by B.

Example 9. Let $C = \{c_0, c_1, c_2, c_3, c_4\}$, $\mathscr{F}_0 = \{\{c_0, c_1\}, \{c_2, c_3, c_4\}, \{c_2, c_3\}, \{c_2, c_4\}, \{c_3\}, \{c_4\}\}$, $\mathscr{F}_1 = \mathscr{P}_0(C) \setminus \mathscr{F}_0$. The split tree for $(\mathscr{F}_0, \mathscr{F}_1)$ is presented on Fig. 5.

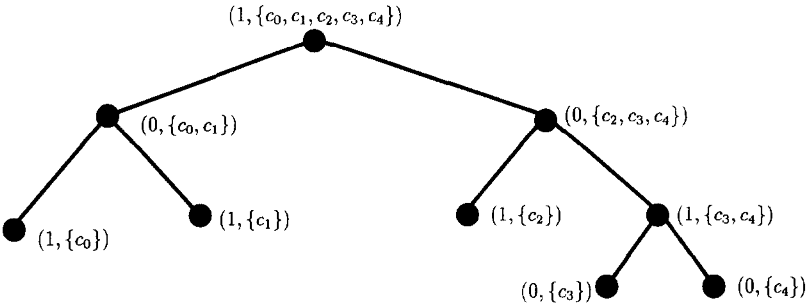

Fig. 5.

Let us note that not only each condition $(\mathscr{F}_0,\mathscr{F}_1)$ determines a unique split tree but also conversely, given a split tree T we can recover the corresponding condition $(\mathscr{F}_0,\mathscr{F}_1)$ thanks to the following property:

Remark 10. Let $\alpha\in\{0,1\}$. A nonempty subset Y of C belongs to $\mathscr{F}_\alpha$ if and only if there exists in the split tree T an α-vertex labelled B such that $Y\subseteq B$ and if this α-vertex has children labelled by B_i, $0\leqslant i<m$ (all these children are $1-\alpha$-vertices) then $Y\nsubseteq B_i$ for all i.

Proof. The right to left implication is obvious. The inverse implication can be established easily by induction on the height of T. □

It is clear now that the procedure of calculating winning sets and winning strategies presented in Section 4.3 can be viewed more adequately as an induction on the height of the split tree rather than induction on the number of colours. Indeed each time the induction hypothesis is applied in the proof of Theorem 7 it is used to solve games on subarenas labelled by $C_t\in\max(\mathscr{F}_{1-\sigma})$ with the winning condition $(\mathscr{F}_0^t,\mathscr{F}_1^t)$ given by (12). But, if T is the split tree for $(\mathscr{F}_0,\mathscr{F}_1)$ then the root r of T has children r_t labelled by $(1-\sigma,C_t)$, $0\leqslant t<k$, and a subtree of T starting at the child r_t is the split tree for $(\mathscr{F}_0^t,\mathscr{F}_1^t)$. The same situation repeats at all induction levels, each (sub)game solved during the induction process corresponds to some vertex in the split tree, the immediate subgames of this game correspond to the children of this vertex.

In particular we can see that not all splits in $\mathscr{F}_{1-\sigma}$ are relevant for player α. If a split X_1,X_2 in $\mathscr{F}_{1-\alpha}$ does not label some siblings in the split tree then during the induction process we never meet a subgame where we need to distinguish these two sets.

Due to this observation we can redefine the set $\mathfrak{Usf}(\mathscr{F}_\alpha)$ of useful colours for player α to be the union of all $B_1\triangle B_2$ such that there exist siblings in the split tree T labelled $(1-\alpha,B_1)$, $(1-\alpha,B_2)$. This modification does not change anything in the proof of Theorem 7. It is not difficult to show an example of a winning condition where this new set of useful colours is smaller than the one calculated with all splits in $\mathscr{F}_{1-\sigma}$. However, similar simple methods reducing the memory size of LAR-strategies are of very limited interest since they are not powerful enough to produce the minimal memory strategies. In fact our main motivation in introducing split trees is quite different: they allow to construct simple non-LAR strategies. We discuss this topic in the next subsection.

4.5. Direct goal strategies

Let T be the split tree for $(\mathscr{F}_0, \mathscr{F}_1)$. We order the children of each vertex x of T and number them by consecutive integers from 0 to $l-1$, where l is the number of children of the vertex x. Then any path from the root r of T to a vertex x is determined in a unique way by the sequence $t_1, t_2, \ldots, t_n$ of integers that indicate the successive directions we should take to go from r to x. Now the idea is to maintain explicitly such a list of goals in player's memory rather than to calculate goals from LAR memory. We shall call this strategy *direct goal strategy*.

In this section we assume again that, as in the proof of Theorem 7, the split tree T has the root labelled by (σ, C) with k children labelled $(1-\sigma, C_t)$, $0 \leqslant t < k$. The winning strategy for player σ is almost trivial when he uses the paths in the split tree as his memory. Suppose that $m = t_1, t_2, \ldots, t_n$ is his current memory state, i.e. t_1 is the topmost level goal of σ, $0 \leqslant t_1 < k$. (If the memory state of player σ is the empty sequence ε then he takes $m' = 0$ as his new memory state, i.e. he (re)initializes his memory with the first goal 0 at the topmost level). Describing the strategy of player σ on W^σ and how he updates his memory we should distinguish three cases (compare with Eq. (18) for $t = t_1$):

(i) If the token is in Z_{t_1} then σ plays according to his wining strategy $z^\sigma_{t_1}$ on Z_{t_1}, passing the subsequence $t_2, \ldots, t_n$ composed of lower level subgoals as the memory to the substrategy $z^\sigma_{t_1}$. (This subsequence identifies the path starting from the vertex $(1-\sigma, C_t)$ of the split tree).

(ii) If the token is $\mathrm{Attr}^\sigma(G[W^\sigma], N_{t_1}) \backslash N_{t_1}$ then σ attracts it in a finite number of steps to N_{t_1}. He can also forget the subgoals since they are only useful on Z_{t_1}. Therefore entering $\mathrm{Attr}^\sigma(G[W^\sigma], N_{t_1}) \backslash N_{t_1}$ player σ sets his memory state to $m' = t_1$ and this memory remains unchanged until the token hits N_{t_1}.

(iii) If the token finally enters N_{t_1} then σ can move it to any successor vertex inside of W^σ. Simultaneously he modifies his memory by setting $m' = t'_1$ as his new memory state, where $t'_1 = (t_1 + 1) \bmod k$ is the subsequent top level goal.

Suppose that $h = (v_0, m_0), (v_1, m_1), \ldots$ is a history consistent with the strategy described above. Two cases are possible.

If there is a moment n such that for all $i \geqslant n$ the topmost goal of m_i constantly equals t for some fixed $0 \leqslant t < k$ then from this moment onwards the token remains always in Z_t (otherwise, leaving Z_t the token would hit eventually N_t provoking the modification of the topmost level goal). Thus σ plays from the moment n onwards according to his winning substrategy z^σ_t and he wins.

The other possibility is when the topmost level goal never stabilizes in h. Then it takes cyclically again and again all values t, $0 \leqslant t < k$. However, each transition of the topmost level goal from t to $(t+1) \bmod k$ takes place iff the token visits a colour of $C \setminus C_t$. If this happens infinitely often for each $0 \leqslant t < k$ then there exists a colour of $C \setminus C_t$ visited infinitely often and therefore $\inf(p)$ is not a subset of any $C_t \in \max(\mathscr{F}_{1-\sigma})$, yielding $\inf(p) \in \mathscr{F}_\sigma$ and σ wins also in this case.

The strategy described above is a finite memory strategy in the sense of the definition of Section 2.1, however its nature is much different from LAR strategies. The main feature of LAR strategies is that the new memory state depends on the current memory state and on the *colour* of the visited vertex. Therefore, since the number of colours is finite, the memory updates are described by a *finite* transition system. In the direct goal strategy the new memory depends on the previous memory and on the new token *position*. For infinite graphs the memory updates cannot be described any more by a finite transition system.

Up to now we have explored only the topmost level goal of player σ and it is still vague what happens at lower levels.

Let us examine player $1-\sigma$. Let us recall that when playing on $G[W^{1-\sigma}]$ he does not use the memory directly at the topmost level (cf. the remarks at the end of Section 4.3.1). If the token visits a subarena $G[Z^{1-\sigma}_{t,\xi}]$ of $G[W^{1-\sigma}]$ then $1-\sigma$ uses the corresponding winning substrategy $z^{1-\sigma}_{t,\xi}$ (together with the associated function describing how he updates his memory, which should be constructed at the same time as $z^{1-\sigma}_{t,\xi}$). If the token visits a vertex of $W^{1-\sigma}$ not belonging to any of the sets $Z^{1-\sigma}_{t,\xi}$ then player $1-\sigma$ plays according to a memoryless attracting strategy, in this case he takes the empty sequence ε as his goal list. In conclusion, we can see that in some sense the topmost level goal of player $1-\sigma$ on $G[W^{1-\sigma}]$ is determined completely by the current token position, inside of some $Z^{1-\sigma}_{t,\xi}$ the "goal" is t, outside there is no goal and memoryless attracting strategy is used.

