

Fortschr. Phys. **40** (1992) 1, 1–30**Mathematical Structures of Space-Time**

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**Abstract**

At first we introduce the space-time manifold and we compare some aspects of Riemannian and Lorentzian geometry such as the distance function and the relations between topology and curvature. We then define spinor structures in general relativity, and the conditions for their existence are discussed. The causality conditions are studied through an analysis of strong causality, stable causality and global hyperbolicity. In looking at the asymptotic structure of space-time, we focus on the asymptotic symmetry group of Bondi, Metzner and Sachs, and the  $b$ -boundary construction of Schmidt. The Hamiltonian structure of space-time is also analyzed, with emphasis on Ashtekar's spinorial variables.

Finally, the question of a rigorous theory of singularities in space-times with torsion is addressed, describing in detail recent work by the author. We define geodesics as curves whose tangent vector moves by parallel transport. This is different from what other authors do, because their definition of geodesics only involves the Christoffel symbols, though studying theories with torsion. We then prove how to extend Hawking's singularity theorem without causality assumptions to the space-time of the ECSK theory. This is achieved studying the generalized Raychaudhuri equation in the ECSK theory, the conditions for the existence of conjugate points and properties of maximal timelike geodesics. Our result can also be interpreted as a no-singularity theorem if the torsion tensor does not obey some additional conditions. Namely, it seems that the occurrence of singularities in closed cosmological models based on the ECSK theory is less generic than in general relativity. Our work is to be compared with important previous papers. There are some relevant differences, because we rely on a different definition of geodesics, we keep the field equations of the ECSK theory in their original form rather than casting them in a form similar to general relativity with a modified energy-momentum tensor, and we emphasize the role played by the full extrinsic curvature tensor and by the variation formulae.

**1. Introduction**

The space-time manifold plays still a vital role in modern relativity theory, and we are going to examine it in detail through an analysis of its mathematical structures. Our first aim is to present an unified description of some aspects of Lorentzian and Riemannian geometry, of the theory of spinors, and of causal, asymptotic and Hamiltonian structure. This review paper is aimed both at theoretical and mathematical physicists interested in relativity and gravitation, and it tries to present together several topics which are treated in greatly many more books and original papers. Thus in section 2, after defining the space-time manifold, following [1] we discuss the distance function and the relations between topology and curvature in Lorentzian and Riemannian geometry. In section 3 we use two-component spinor language which is more familiar to relativists. At first we define spin space and Infeld-van der Waerden symbols, and then we present the results of Geroch on spinor structures. This section can be seen in part as complementary to important recent work

appeared in [2], and we hope it can help in improving the understanding of the foundational points of a classical treatise such as [3]. In section 4, after some basic definitions, we study three fundamental causality conditions such as strong causality, stable causality and global hyperbolicity. In section 5, the asymptotic structure of space-time is studied focusing on the asymptotic symmetry group of Bondi-Metzner-Sachs (hereafter referred to as BMS) and on the boundary of space-time. This choice of arguments is motivated by the second part of our paper, where the singularity theory in cosmology for space-times with torsion is studied. In fact the Poincaré group can be seen as the subgroup of the BMS group which maps good cuts into good cuts, and it is also known that the gauge theory of the Poincaré group leads to theories with torsion [4–7]. Thus it appears important to clarify these properties. The boundary of space-time is studied in section 5.2 defining the  $b$ -boundary of Schmidt [8], discussing its construction and related questions [9]. In section 6 we present ASHTEKAR’s spinorial variables for canonical gravity [10]. In agreement with the aims of our paper, we only emphasize the classical aspects of Ashtekar’s theory. This section presents a striking application of the concepts defined in section 3, and it illustrates the modern approach to the Hamiltonian formulation of general relativity. Finally, section 7 is devoted to the clarification of recent work of the author [11] on the singularity problem for space-times with torsion, using also concepts defined in sections 2, 4 and 5.

So far, the singularity problem for theories with torsion had been studied defining geodesics as extremal curves. However, a rigorous theory of geodesics in general relativity can be based on the concept of autoparallel curves [1, 12]. Thus it appears rather important to develop the mathematical theory of singularities when geodesics are defined as curves whose tangent vector moves by parallel transport. This definition involves the full connection with torsion, whereas extremal curves just involve the Christoffel symbols. In so doing one appreciates the role of the full extrinsic curvature tensor and of the variation formulae, two important concepts which were not considered in [13]. One can also see that one can keep the field equations of the Einstein-Cartan-Sciama-Kibble (hereafter referred to as ECSK) theory in their original form, rather than casting them (as done in [13]) in a form similar to general relativity but with a modified energy momentum tensor. We then follow and clarify [11] in proving how to extend Hawking’s singularity theorem without causality assumptions to the space-time of the ECSK theory. At the end, our concluding remarks are presented in section 8.

