On the nature of Bondi-Metzner-Sachs transformations

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SIGRAV Conference, SISSA, September 2023

(based on Symmetry 15, 947 (2023))

Outline

- 1. Fractional linear maps and BMS
- 2. The associated Segre manifold

1.1 Cuts of null infinity and FLT

$$\zeta' = f(\zeta) = \frac{(a\zeta + b)}{(c\zeta + d)} = f_\Lambda(\zeta),$$

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

1.2 The group PSL(2,C)

$$\begin{aligned}\mathrm{PSL}(2,\mathbb{C}) &= \left\{(f,\Lambda)|\ f:\zeta\in\mathbb{C}\to f(\zeta)=\frac{(a\zeta+b)}{(c\zeta+d)},\ ad-bc=1\right\}\\ &= \mathrm{SL}(2,\mathbb{C})/\delta,\end{aligned}$$

$$\delta(a,b,c,d)=(-a,-b,-c,-d).$$

$$f(\infty)=\frac{a}{c},\ f\left(-\frac{d}{c}\right)=\infty.$$

1.3 A cut remains the unit 2-sphere under FLT provided its metric is conformally rescaled

$$K^2(\Lambda,\zeta)(d\theta\otimes d\theta+\sin^2\theta d\phi\otimes d\phi),$$

$$K(\Lambda,\zeta)=K_\Lambda(\zeta)=\frac{1+|\zeta|^2}{|a\zeta+b|^2+|c\zeta+d|^2},$$

1.4 Lengths along the generators of null infinity scale according to

$$du' = K_\Lambda(\zeta)du,$$

$$u' = K_\Lambda(\zeta)\Big[u + \alpha(\zeta, \overline{\zeta})\Big],$$

1.5 BMS transformations

$$T(\zeta) = f_\Lambda(\zeta) = \frac{(a\zeta + b)}{(c\zeta + d)},$$

$$T(u) = K_\Lambda(\zeta)\Big[u + \alpha(\zeta, \overline{\zeta})\Big].$$

1.6 Only 4 families of FLT exist (parabolic, elliptic here displayed)

$$\Lambda = A_P = \begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix},$$

$$f_\Lambda(\zeta) = f_P(\zeta) = \zeta \pm \beta.$$

$$\Lambda = A_E = \begin{pmatrix} e^{i\frac{\chi}{2}} & 0 \\ 0 & e^{-i\frac{\chi}{2}} \end{pmatrix},$$

$$f_\Lambda(\zeta) = f_E(\zeta) = e^{i\chi}\zeta.$$

1.7 Hyperbolic and Loxodromic here displayed

$$\Lambda = A_H = \begin{pmatrix} \sqrt{|\kappa|} & 0 \\ 0 & \frac{1}{\sqrt{|\kappa|}} \end{pmatrix},$$

$$f_\Lambda(\zeta) = f_H(\zeta) = |\kappa|\zeta.$$

$$\Lambda = A_L = \begin{pmatrix} \sqrt{\rho}e^{i\frac{\sigma}{2}} & 0 \\ 0 & \frac{1}{\sqrt{\rho}}e^{-i\frac{\sigma}{2}} \end{pmatrix},$$

$$f_\Lambda(\zeta) = f_L(\zeta) = \rho e^{i\sigma}\zeta.$$

1.8 The resulting conformal factors

$$K_P(\zeta) = \frac{1 + |\zeta|^2}{\left(1 + |\pm\beta + \zeta|^2\right)},$$

$$K_E(\zeta) = 1,$$

$$K_H(\zeta) = \frac{|\kappa|(1 + |\zeta|^2)}{\left(1 + |\kappa|^2|\zeta|^2\right)},$$

$$K_L(\zeta) = \frac{\rho(1 + |\zeta|^2)}{\left(1 + \rho^2|\zeta|^2\right)},$$

2.1 The complex variable for FLT is the ratio of complex variables

$$z_0 \equiv e^{i\frac{\phi}{2}} \cos\frac{\theta}{2},\ \ z_1 \equiv e^{-i\frac{\phi}{2}} \sin\frac{\theta}{2},$$

$$\begin{pmatrix} z_0' \\ z_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$$

$$u' = K_\Lambda(z_0, z_1)\Big[u + \alpha(z_0, z_1; \bar{z}_0, \bar{z}_1)\Big],$$

$$|z_0| \leq 1,\ \ |z_1| \leq 1,$$

2.2 BMS transformations are restriction to the unit circle of the maps

$$\begin{pmatrix} w_0' \\ w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix}$$

$$u' = \widetilde{K}_\Lambda(W)\Big[u + \widetilde{\alpha}(W, \overline{W})\Big]$$

2.2’ where

$$w_1|_\Gamma = z_0, \quad w_2|_\Gamma = z_1$$

$$\widetilde{K}_\Lambda\Big|_\Gamma = K_\lambda(\zeta), \quad \widetilde{\alpha}|_\Gamma = \alpha(\zeta, \overline{\zeta})$$

2.3 Let us now consider points P and P’ in complex projective planes with homogeneous coordinates

(w_0, w_1, w_2) (s_0, s_1, s_2)

$$Z_{hk} \equiv w_h s_k, \quad h, k = 0, 1, 2$$

2.4 The Segre manifold associated to BMS

- The last equation is the coordinate description of the projective image of the product of projective spaces.
- As the points P and P' are varying in their own plane, the point Z describes a four-complex-dimensional manifold, since both P and P' are varying on a two-complex-dimensional geometric object. We deal therefore with a four-complex-dimensional manifold V_{4}, the Segre manifold. If we fix P, the Eq. for Z becomes linear homogeneous in the s_{k} coordinates. Thus, to every point of the first plane, there corresponds a plane on V_{4}. Similarly, we can fix a point P' in the second plane and then let P vary in the first plane. Each of these ∞^1

2.5 The Segre manifold

- Systems of planes is an array (=schiera) by virtue of the correspondence between elements of the system and points of a plane. Thus, the Segre manifold contains two arrays of planes. Two planes of the same array do not have common points, whereas two planes belonging to different arrays have one and only one common point.
- One can also fix P and let P’ vary only on a line in its plane. One therefore obtains a curve V_{1} in the Segre manifold V_{4}. Moreover, if both P and P’ describe a line in their own plane, one obtains on the Segre manifold a V_{2} subset, i.e. a quadric.

2.6 The Segre manifold

- Thus, to every pair of lines, there corresponds a quadric. Since there exist ∞^2
- lines in a plane, the Segre manifold contains a complex fourfold infinity of quadrics.