Note finally that at the level just below the topmost one the roles of players σ and $1-\sigma$ exchange, for example player $1-\sigma$ needs explicitly the topmost goal when playing on $G[Z^{1-\sigma}_{t,\xi}]$ (topmost relatively to the corresponding subtree of the split tree T starting at the child $(1-\sigma, C_t)$ of the root).

Thus in fact player σ and his adversary do not need to maintain the complete path $t_1, t_2, \ldots, t_n$ in T in their memory, each second element of such a path is determined by the current token position. More precisely, for each path $u = t_1, t_2, \ldots, t_n$ starting at the root of the split tree let $\Pi_\sigma(u) = t_1, t_3, t_5, \ldots$ and $\Pi_{1-\sigma}(u) = t_2, t_4, t_6, \ldots$, i.e. we erase from u either all elements on even or on odd positions. All such sequences $\Pi_\sigma(u)$ constitute the memory states of player σ for his winning direct goal strategy on W^σ, while sequences $\Pi_{1-\sigma}(u)$ constitute the memory of his adversary when he plays on $W^{1-\sigma}$ his winning direct goal strategy.

5. Memoryless strategies

In this section we examine the question of sufficient and necessary conditions for $(\mathscr{F}_0, \mathscr{F}_1)$ under which one or both players have memoryless winning strategies in all games $(G, (\mathscr{F}_0, \mathscr{F}_1))$.

Suppose that for a player $\sigma \in \{0, 1\}$ the set $\mathfrak{Usf}(\mathscr{F}_\sigma)$ of his useful colours is empty. Then his winning strategy w^σ constructed in Theorem 7 is a $\mathrm{LAR}_\emptyset$ strategy. Since $M_\emptyset = \{\varepsilon\}$, this means that player σ has only one possible memory state, i.e. his memory is in fact useless and we can discard the memory component of w^σ. (Formally, we define a memoryless strategy f^σ by setting $f^\sigma(v) = w^\sigma(v, \varepsilon)$ for $v \in V_\sigma$.) Since by Lemma 1 the emptiness of $\mathfrak{Usf}(\mathscr{F}_\sigma)$ is also necessary for memoryless strategies we get

Corollary 11. *Let $(\mathscr{F}_0, \mathscr{F}_1)$ be a Muller condition. Player $\sigma \in \{0, 1\}$ has a winning memoryless strategy (on the set W^σ of his winning positions) in all games $\mathscr{G} = (G, (\mathscr{F}_0, \mathscr{F}_1))$ iff $\mathscr{F}_{1-\sigma}$ is closed under union.*

The fact that $\mathscr{F}_{1-\sigma}$ does not contain splits is reflected by the shape of the split tree:

Lemma 12. *The family $\mathscr{F}_{1-\sigma}$ is closed under union iff each σ-vertex of the split tree T of $(\mathscr{F}_0, \mathscr{F}_1)$ has at most one child.*

Proof. Left to right implication follows from the fact that two different children of a σ-vertex of T would constitute a split in $\mathscr{F}_{1-\sigma}$.

Assume now that all σ-vertices of T have at most one child. Suppose that $\mathscr{F}_{1-\sigma}$ contains a split: $X, Y \in \mathscr{F}_{1-\sigma}$ and $X \cup Y \in \mathscr{F}_\sigma$. Then, by Remark 10, there exists a vertex of T labelled by (σ, B') with the only child labelled by $(1 - \sigma, B'')$ such that $X \cup Y \subseteq B'$ but $X \cup Y \not\subseteq B''$. However, B'' is the greatest subset of B' belonging to $\mathscr{F}_{1-\sigma}$, thus each subset of B' belonging to $\mathscr{F}_{1-\sigma}$ is a subset of B'', in particular X and Y should be subsets of B''. This would imply $X \cup Y \subseteq B''$. This contradiction ends the proof. $\square$

A pairs family over the set C of colours is a family $\mathscr{P} = \{(L_i, R_i)\}_{i \in I}$ of pairs of subsets of C (we allow also the empty pairs family $\mathscr{P} = \{\ \}$). We associate with $\mathscr{P}$ two conditions: Rabin condition $(\mathsf{R}, \mathscr{P})$ and Streett condition $(\mathsf{S}, \mathscr{P})$.

A nonempty subset X of C is said to satisfy *Rabin condition* $(\mathsf{R}, \mathscr{P})$ if $\exists i \in I$, $L_i \cap X \neq \emptyset$ and $R_i \cap X = \emptyset$. Streett condition is the complement of Rabin condition: a nonempty subset X of C satisfies *Streett condition* $(\mathsf{S}, \mathscr{P})$ if $\forall i \in I$, $L_i \cap X = \emptyset$ or $R_i \cap X \neq \emptyset$.

Since Rabin/Streett conditions are mutually complementary they can be used to express winning conditions of a player and his adversary. For instance, if player σ uses Rabin condition then he wins a play p if $\mathrm{inf}(p)$ satisfies $(\mathsf{R}, \mathscr{P})$, otherwise $\mathrm{inf}(p)$ satisfies $(\mathsf{S}, \mathscr{P})$ and player $1 - \sigma$, who uses Streett condition, wins p.

A chain is a pairs family $\mathscr{P} = \{(L_i, R_i)\}_{1 \leqslant i \leqslant l}$ such that $R_1 \subset L_1 \subset R_2 \subset L_2 \subset \cdots R_l \subset L_l$. A chain condition $(\mathsf{C}, \mathscr{P})$ is simply a Rabin condition with a chain $\mathscr{P}$.

If the set family $\mathscr{F}$ consists of sets satisfying a Rabin/Streett/chain condition then we say that $\mathscr{F}$ is expressible by the corresponding condition.

All three conditions defined above are frequently used as acceptance conditions for automata on infinite objects. The following lemma characterizes collections $\mathscr{F} \subseteq \mathscr{P}_0(C)$

that can be expressed by means of Rabin and chain conditions (although this lemma is a simple observation, maybe folklore, we failed to find this statement in the literature).

Lemma 13. *Let* $\mathscr{F} \subseteq \mathscr{P}_0(C)$ *and* $\overline{\mathscr{F}} = \mathscr{P}_0(C) \setminus \mathscr{F}$.
(1) $\mathscr{F}$ *is expressible by a Rabin condition iff* $\overline{\mathscr{F}}$ *is closed under union.*
(2) $\mathscr{F}$ *is expressible by a chain condition iff both* $\mathscr{F}$ *and* $\overline{\mathscr{F}}$ *are closed under union.*

Proof. We leave to the reader the direct verification that the family of sets satisfying a given Streett condition is always closed under union and similarly that the family $\mathscr{F}$ satisfying a chain condition and its complement are both closed under union. This gives the left to right implication of both statements.

Now suppose that $\overline{\mathscr{F}}$ is closed under union. If $\mathscr{F} = \emptyset$ then it suffices to take $\mathscr{P} = \{\ \}$. Thus assume that $\mathscr{F} \neq \emptyset$. Let T be the split tree onstructed for the condition $(\mathscr{F}, \overline{\mathscr{F}})$ and let $H_1, \ldots, H_l$ be the sets labelling 0-vertices of T (obviously all these sets belong to $\mathscr{F}$ but it is possible that some elements of $\mathscr{F}$ are not on this list).

For each i, $1 \leqslant i \leqslant l$, we define the set Q_i: if all subsets of H_i belong to $\mathscr{F}$ then $Q_i = \emptyset$, otherwise Q_i is the maximal subset of H_i belonging to $\overline{\mathscr{F}}$ (Q_i is unique since $\overline{\mathscr{F}}$ does not contain splits). In other words, Q_i is either empty if the vertex of T labelled $(0, H_i)$ does not have children, otherwise Q_i is the label of the only child of this vertex.