## 2. Lorentzian and Riemannian Geometry

### 2.1. The space-time manifold

A space-time  $(M, g)$  is the following collection of mathematical entities [1, 12]:

- (1) A connected four-dimensional Hausdorff  $C^\infty$  manifold  $M$ ;
- (2) A Lorentz metric  $g$  on  $M$ , namely the assignment of a nondegenerate bilinear form  $g|_p: T_pM \times T_pM \rightarrow \mathbb{R}$  with diagonal form  $(-, +, +, +)$  to each tangent space. Thus  $g$  has signature  $+2$  and is not positive-definite;
- (3) A time orientation, given by a globally defined timelike vector field  $X: M \rightarrow TM$ . A timelike or null tangent vector  $v \in T_pM$  is said to be future-directed if  $g(X(p), v) < 0$ , or past-directed if  $g(X(p), v) > 0$ .

Some important remarks are now in order:

- (a) The condition (1) can be formulated for each number of space-time dimensions  $\geq 2$ ;
- (b) Also the convention  $(+, -, -, -)$  for the diagonal form of the metric can be chosen [14]. This convention seems to be more useful in the study of spinors, and can be adopted

also in using tensors as Penrose does so as to avoid a change of conventions. The definitions of timelike and spacelike will then become opposite to our definitions:  $X$  is timelike if  $g(X(p), X(p)) > 0 \forall p \in M$ , and  $X$  is spacelike if  $g(X(p), X(p)) < 0 \forall p \in M$ ;

(c) The pair  $(M, g)$  is only defined up to equivalence. Two pairs  $(M, g)$  and  $(M', g')$  are equivalent if there is a diffeomorphism  $\alpha: M \rightarrow M'$  such that:  $\alpha_*g = g'$ . Thus we are really dealing with an equivalence class of pairs [12].

The fact that the metric is not positive-definite is the source of several mathematical problems. This is why mathematicians generally focused their attention on Riemannian geometry. We are now going to sum up some basic results of Riemannian geometry, and to formulate their counterpart (when possible) in Lorentzian geometry. This comparison is also very useful for gravitational physics. In fact Riemannian geometry is related to the Euclidean path-integral approach to quantum gravity [15], whereas Lorentzian geometry is the framework of general relativity.

## 2.2. Riemannian geometry versus Lorentzian geometry

A Riemannian metric  $g_0$  on a manifold  $M$  is a smooth and positive-definite section of the bundle of symmetric bilinear 2-forms on  $M$ . A fundamental result in Riemannian geometry is the Hopf-Rinow theorem. It can be formulated as follows [1]:

*Theorem 2.1.* For any Riemannian manifold  $(M, g_0)$  the following properties are equivalent:

- (1) Metric completeness:  $M$  together with the Riemannian distance function (see section 2.2.1.) is a complete metric space;
- (2) Geodesic completeness:  $\forall v \in TM$ , the geodesic  $c(t)$  in  $M$  such that  $c'(0) = v$  is defined  $\forall t \in \mathbb{R}$ ;
- (3) For some  $p \in M$ , the exponential map  $\exp_p$  is defined on the entire tangent space  $T_pM$ ;
- (4) Finite compactness: any subset  $K$  of  $M$  such that  $\sup \{d_0(p, q) : p, q \in K\} < \infty$  has compact closure.

Moreover, if any of these properties holds, we also know that:

- (5)  $\forall p, q \in M$ , there exists a smooth geodesic segment  $c$  from  $p$  to  $q$  with  $L_0(c) = d_0(p, q)$  (namely any two points can be joined by a minimal geodesic).

In Lorentzian geometry there is no sufficiently strong analogue to the Hopf-Rinow theorem. However, one can learn a lot comparing the definitions of distance function and the relations between topology and curvature in the two cases.

### 2.2.1. The distance function in Riemannian geometry

Let  $\Omega_{pq}$  be the set of piecewise smooth curves in  $M$  from  $p$  to  $q$ . Given  $c: [0, 1] \rightarrow M$  and belonging to  $\Omega_{pq}$ , there is a finite partition of  $[0, 1]$  such that  $c$  restricted to the subinterval  $[t_i, t_{i+1}]$  is smooth  $\forall i$ . The Riemannian arc length of  $c$  with respect to  $g_0$  is defined by:

$$L_0(c) \equiv \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \sqrt{g_0(c'(t), c'(t))} dt. \tag{2.1}$$

The Riemannian distance function  $d_0: M \times M \rightarrow [0, \infty)$  is then defined by [1]:

$$d_0(p, q) \equiv \inf \{L_0(c) : c \in \Omega_{pq}\}. \tag{2.2}$$

Thus  $d_0$  has the following properties:

- (1)  $d_0(p, q) = d_0(q, p) \quad \forall p, q \in M$ ;
- (2)  $d_0(p, q) \leq d_0(p, r) + d_0(r, q) \quad \forall p, q, r \in M$ ;
- (3)  $d_0(p, q) = 0$  if and only if  $p = q$ ;
- (4)  $d_0$  is continuous and,  $\forall p \in M$  and  $\varepsilon > 0$ , the family of metric balls  $B(p, \varepsilon) = \{q \in M : d_0(p, q) < \varepsilon\}$  is a basis for the manifold topology.