Obviously $Y \in \mathscr{F}$ iff there exists i, $1 \leqslant i \leqslant l$, such that $Y \subseteq H_i$ and $Y \not\subseteq Q_i$, which is equivalent with $Y \cap (C \setminus H_i) = \emptyset$ and $Y \cap (C \setminus Q_i) \neq \emptyset$. Therefore, setting $L_i = C \setminus Q_i$ and $R_i = C \setminus H_i$ we get a pairs family $\mathscr{P} = \{(L_i, R_i)\}_{1 \leqslant i \leqslant l}$ such that $\mathscr{F}$ consists of sets satisfying $(\mathsf{R}, \mathscr{P})$. Note that if both $\mathscr{F}$ and $\overline{\mathscr{F}}$ are closed under union then the corresponding split tree has no forks (Lemma 12) and the pairs family constructed above forms a chain. □

From Corollary 11 and Lemma 13 we obtain

Corollary 14. *Let* $(\mathscr{F}_0, \mathscr{F}_1)$ *be a Muller condition.*
(1) *Player* σ *has a winning memoryless strategy (over the set* W^σ *of his winning positions) in all games* $\mathscr{G} = (G, (\mathscr{F}_0, \mathscr{F}_1))$ *iff* $\mathscr{F}_\sigma$ *can be expressed as a Rabin condition.*
(2) *Both players have memoryless winning strategies (on their respective sets* W^0, W^1 *of winning positions) in all games* $\mathscr{G} = (G, (\mathscr{F}_0, \mathscr{F}_1))$ *iff* $\mathscr{F}_0$ *(and thus also* $\mathscr{F}_1$*) is expressible by a chain condition.*

The difficult part of Corollary 14(1) asserting that in order to have memoryless winning strategy for a player σ it is sufficient that his winning condition be in Rabin form was first proved by Klarlund [14]. His method is however quite different. It is based on a result of Klarlund and Kozen [15] asserting that if in a coloured graph G all infinite paths satisfy Rabin condition then we can associate with G the so-called Rabin progress measure. As Klarlund writes in [14]: “Intuitively, the value of the progress measure represents a prioritized list of hypotheses about which pair[3] is going to be satisfied in the limit". Invoking the result of Martin [18] about the determinacy of Borel games (a simpler result of Davis [5] would be sufficient) Klarlund establishes that for each initial position one of the two players has a winning strategy (with unbounded memory). The final step of Klarlund's proof consists in, roughly speaking, transformation of such a perfect memory strategy into a memoryless strategy (this transformation is carried out only for the player using Rabin condition, the adversary player using Streett condition may need some memory, however neither his strategy nor his memory requirements are examined in [14]). Properties of the Rabin measure are used to show that the resulting memoryless strategy is winning for the player with Rabin condition if the initial unbounded memory strategy was winning for him.

Let us note that, for games on coloured graphs, chain conditions and parity conditions can be considered as two different forms of the same winning condition. Thus Corollary 14(2) is in fact equivalent with Theorem 6. Let us note also that recently Thomas [28, 29] proposed a simplified transformation of Muller automata to chain automata (remarkably but not surprisingly Thomas's transformation uses a variant of LAR due to Büchi [4]). The reader may note also that if the winning condition is a chain condition then the proof of Theorem 7 reduces in fact to the first proof of Theorem 6.

6. Games on totally coloured arenas

An arena $G = (V_0, V_1, E, \varphi, C)$ is *totally coloured* if φ is a total mapping from V into C.

Since arenas that appear in applications are totally coloured they deserve a special attention. Quite surprisingly the fact that all vertices of G are coloured has an impact on the memory requirements for winning strategies. First of all note that Lemma 1 does not hold any more. The arena of Fig. 1 contains an uncoloured vertex u and when we try to colour it then we discover quickly that we are able to reproduce the result of Lemma 1 only if $\mathscr{F}_{1-\sigma}$ contains a special type of splits.

For $\mathscr{F} \subseteq \mathscr{P}_0(C)$, a pair of sets X_1, X_2 in $\mathscr{F}$ is called a *strong split* in $\mathscr{F}$ if $X_1 \cap X_2 \neq \emptyset$ and $X_1 \cup X_2 \notin \mathscr{F}$.

Lemma 15. *Let $(\mathscr{F}_0, \mathscr{F}_1)$ be a Muller condition. If $\mathscr{F}_{1-\sigma}$ contains a strong split then there exists a game $(G, (\mathscr{F}_0, \mathscr{F}_1))$ on a totally coloured arena G such that player σ has a winning strategy but no memoryless winning strategy.*

Proof. Take a strong split X_1, X_2 in $\mathscr{F}_{1-\sigma}$ and the graph of Fig. 1. Colour vertices v_i and w_i as in the proof of Lemma 1 and colour the vertex u by any colour $c \in X_1 \cap X_2$. The same reasoning as in Lemma 1 shows that σ has a winning strategy but no memoryless winning strategy. $\square$

[3] From the pairs family of the Rabin condition.

For Muller condition $(\mathscr{F}_0, \mathscr{F}_1)$ the set $\mathfrak{Susf}(\mathscr{F}_\sigma)$ of *strongly useful colours* for player σ is defined as the union of all symmetric differences $X_1 \triangle X_2$, where X_1, X_2 range over all strong splits in $\mathscr{F}_{1-\sigma}$. (As in the case of splits we can restrict the set of strongly useful colours by taking into account only the strong splits in $\mathscr{F}_{1-\sigma}$ that appear labelling some siblings in the split tree.) Then it is possible to prove the counterpart of Theorem 7:

Proposition 16. *For any game $(G, (\mathscr{F}_0, \mathscr{F}_1))$ on a totally coloured arena G the set V of vertices can be partitioned on two sets W^0 and W^1, such that, for each $\sigma \in \{0,1\}$, W^σ is an $(1-\sigma)$-trap and player σ has a winning LAR_{U_σ} strategy on W^σ, where $U_\sigma = \mathfrak{Susf}(\mathscr{F}_\sigma)$ is his set of strongly useful colours.*

Proposition 16 allows to deduce Theorem 7 quite directly. Namely consider a game $\mathscr{G} = (G, (\mathscr{F}_0, \mathscr{F}_1))$ on a partially coloured arena G. Let α be a new colour not belonging to C. Colouring the uncoloured vertices of G with the colour α we get a totally coloured arena G'. The winning condition $(\mathscr{F}_0', \mathscr{F}_1')$ on G' is defined in the following way: $X \in \mathscr{P}_0(C \cup \{\alpha\})$ belongs to $\mathscr{F}_\sigma'$ iff $X \cap C \in \mathscr{F}_\sigma$, $\sigma \in \{0,1\}$. Note that every play p has the same winner in both games $\mathscr{G} = (G, (\mathscr{F}_0, \mathscr{F}_1))$ and $\mathscr{G}' = (G', (\mathscr{F}_0', \mathscr{F}_1'))$. However, X_1, X_2 is a split in $\mathscr{F}_\sigma$ iff $X_1 \cup \{\alpha\}$, $X_2 \cup \{\alpha\}$ is a strong split in $\mathscr{F}_\sigma'$. Therefore for each player the useful colours in the game $\mathscr{G}$ are the same as the strongly useful colours in the game $\mathscr{G}'$. Therefore Proposition 16 applied to $\mathscr{G}'$ implies Theorem 7 for $\mathscr{G}$.

On the other hand, we failed to establish the implication in the other direction, in fact Proposition 16 seems to be stronger than Theorem 7. Nevertheless, we have decided to leave Proposition 16 as an unproven claim and to content ourselves with the weaker result of Section 4. The reason is that the proof of Proposition 16 is more obscure and technically involved and the additional clumsiness seems to be to high a price to pay for a bit more generality.

However, the most important instance of Proposition 16 concerning necessary and sufficient conditions for memoryless winning strategies for one or both players on totally coloured arenas is easy enough to be worked out completely. Since these conditions turn out to be weaker than Rabin and chain conditions and the arenas appearing in complementation lemma *are* totally coloured the results of this section improve on the previous results of Emerson and Jutla [9], Mostowski [21] and Klarlund [14].

Theorem 17. *Player $\alpha \in \{0,1\}$ has a memoryless winning strategy on his winning set W^α for all games $(G, (\mathscr{F}_0, \mathscr{F}_1))$ on totally coloured arenas G if and only if the set $\mathscr{F}_{1-\alpha}$ does not contain strong splits.*

Proof. That the absence of strong splits in $\mathscr{F}_{1-\alpha}$ is necessary follows from Lemma 15. To prove that this is also sufficient we reconsider the proof of Theorem 7 and indicate the necessary modifications. Assume again that σ is the player for which $C \in \mathscr{F}_\sigma$

and that $\max(\mathscr{F}_{1-\sigma}) = \{C_0, \ldots, C_{k-1}\}$ are the maximal elements of $\mathscr{F}_{1-\sigma}$. The winning sets $W^{1-\sigma}$ and W^{σ} of both players are constructed exactly as in Theorem 7, the modifications will concern only the winning strategies.

There are two cases to examine depending on whether C belongs to the winning set of the player without strong splits.

Case I. Suppose that $\mathscr{F}_{\sigma}$ does not contain strong splits.

Then we should exhibit a winning memoryless strategy for player $1-\sigma$. Let us recall that in Section 4.3.1 we have constructed an increasing sequence of strategies $w_{\xi}^{1-\sigma}$, the strategy $w^{1-\sigma}$ was the last of them. Thus we should now modify the strategies $w_{\xi}^{1-\sigma}$ in order to make them memoryless. First note that if $\mathscr{F}_{\sigma}$ does not contain strong splits then also the families $\mathscr{F}_{\sigma}^{t}$ defined by Eq. (12) do not contain strong splits for $0 \leqslant t < k$ and therefore, by the induction hypothesis, the winning strategies $z_{t,\xi}^{1-\sigma}$ of player $1-\sigma$ in the games (13) are memoryless. Thus in Eq. (16) and (11) defining the strategy $w_{\xi+1}^{1-\sigma}$ we can discard the memory components from x_{ξ}, $z_{t,\xi}^{1-\sigma}$, $w_{\xi}^{1-\sigma}$ and $w_{\xi+1}^{1-\sigma}$.

For a limit ordinal ξ, $w_{\xi}^{1-\sigma}$ becomes now the union of an ascending sequence of memoryless strategies $w_{\eta}^{1-\sigma}$, $\eta < \xi$, and thus is also memoryless. In this way, making all strategies $w_{\xi}^{1-\sigma}$ memoryless, also the global winning strategy $w^{1-\sigma}$ becomes memoryless.

Case II. Suppose that $\mathscr{F}_{1-\sigma}$ does not contain strong splits.

Now we should exhibit a winning memoryless strategy w^{σ} for player σ. This is in fact the only interesting case – after all player $1-\sigma$ does not use his memory at the topmost induction level, thus we cannot see directly if he needs some memory or not.

To simplify the notation we can assume without loss of generality that $W^{\sigma} = V$, i.e. that $G[W^{\sigma}]$ is the whole arena G. Therefore, for any set $B \subseteq C$ of colours $\varphi^{-1}(B)$ will denote the set of all vertices of W^{σ} that are coloured by elements of B.

Let us recall that for all $0 \leqslant t < k$ the set W^{σ} is partitioned on two sets (cf. Fig. 4): Z_t, where player σ has a winning strategy z_t^{σ} and $\mathrm{Attr}^{\sigma}(G[W^{\sigma}], N_t)$, where he has a strategy to attract the token to the set $N_t = \varphi^{-1}(C \setminus C_t)$ of vertices coloured by $C \setminus C_t$. The strategies z_t^{σ}, obtained by solving the games (17), are now memoryless by the induction hypothesis (if $\mathscr{F}_{1-\sigma}$ does not contain strong splits then the same holds for $\mathscr{F}_{1-\sigma}^{t} = \mathscr{F}_{1-\sigma} \cap \mathscr{P}_0(C_t)$).

Moreover, since G is totally coloured, $W^{\sigma} \setminus N_t = \varphi^{-1}(C_t)$.

The union of two different elements C_i and C_j of $\max(\mathscr{F}_{1-\sigma})$ belongs always to $\mathscr{F}_{\sigma}$, therefore $C_i \cap C_j = \emptyset$ (otherwise C_i, C_j would constitute a strong split in $\mathscr{F}_{1-\sigma}$ contradicting our assumption that $\mathscr{F}_{1-\sigma}$ does not contain such splits). This implies that also the sets $\varphi^{-1}(C_t)$ are pairwise disjoint for $0 \leqslant t < k$.

Finally observe also the inclusion $Z_t = W^{\sigma} \setminus \mathrm{Attr}^{\sigma}(G[W^{\sigma}], N_t) \subseteq W^{\sigma} \setminus N_t = \varphi^{-1}(C_t)$ and the equality $\varphi^{-1}(C_t) \setminus Z_t = \mathrm{Attr}^{\sigma}(G[W^{\sigma}], N_t) \setminus N_t$ (this equality results directly from the following two equivalences holding for all $v \in W^{\sigma}$: (1) $v \in \varphi^{-1}(C_t)$ iff $v \notin N_t$ and (2) $v \notin Z_t$ iff $v \in \mathrm{Attr}^{\sigma}(G[W^{\sigma}], N_t)$).

All these observations are summarized on Fig. 6.

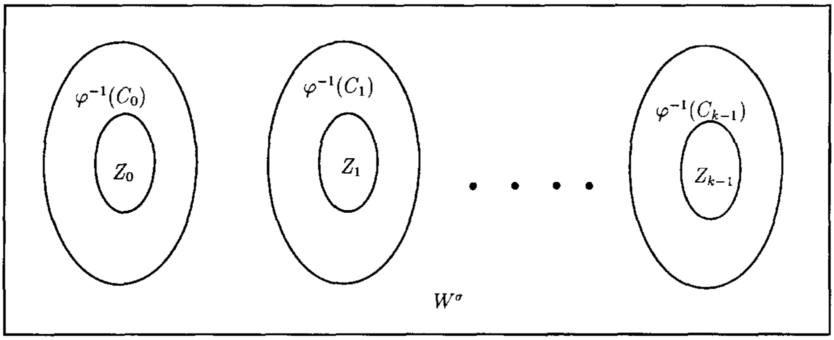

Fig. 6. For all $0 \leqslant t < k$, $\varphi^{-1}(C_t)$ consists of vertices coloured by the colours from C_t. The sets $\varphi^{-1}(C_t)$ are pairwise disjoint, each of them contains a subset Z_t, where player σ has a memoryless winning strategy z_t^σ. If the token is in $\varphi^{-1}(C_t) \setminus Z_t = \mathrm{Attr}^\sigma(G[W^\sigma], N_t) \setminus N_t$ then player σ can use the attracting strategy $\mathrm{attr}^\sigma(G[W^\sigma], N_t)$ to force the token outside of $\varphi^{-1}(C_t)$, i.e. to $N_t = W^\sigma \setminus \varphi^{-1}(C_t)$.

The definition of the memoryless strategy w^σ is now obvious:
- if the token is inside of one of the sets Z_t then player σ plays according to the strategy z_t^σ,
- if the token visits one of the sets $\varphi^{-1}(C_t) \setminus Z_t$ then player σ attracts it to the set $W^\sigma \setminus \varphi^{-1}(C_t) = N_t$ in a finite number of steps,
- if the token is in $W^\sigma \setminus \bigcup_{0 \leqslant t < k} \varphi^{-1}(C_t)$ then player σ can take any successor vertex inside of W^σ (since W^σ is a $1 - \sigma$-trap such successors always exist).

Let us take any play $p = v_0 v_1 v_2 \ldots$ starting at a vertex $v_0 \in W^\sigma$ and consistent with the strategy w^σ described above. Two cases arise.

Case 1. If there exists $0 \leqslant t < k$ such that $\inf(p) \subseteq C_t$ then from some moment n onwards the token visits only vertices of $\varphi^{-1}(C_t)$. However, this implies that from that moment n onwards the token is in fact always inside of Z_t; otherwise, had it visited $\varphi^{-1}(C_t) \setminus Z_t$ it would have been forced to visit a vertex of N_t, i.e. to visit a colour not in C_t.

Now, when in the suffix play $v_n v_{n+1} v_{n+2} \ldots$ of p all vertices belong to Z_t then this play is consistent with z_t^σ and winning for σ. Thus the whole play p is winning for σ as well.

Case 2. The second possibility is that $\inf(p)$ is not a subset of any C_t, $0 \leqslant t < k$. Then however $\inf(p)$ cannot belong to $\mathscr{F}_{1-\sigma}$, i.e. player σ wins also in this case.

This terminates the proof that w^σ is winning for σ. □

It is interesting to note why the proof given above fails to work for partially coloured arenas. The reason is that, even if the sets C_t are pairwise disjoint, the sets $W^\sigma \setminus N_t$ are not since they consist not only of all vertices coloured by C_t by also of all uncoloured vertices of W^σ.

Similarly as for splits (Lemma 12), the absence of strong splits is reflected by the form of the split tree:

Lemma 18. *Let T be the split tree associated with $(\mathscr{F}_0, \mathscr{F}_1)$. Then $\mathscr{F}_{1-\sigma}$ does not contain strong splits iff the labels of any two sibling $(1-\sigma)$-vertices in T are disjoint.*

To compare different classes of winning conditions examined in this paper it is convenient to introduce a notation allowing to express them in an uniform way.

Let X_i, $Y_{i,j}$ be subsets of C. For any nonempty subset B of C we write $B \models \bigvee_{i=1}^{n}(X_i \wedge \neg Y_{i,1} \wedge \cdots \wedge \neg Y_{i,k_i})$ to denote that there exists $1 \leqslant i \leqslant n$ such that $B \subseteq X_i$ and, for all $1 \leqslant j \leqslant k_i$, $B \nsubseteq Y_{i,j}$. Conditions in this form will be called *inclusion conditions.*

For each family $\mathscr{F} \subseteq \mathscr{P}_0(C)$ we can construct in a canonical way an inclusion condition expressing $\mathscr{F}$. To this end we take the split tree T of $(\mathscr{F}, \overline{\mathscr{F}})$ and the inclusion condition is obtained as the disjunction of all $X_i \wedge \neg Y_{i,1} \wedge \cdots \wedge \neg Y_{i,k_i}$, where X_i labels a 0-vertex of T and $Y_{i,1}, \ldots, Y_{i,k_i}$ are labels of the children of this 0-vertex.

This method applied to the family $\mathscr{F}_0$ from Example 9 yields the following inclusion condition: $(\{c_0, c_1\} \wedge \neg\{c_0\} \wedge \neg\{c_1\}) \vee (\{c_2, c_3, c_4\} \wedge \neg\{c_2\} \wedge \neg\{c_3, c_4\}) \vee \{c_3\} \vee \{c_4\}$.

Using Lemmas 12 and 18 in one direction and the direct verification in the other we obtain the following complete characterization of four special classes of inclusion conditions:

(Rabin) $\bigvee_{i=1}^{n}(X_i \wedge \neg Y_i)$.

Such inclusion condition is equivalent with the usual Rabin conditions.

(chain) $\bigvee_{i=1}^{n}(X_i \wedge \neg Y_i)$, where $X_1 \supset Y_1 \supset \cdots X_n \supset Y_n$.

These inclusion conditions are equivalent with the usual chain conditions.

(extended Rabin) $\bigvee_{i=1}^{n}(X_i \wedge \neg Y_{i,1} \wedge \cdots \wedge \neg Y_{i,k_i})$, where for each $1 \leqslant i \leqslant n$ and all $1 \leqslant j < l \leqslant k_i$, $Y_{i,j} \cap Y_{i,l} = \emptyset$.

Such condition expresses a family $\mathscr{F}$ if and only if the complement of $\mathscr{F}$ does not contain strong splits. Therefore a player has a memoryless winning strategy in all games over totally coloured arenas iff his winning condition can be expressed in this form.

(strongly branching) $\bigvee_{i=1}^{n}(X_i \wedge \neg Y_{i,1} \wedge \cdots \wedge \neg Y_{i,k_i})$, where

- for all $1 \leqslant i < j \leqslant n$, either X_i and X_j are disjoint or one of them is included in the other,
- for all $1 \leqslant i, j \leqslant n$ and all $1 \leqslant l \leqslant k_i$, $1 \leqslant m \leqslant k_j$, $Y_{i,l}$ and $Y_{j,m}$ are either disjoint or one of them is included in the other.

Such a condition expresses a family $\mathscr{F}$ iff neither $\mathscr{F}$ nor its complement contain strong splits. Therefore both players have memoryless winning strategies on totally coloured arenas iff their winning conditions are of this form.

Note that the family $\mathscr{F}_0$ of Example 9, as well as its complement, are expressible by a strongly branching inclusion condition but not by a Rabin condition: both $\mathscr{F}_0$ and $\mathscr{F}_1$ contain splits but do not contain strong splits. Therefore there are games on partially coloured arenas using this winning condition where both players need memory for their winning strategies. But when we play with the winning condition of Example 9 on totally coloured arenas then both players do not need any memory.

Other similar examples – their construction is left to the reader – show that the diagram below presents all possible inclusions (denoted by arrows) among different set families, all these inclusions are strict (if the set C of all colours is large enough).

$$\begin{tikzcd}
& \text{Rabin} \arrow[dr] & & \\
\text{chain} \arrow[ur] \arrow[dr] & & \text{extended Rabin} \arrow[r] & \mathscr{P}_0(C) \\
& \text{strongly branching} \arrow[ur] & &
\end{tikzcd}$$

An interesting question is how much memory we can save playing on arenas that are totally coloured rather than only partially coloured. Comparing the memoryless conditions for such arenas established in this section with the optimal memory bounds for games on partially coloured arenas from [7] we can deduce easily the following facts. Let $(\mathscr{F}_0, \mathscr{F}_1)$ be a winning condition such that player σ needs no memory when playing on totally coloured arenas, i.e. $\mathscr{F}_\sigma$ satisfies the extended Rabin condition. Then playing on partially coloured arenas with the same winning condition the same player needs the memory of the size at most $O(|C|)$. On the other hand, this bound may be tight, i.e. there are winning conditions where σ needs $\Omega(|C|)$ memory on some partially coloured arenas and no memory on all totally coloured arenas.

7. Applications to automata on infinite trees

The winning strategies constructed in Sections 3 and 4 are in general non-deterministic. In this section we assume that all considered strategies are deterministic, i.e. a strategy for player σ is a partial mapping from $V_\sigma \times M_B$ into $V_{1-\sigma}$, where M_B is the corresponding LAR memory. Obviously, for each non-deterministic winning strategy w^σ for player σ we can always obtain a deterministic one by choosing an element of $w^\sigma(v, m)$ for each $v \in V_\sigma$ and each $m \in M_B$.

7.1. *Infinite trees and automata*

The full binary tree is formed by the set $\{0,1\}^*$ of all binary words, the elements of $\{0,1\}^*$ are called vertices. For any $x \in \{0,1\}^*$, $x0$ and $x1$ are respectively the left and the right successor of the vertex x, the empty word ε is the root of the tree.

For any alphabet Σ, a Σ-tree τ is a complete binary tree with vertices labelled by elements of Σ, i.e. τ is a mapping from the set $\{0,1\}^*$ of vertices into Σ. Any set of Σ-trees is called a Σ-forest, the Σ-forest consisting of all Σ-trees being denoted by $\mathscr{T}_\Sigma$ For each $\tau \in \mathscr{T}_\Sigma$ and each vertex $x \in \{0,1\}^*$, by $\tau_x \in \mathscr{T}_\Sigma$ we note the subtree of τ starting at the vertex x: $\forall y \in \{0,1\}^*$, $\tau_x(y) = \tau(xy)$.

A finite tree automaton is a tuple $\mathscr{A} = (\Sigma, C, c_0, \Delta, \mathfrak{C})$, where

- Σ is an alphabet,

- C is a finite set of colours (usually elements of C are called states, we prefer the term "colours" for the sake of consistency with the terminology used for games),
- $c_0 \in C$ is the initial colour,
- Δ is the set of transitions, $\Delta \subseteq C \times \Sigma \times C \times C$,
- $\mathfrak{C}$ is an acceptance condition.

A run of $\mathscr{A}$ over a Σ-tree τ is a C-tree $r:\{0,1\}^* \to C$ (i.e. a binary tree coloured by elements of C) such that $\forall x \in \{0,1\}^*$, $(r(x), \tau(x), r(x0), r(x1)) \in \Delta$. Infinite words of $\{0,1\}^\omega$ are called paths in the tree. For each C-tree $r \in \mathscr{T}_C$ and each path $p = p_1 p_2 p_3 \ldots$, where $p_i \in \{0,1\}$, we define

$$\inf_r(p) = \{c \in C \mid r(p_1 \ldots p_i) = c \text{ for infinitely many } i\}$$

to be the set of colours occurring infinitely often on the path p in r.

A run r over a Σ-tree τ *accepts* τ if
- the root of r is coloured by the initial colour, $r(\varepsilon) = c_0$, and
- for each path $p \in \{0,1\}^\omega$, the set $\inf_r(p)$ of colours occurring infinitely often satisfies $\mathfrak{C}$.

A Σ-tree τ is recognized by an automaton $\mathscr{A}$ if there is a run r of $\mathscr{A}$ over τ accepting τ. The forest of all Σ-trees recognized by $\mathscr{A}$ is denoted by $T(\mathscr{A})$.

Remarks. From this moment on we assume that $\mathfrak{C}$ is either Streett or chain (or parity) condition, its complement will be noted as $\mathfrak{C}^c$. (This will save notation without loss of generality since other types of automata like Rabin or Muller can be transformed to chain automata.) In the sequel we shall consider several games between two players that are called, after Gurevich and Harrington [11], Automaton and Pathfinder. Always $\mathfrak{C}$ will be used as Automaton winning condition while $\mathfrak{C}^c$ will be Pathfinder winning condition, i.e. $\mathfrak{C}^c$ is always either in Rabin or in chain form. In fact what we need is the Pathfinder winning condition $\mathfrak{C}^c$ in the form assuring for him a *memoryless* winning strategy, in particular since arenas constructed here are totally coloured we can also admit $\mathfrak{C}^c$ in extended Rabin or strongly branching form from Section 6.

To assure that in constructed arenas each vertex has at least one successor we assume in the sequel that tree automata are always complete, i.e. for all $c \in C$ and $a \in \Sigma$ there is at least one transition in Δ of the form (c, a, c', c'') (each tree automaton can be completed easily if necessary).

7.2. Complementation of tree automata

Let $\mathfrak{Rec}$ be the class of all forests recognized by finite tree automata (recall that this class is independent of a particular acceptance condition since all of them are equivalent with chain condition). It is an elementary exercise from automata theory to show that $\mathfrak{Rec}$ is closed under union and intersection if the acceptance condition is in Muller form. The closure of $\mathfrak{Rec}$ under complement is however highly nontrivial and constitutes the main technical achievement of Rabin's paper [25].

Theorem 19. *The class* $\mathfrak{Rec}$ *of recognizable* Σ*-forests is closed under complement.*

Let τ be a Σ-tree. We consider the following *colouring game* played on τ by Automaton and Pathfinder. Initially, only the root ε of τ is coloured by the initial colour c_0 of $\mathscr{A}$ and ε is also the initial game position. The players play by rounds, each round consists in one step of Automaton followed by one step of Pathfinder. If the current play position is a vertex $x \in \{0,1\}^*$ of τ and x is coloured by $c \in C$ then Automaton chooses a transition of the form $(c, \tau(x), c_0', c_1') \in \Delta$ and colours the left and the right successor of x by c_0' and c_1' respectively. In the next step Pathfinder chooses a direction $i \in \{0,1\}$ and moves the current play position to the successor xi of x. After an infinite number of rounds the infinite sequence $p = i_1 i_2 i_3 \ldots, i_k \in \{0,1\}$, of Pathfinder moves gives a path in τ, this path was completely coloured by Automaton during the play. If the set of colours that occur infinitely often along the path p satisfies $\mathfrak{C}$ then Automaton wins, otherwise Pathfinder wins. The notions of strategies, winning strategies, playing consistent with a given strategy can be adapted directly to the colouring games. We can also observe directly that winning deterministic Automaton strategies can be identified with accepting runs of $\mathscr{A}$ on τ: given such a run r, Automaton's strategy consists in choosing the transition $(r(x), \tau(x), r(x0), r(x1))$ whenever the current game position is the vertex $x \in \{0,1\}^*$. On the other hand, the colouring game over τ can also be represented as a game over an arena, i.e. it has a representation that we have used for games up to now. Such arena $G_\tau = (V_{\text{Automaton}}, V_{\text{Pathfinder}}, E_\tau, \varphi, C)$ is defined in the following way:

$$V_{\text{Automaton}} = \{(x, c, \tau(x)) \mid x \in \{0,1\}^* \quad \text{and} \quad c \in C\}$$

and

$$V_{\text{Pathfinder}} = \{(x, c, \tau(x), c_0', c_1') \mid x \in \{0,1\}^* \quad \text{and} \quad (c, \tau(x), c_0', c_1') \in \Delta\}$$

are Automaton's and Pathfinder's vertices, respectively.

The token visiting a vertex $v = (x, c, \tau(x)) \in V_{\text{Automaton}}$ of G_τ corresponds in the colouring game to the situation when the vertex x of τ is the current play position and this x is coloured by c. In the colouring game Automaton chooses a transition of the form $(c, \tau(x), c_0', c_1') \in \Delta$ to colour two successors of x. Playing on G_τ this step is represented by automaton advancing the token to $w = (x, c, \tau(x), c_0', c_1') \in V_{\text{Pathfinder}}$, in other words such pairs (v, w) constitute the edges of E_τ representing valid Automaton moves. If the token visits a Pathfinder vertex $w = (x, c, \tau(x), c_0', c_1') \in V_{\text{Pathfinder}}$ then Pathfinder should choose a direction $i \in \{0,1\}$, which is represented in G_τ by the token moving from w to $w_i = (xi, c_i', \tau(xi)) \in V_{\text{Automaton}}$, $i \in \{0,1\}$, i.e. such pairs (w, w_i) are the edges of E_τ describing possible Pathfinder moves.

The colouring mapping φ of G_τ is trivial, vertices $(x, c, \tau(x)) \in V_{\text{Automaton}}$ and $(x, c, \tau(x), c_0', c_1') \in V_{\text{Pathfinder}}$ are coloured by c. (Note that leaving Pathfinder's vertices uncoloured would not change the set of infinitely visited colours for any play in G_τ. However, totally coloured arenas are preferable to reduce memory requirements, cf. Section 6). The acceptance condition $\mathfrak{C}$ of $\mathscr{A}$ is the winning Automaton condition (and its complement $\mathfrak{C}^c$ is the Pathfinder winning condition). Let $\mathscr{G}_\tau$ be the resulting game.

From the presentation above it should be clear that the colouring game on τ and the game on the arena G_τ with the starting token position $v_0 = (\varepsilon, c_0, \tau(\varepsilon)) \in V_{\text{Automaton}}$ are just two different representations of the same game.

As noted previously, τ is accepted by $\mathscr{A}$ iff Automaton has a winning strategy in the colouring game iff Automaton has a winning strategy for plays starting at v_0 in G_τ. Conversely, $\tau \notin T(\mathscr{A})$ iff Pathfinder has a winning strategy in such games. By our assumption about the form of the winning condition, Pathfinder's winning strategy can be chosen memoryless. It is easy to see now that the definition given below captures correctly the notion of a memoryless Pathfinder strategy in the colouring game on τ.

Definition 20. For any $a \in \Sigma$ let $\Delta_a = \{\delta \in \Delta \mid \exists c, c'_0, c'_1 \in C,\ \delta = (c, a, c'_0, c'_1)\}$ be the set of transitions that are applicable at a vertex labelled by a. A memoryless Pathfinder strategy in the colouring game over τ is a family $\mathbf{s} = \{s_x\}_{x \in \{0,1\}^*}$ of mappings such that $\forall x \in \{0,1\}^*$, $s_x : \Delta_{\tau(x)} \rightarrow \{0,1\}$.

Pathfinder uses such a strategy in the following way. If the current play position in τ is a vertex $x \in \{0,1\}^*$ and in the first step of the current round Automaton has chosen a transition δ (note that he could choose only some δ from $\Delta_{\tau(x)}$) then Pathfinder playing according to $\mathbf{s}$ moves the current play position in the direction $i = s_x(\delta)$ to the successor xi of x. Thus intuitively each $s_x, x \in \{0,1\}^*$, describes locally Pathfinder's strategy at the vertex x, this strategy is memoryless since Pathfinder move is determined uniquely by the current position in the colouring game and by the transition chosen by Automaton at this positions and it does not depend on the previous transitions chosen by Automaton. From our previous discussion it follows that

Lemma 21. *For any tree $\tau \in \mathscr{T}_\Sigma$, $\tau \notin T(\mathscr{A})$ iff Pathfinder has a memoryless winning strategy, in the sense of Definition* 20, *in the colouring game on τ.*

Our construction of an automaton recognizing $\mathscr{T}_\Sigma \setminus T(\mathscr{A})$ will take four steps:

(1) We define a new finite alphabet Ω and a finite non-deterministic automaton $\mathscr{B}_\Omega$ recognizing a subset of Ω^ω (thus $\mathscr{B}_\Omega$ is an automaton on infinite words),

(2) With each $\tau \in \mathscr{T}_\Sigma$ and each total memoryless Pathfinder strategy $\mathbf{s}$ in the colouring game on τ we associate a language $L_{\tau,\mathbf{s}} \subseteq \Omega^\omega$ of infinite words. We prove that $\mathbf{s}$ is winning for Pathfinder iff $L_{\tau,\mathbf{s}} \cap L(\mathscr{B}_\Omega) = \emptyset$, where $L(\mathscr{B}_\Omega)$ stands for the language recognized by $\mathscr{B}_\Omega$.

(3) From $\mathscr{B}_\Omega$ find a deterministic finite automaton $\mathscr{B}_\Omega^c$ recognizing $\Omega^\omega \setminus L(\mathscr{B}_\Omega)$.

(4) The automaton $\mathscr{B}_\Omega^c$ is transformed to a nondeterministic automaton $\mathscr{A}^c$ recognizing $\mathscr{T}_\Sigma \setminus T(\mathscr{A})$.

Let, for $a \in \Sigma$, $\mathscr{STR}_a$ denote the set of all mappings from Δ_a to $\{0,1\}$. Then we set

$$\Omega = \{(a,s,i) \mid a \in \Sigma,\ s \in \mathscr{STR}_a \quad \text{and} \quad i \in \{0,1\}\}$$

to be the new alphabet. Obviously, since Σ is finite, $\mathscr{STR}_a$ and Ω are finite as well.

Let $\tau \in \mathscr{T}_\Sigma$ and $\mathbf{s} = \{s_x\}_{x \in \{0,1\}^*}$ be a total memoryless Pathfinder strategy in the colouring game on τ. For each infinite binary sequence $p = p_1 p_2 p_3 \ldots \in \{0,1\}^\omega$, where $p_i \in \{0,1\}$, we construct the infinite word

$$w^p_{\tau,\mathbf{s}} = (\tau(\varepsilon), s_\varepsilon, p_1), (\tau(p_1), s_{p_1}, p_2), (\tau(p_1 p_2), s_{p_1 p_2}, p_3) \ldots$$

over the alphabet Ω (note that Definition 20 implies that $s_{p_1 \ldots p_{i-1}} \in \mathscr{STR}_{\tau(p_1 \ldots p_{i-1})}$). Intuitively, $w^p_{\tau,\mathbf{s}}$ codes the letters of Σ and the local strategies s_x for vertices x occurring along the path p (as well as the path p itself).

Let $L_{\tau,\mathbf{s}} \subseteq \Omega^\omega$ be the language consisting of all words $w^p_{\tau,\mathbf{s}}$, with p ranging over $\{0,1\}^\omega$. Now we define an automaton (on infinite words) $\mathscr{B}_\Omega = (\Omega, C, c_0, \gamma, \mathfrak{C})$ – the set of states C, the initial state c_0 and the acceptance condition $\mathfrak{C}$ are the same as for $\mathscr{A}$. The transition relation $\gamma \subseteq C \times \Omega \times C$ of $\mathscr{B}_\Omega$ is defined in the following way: for $c, c' \in C$ and $(a,s,i) \in \Omega$, $(c,(a,s,i),c'_i) \in \gamma$ iff there exists $\delta = (c, a, c'_0, c'_1) \in \Delta_a$ such that $s(\delta) = i$.

Lemma 22. *A total memoryless Pathfinder strategy* $\mathbf{s}$ *in the colouring game on* τ *is winning iff* $L(\mathscr{B}_\Omega) \cap L_{\tau,\mathbf{s}} = \emptyset$.

Proof. Suppose that $L(\mathscr{B}_\Omega) \cap L_{\tau,\mathbf{s}} \neq \emptyset$. Thus there exists a path p in τ such that $\mathscr{B}_\Omega$ accepts the word $w^p_{\tau,\mathbf{s}} \in L_{\tau,\mathbf{s}}$. Let $c_0 c_1 c_2 \ldots$ be the sequence of states in a run accepting $w^p_{\tau,\mathbf{s}}$. Note that the successive transitions executed by $\mathscr{B}_\Omega$ in this run have the form

$$(c_i, (\tau(p_1 \ldots p_i), s_{p_1 \ldots p_i}, p_{i+1}), c_{i+1}) \in \gamma \tag{24}$$

However, from the definition of $\mathscr{B}_\Omega$ it follows that (24) holds only if there exists a transition $\delta_i \in \Delta_{\tau(p_1 \ldots p_i)}$ of the form

$$\delta_i = (c_i, \tau(p_1 \ldots p_i), c'_0, c'_1) \tag{25}$$

and such that $s_{p_1 \ldots p_i}(\delta_i) = p_{i+1} \in \{0,1\}$ and either $c_{i+1} = c'_0$ if $p_{i+1} = 0$ or $c_{i+1} = c'_1$ if $p_{i+1} = 1$.

Suppose that in the colouring game on τ Automaton plays in such a way that at the vertex $p_1 \ldots p_i$ he chooses the transition δ_i given by (25). We have just seen that playing according to $\mathbf{s}$ Pathfinder will choose the direction p_{i+1}. The sequence of colours visited during this play will be $c_0 c_1 c_2 \ldots$ recognized by $\mathscr{B}_\Omega$, i.e. $\inf_C(c_0 c_1 c_2 \ldots)$ satisfies $\mathfrak{C}$ and Pathfinder loses this play. We conclude that strategy $\mathbf{s}$ cannot be winning for Pathfinder.

Suppose now that $L(\mathscr{B}_\Omega) \cap L_{\tau,\mathbf{s}} = \emptyset$. Let r be any run of $\mathscr{A}$ over τ such that $r(\varepsilon) = c_0$. Suppose that Automaton uses r as his strategy in the colouring game on τ playing $(r(x), \tau(x), r(x0), r(x1)) \in \Delta$ whenever the current play position is x. Suppose that Pathfinder plays against this strategy using the strategy $\mathbf{s}$. The resulting play will give an infinite path p in r. It is easy to verify that the sequence $r(\varepsilon), r(p_1), r(p_1 p_2) \ldots$ of states along the path p is a run of $\mathscr{B}_\Omega$ over the word $w^p_{\tau,\mathbf{s}} \in L_{\tau,\mathbf{s}}$ associated with this path and with $\mathbf{s}$. The fact that $w^p_{\tau,\mathbf{s}} \notin L(\mathscr{B}_\Omega)$ implies that this sequence of states does not satisfy $\mathfrak{C}$, i.e. r is not an accepting run. Since the same holds for any run, τ is not recognized by $\mathscr{A}$. $\square$

As it is known from the theory of automata over infinite words [24] it is possible to construct a deterministic finite automaton $\mathscr{B}^c_\Omega = (\Omega, Q, q_0, \gamma', \mathfrak{C}')$ recognizing the language $\Omega \backslash L(\mathscr{B}_\Omega)$, where $\mathfrak{C}'$ is any of the conditions (Muller, Streett, Rabin, chain) used in this paper. (However the efficiency of such a construction depends heavily on the form of the condition. We have assumed in our discussion that the acceptance condition $\mathfrak{C}$ is a Streett or chain condition. In this case we can use the determinization method due to Safra [27] that for a nondeterministic Streett automaton with n states and h acceptance pairs gives an equivalent deterministic Rabin automaton with $O(2^{nh\log(nh)})$ states and nh acceptance pairs.)

The tree automaton $\mathscr{A}^c = (\Sigma, Q, q_0, \Delta', \mathfrak{C}')$ recognizing $\mathscr{T}_\Sigma \backslash T(\mathscr{A})$ is obtained directly from $\mathscr{B}^c_\Omega$ – we set $(q, a, q', q'') \in \Delta'$ iff there exists $s \in \mathscr{S}\mathscr{T}\mathscr{R}_a$ such that $\gamma'(q, (a, s, 0)) = q'$ and $\gamma'(q, (a, s, 1)) = q''$.

To show that $\mathscr{A}^c$ recognizes the complement of $T(\mathscr{A})$ we note the equivalence of the following statements:

- $\tau \in T(\mathscr{A}^c)$ iff
- there exists a run r of $\mathscr{A}^c$ accepting τ iff
- (by definition of $\mathscr{A}^c$) there exists a family $\mathbf{s} = \{s_x\}_{x \in \{0,1\}^*}$ of mappings, $s_x : \Delta_{\tau(x)} \to \{0,1\}$, such for all $x \in \{0,1\}^*$

  $$\gamma'(r(x), (\tau(x), s_x, 0)) = r(x0) \quad \text{and} \quad \gamma'(r(x), (\tau(x), s_x, 1)) = r(x1)$$

  and for each $p \in \{0,1\}^\omega$, $\inf_r(p)$ satisfies $\mathfrak{C}'$, iff
- all words $w^p_{\tau,\mathbf{s}}$, for $p \in \{0,1\}^\omega$, are recognized by $\mathscr{B}^c_\Omega$, i.e. $L_{\tau,\mathbf{s}} \subseteq L(\mathscr{B}^c_\Omega)$, iff
- $L_{\tau,\mathbf{s}} \cap L(\mathscr{B}_\Omega) = \emptyset$ iff (cf. Lemma 22)
- $\mathbf{s}$ is a winning Pathfinder strategy in the colouring game on τ iff
- $\tau \notin T(\mathscr{A})$.

7.3. *Regular trees and decidability of the emptiness problem for tree automata*

Each Σ-tree induces an equivalence relation $\sim_\tau$ over the set $\{0,1\}^*$ of vertices that identifies these vertices that are roots of isomorphic subtrees: $\forall x, y \in \{0,1\}^*$, $x \sim_\tau y$ iff $\tau_x = \tau_y$. A Σ-tree is said to be *regular* if the relation $\sim_\tau$ is of finite index.

Obviously a Σ-tree τ is regular iff there exists a finite labelled graph G with a distinguished vertex $v \in V(G)$ (the root) such that unravelling of G starting at v gives τ.

Theorem 23 (Rabin [26]). *For any tree automaton $\mathscr{A}$, if $T(\mathscr{A}) \neq \emptyset$ then $T(\mathscr{A})$ contains a regular tree. In fact, we can effectively decide if $T(\mathscr{A})$ is empty or not and in the last case we can effectively find a finite representation of a regular tree in $T(\mathscr{A})$.*

For a tree automaton $\mathscr{A} = (\Sigma, C, c_0, \Delta, \mathfrak{C})$ let $\overline{\Delta} = \{(c, c_1, c_2) \in C^3 \mid \exists a \in \Sigma, (c, a, c_1, c_2) \in \Delta\}$. Consider the colouring game on the infinite binary tree $\{0,1\}^*$, where initially the root ε is coloured by c_0 and other vertices are uncoloured. The players play exactly as in the colouring game from Section 7.2, the only difference is that now Automaton uses the elements of $\overline{\Delta}$ to colour successor vertices. Again, winning Automaton strategies can be identified with accepting runs of $\mathscr{A}$. What is most important however is that this new colouring game can be represented as a game over a finite arena: $G_{\text{colouring}} = (V_{\text{Automaton}}, V_{\text{Pathfinder}}, E, \varphi, C)$, where $V_{\text{Automaton}} = C$ are Automaton vertices, $V_{\text{Pathfinder}} = \overline{\Delta}$ are Pathfinder vertices, the set E of edges is the union of two sets $\{(c, \delta) \mid c \in C \text{ and } \delta = (c, c'_0, c'_1) \in \overline{\Delta}\}$ and $\{(\delta, c'_i) \mid \delta = (c, c'_0, c'_1) \in \overline{\Delta} \text{ and } c'_i \in \{c'_0, c'_1\}\}$. Each element $c \in C = V_{\text{Automaton}}$ is coloured by itself, elements of $V_{\text{Pathfinder}}$ are uncoloured. Thus in fact $G_{\text{colouring}}$ is an arena in the sense of McNaughton [19] – we can view $G_{\text{colouring}}$ as finite graph with the set $V_{\text{Automaton}} = C$ of distinguished vertices. An alternative possibility is to colour also vertices of $\overline{\Delta}$, each $(c, c_1, c_2) \in \overline{\Delta}$ being coloured by c. In this way we obtain a totally coloured arena with the same set $\inf(p)$ for any path p.)

For finite arenas the methods presented in Sections 3 and 4 are in fact recursive algorithms (the construction of winning sets and winning strategies terminates after a finite number of steps) in particular the winning set and a winning finite memory strategy are calculated effectively for each player. Obviously one can apply here also McNaughton's algorithm [19], for finite graphs the difference between these algorithms is in fact minor.

It remains to note that $T(\mathscr{A}) \neq \emptyset$ iff the vertex $c_0 \in V_{\text{Automaton}}$ belongs to Automaton's winning set of vertices (which shows the decidability of the emptiness problem) and Automaton's winning finite memory strategy for plays starting at c_0 is nothing else but a regular accepting run. Given such a run r we can find easily a regular tree $\tau \in \mathscr{T}_\Sigma$ such that r is a run over τ: choose for each $(c, c_1, c_2) \in \overline{\Delta}$ an $a \in \Sigma$ such that $(c, a, c_1, c_2) \in \Delta$ and label all $x \in \{0,1\}^*$ such that $(r(x), r(x0), r(x1)) = (c, c_1, c_2)$ with this symbol a.

We can note that McNaughton algorithm [19] has the complexity exponential in the number of graph vertices for any type of winning conditions examined in this paper. For Rabin tree automata Emerson and Jutla [8] give an algorithm testing nonemptiness in time $O((mn)^{3n})$, where m is the number of states and n the number of pairs, and show that this problem is NP-complete for such automata. On the other hand, the exact complexity of nonemptiness problem for parity (chain) automata seems to be an open problem.

As the last example of game applications we show the following result due to Gurevich and Harrington [11]:

For any tree τ recognized by a tree automaton $\mathscr{A}$ there exists an accepting run r such that, for all pairs of vertices $x, y \in \{0,1\}^*$, if the subtrees τ_x and τ_y are isomorphic and $r(x)=r(y)$ and if the Automaton memory is in the same state when we arrive at x and y in the colouring game then the subruns r_x and r_y are also isomorphic [4] (in particular for Rabin automata when Automaton has a memoryless strategy the equalities $\tau_x = \tau_y$ and $r(x)=r(y)$ imply together that $r_x = r_y$).

To obtain such a run we replace the arena G_τ considered in Section 7.2 by an arena G'_τ, with $V'_{\text{Automaton}} = \{(\tau_x, c) \mid x \in \{0,1\}^* \text{ and } c \in C\}$ as Automaton vertices and $V'_{\text{Pathfinder}} = \{(\tau_x, c, c'_0, c'_1) \mid x \in \{0,1\}^* \text{ and } (c, \tau(x), c'_0, c'_1) \in \Delta\}$ as Pathfinder vertices.

The edges leaving (τ_x, c) end in vertices of the form $(\tau_x, c, c'_0, c'_1) \in V'_{\text{Pathfinder}}$. Conversely, for $(\tau_x, c, c'_0, c'_1) \in V'_{\text{Pathfinder}}$ there are two outgoing edges, one ending in (τ_{x0}, c'_0) and the other in (τ_{x1}, c'_1). As for the games on G_τ, winning Automaton strategies for plays starting at the vertex (τ, c_0) can be identified with runs of $\mathscr{A}$ accepting τ. Now, however, finite memory winning strategies give immediately the runs having the property described by Gurevich and Harrington in virtue of the fact that if τ_x and τ_y are isomorphic then the position x coloured by c and the position y coloured by c are represented by the same vertex $(\tau_x, c) = (\tau_y, c)$ of the arena G'_τ. We can go even one step further in this direction. To fix attention suppose that $\mathscr{A}$ is a Rabin automaton. There exists a family $R = \{(\tau, r)\}_{\tau \in T(\mathscr{A})}$ such that for $(\tau, r) \in R$, r is a successful run of $\mathscr{A}$ on τ and for all $x, y \in \{0,1\}^*$ whenever $(\tau', r'), (\tau'', r'') \in R$ and $\tau'_x = \tau''_y$ and $r'(x) = r''(y)$ then $r'_x = r''_y$. Thus for the family R the property of Gurevich and Harrington holds for isomorphic subtrees of any two trees in $T(\mathscr{A})$ and not just for isomorphic subtrees of each tree separately. To see this fact consider the game played on arena obtained as the union (the usual set union) of all arenas G'_τ. In other words we play on the arena where Automaton vertices are pairs $(\tau, c) \in \mathscr{T}_\Sigma \times C$, Pathfinder vertices are all tuples (τ, c, c'_0, c'_1) where $\tau \in \mathscr{T}_\Sigma$ and $(c, \tau(\varepsilon), c'_0, c'_1) \in \Delta$, with edges going from (τ, c) to (τ, c, c'_0, c'_1) and from (τ, c, c'_0, c'_1) to (τ_0, c'_0) and to (τ_1, c'_1) (τ_0 and τ_1 are the immediate left and the immediate right subtree of the root of τ). Now the result is obvious since in this great arena isomorphic subtrees of all Σ-trees are represented by one vertex.

8. Final remarks

At the end we would like to turn reader's attention to an intriguing paper of Büchi [4], where $F_{\sigma\delta} \cap G_{\delta\sigma}$ games are examined. As it is known from descriptive set theory (cf. [16, p. 358]) $X \in F_{\sigma\delta} \cap G_{\delta\sigma}$ iff there is a (transfinite of length $<\omega_1$) descending sequence of G_δ sets: $Y_0, Y_1, \ldots, Y_\xi, \ldots$ such that $X = \bigcup_{\xi-\text{even}} (Y_\xi \setminus Y_{\xi+1})$ (an ordinal ξ is even if $\xi = \xi' + i$ where ξ' is a limit ordinal and $0 \leqslant i < \omega$ an even integer.) Therefore the games considered by Büchi can be represented as a generalization of parity games. Let G be a totally coloured arena with vertices coloured by ordinals. Player 0 wins p if the minimal element of $\inf_C(p)$ is an odd ordinal, otherwise player

[4] Recall that $r(x)$ is the colour of x in r while r_x is the whole subrun starting at x.

1 wins (note that the minimal element of a well-ordered set is always well-defined). Büchi's result seems to indicate that in this "transfinite" parity game still the winning player has a memoryless winning strategy. However, to demonstrate this conjecture induction on the length of the parity condition is no more applicable and, as noted by Gurevich [10], Büchi's paper is "indeed very hard to understand".

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