### 2.2.2. The distance function in Lorentzian geometry

Let  $\Omega_{pq}$  be the space of all future-directed nonspacelike curves  $\gamma: [0, 1] \rightarrow M$  with  $\gamma(0) = p$  and  $\gamma(1) = q$ . Given  $\gamma \in \Omega_{pq}$  we choose a partition of  $[0, 1]$  such that  $\gamma$  restricted to  $[t_i, t_{i+1}]$  is smooth  $\forall i = 0, 1, \dots, n-1$ . The Lorentzian arc length is then defined as [1]:

$$L(\gamma) \equiv \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sqrt{-g(\gamma'(t), \gamma'(t))} dt. \quad (2.3)$$

The Lorentzian distance function  $d: M \times M \rightarrow R \cup \{\infty\}$  is thus defined as follows. Given  $p \in M$ , if  $q$  does not belong to the causal future of  $p$  (see section 4):  $q \notin J^+(p)$ , we set  $d(p, q) = 0$ . Otherwise, if  $q \in J^+(p)$ , we set [1]:

$$d(p, q) \equiv \sup \{L_\gamma(\gamma) : \gamma \in \Omega_{pq}\}. \quad (2.4)$$

Thus such  $d(p, q)$  may not be finite, if timelike curves from  $p$  to  $q$  attain arbitrarily large arc lengths. It also fails to be symmetric in general, and one has:  $d(p, q) \geq d(p, r) + d(r, q)$  if there are future-directed nonspacelike curves from  $p$  to  $r$  and from  $r$  to  $q$ . Finally, we need to recall the definition of timelike diameter  $\text{diam}(M, g)$  of a space-time  $(M, g)$  [1]:

$$\text{diam}(M, g) \equiv \sup \{d(p, q) : p, q \in M\}. \quad (2.5)$$

### 2.2.3. Topology and curvature in Riemannian geometry

A classical result is the Myers-Bonnet theorem which shows how the properties of the Ricci curvature may influence the topological properties of the manifold. In fact one has [16]:

*Theorem 2.2.* Let  $(M, g)$  be a complete  $n$ -dimensional Riemannian manifold with Ricci curvature  $\text{Ric}(v, v)$  such that:  $\text{Ric}(v, v) \geq \frac{(n-1)}{r}$ . Then  $\text{diam}(M, g) \leq \text{diam}(S^n(r))$ ,  $\text{diam}(M, g) \leq \pi\sqrt{r}$ , and  $M$  is compact. Moreover,  $M$  has finite fundamental homotopy group.

### 2.2.4. Topology and curvature in Lorentzian geometry

The Lorentzian analogue of the Myers-Bonnet theorem can be formulated in the following way [1]:

Let  $(M, g)$  be a  $n$ -dimensional globally hyperbolic space-time (see section 4.3.) such that either:

- (1) All timelike sectional curvatures are  $\leq -l < 0$ , or:
- (2)  $\text{Ric}(v, v) \geq (n-1)l > 0 \forall$  unit timelike vectors  $v \in TM$ .

Then  $\text{diam}(M, g) \leq \frac{\pi}{\sqrt{l}}$ .

The proof of this theorem, together with the discussion of the Lorentzian analogue of the index and Rauch I, II comparison theorems can be found in chapter 10 of [1]. For another recent treatise on Riemannian geometry, see [17]. An enlightening comparison of Riemannian and Lorentzian geometry can also be found in [18].

### 3. Spinor Structure

A full account of two-component spinor calculus may be found in [3, 19–20]. Here we just wish to recall the following definitions.

Spin space [21] is a pair  $(\Sigma, \epsilon)$ , where  $\Sigma$  is a two-dimensional vector space over the complex or real numbers and  $\epsilon$  a symplectic structure on  $\Sigma$ . Such an  $\epsilon$  provides an isomorphism between  $\Sigma$  and the dual space  $\Sigma^*$ . One has:  $\lambda^A \in \Sigma, \lambda_A \in \Sigma^*$ . Unprimed (primed) spinor indices take the values 0 and 1 (0' and 1'). They can be raised and lowered by means of  $\epsilon^{AB}, \epsilon_{AB}, \epsilon^{A'B'}, \epsilon_{A'B'}$ , which are all given by:  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , according to the rules:  $q^A = \epsilon^{AB} q_B, q_A = \epsilon^B{}_{BA} q^B, q^{A'} = \epsilon^{A'B'} q_{B'}, q_{A'} = \epsilon^{B'}{}_{B'A'} q_{B'}$ . An isomorphism exists between the tangent space  $T$  at a point of space-time and the tensor product of the unprimed spin space  $S$  and the primed spin space  $S'$ :  $T \cong S \otimes S'$ . The Infeld-van der Waerden symbols  $\sigma^a{}_{AA'}$  and  $\sigma_a{}^{AA'}$  express this isomorphism, and the correspondence between a vector  $v^a$  and a spinor  $v^{AA'}$  is given by [22]:

$$v^{AA'} = \sigma_a{}^{AA'} v^a, \tag{3.1}$$

$$v^a = \sigma^a{}_{AA'} v^{AA'}. \tag{3.2}$$

The  $\sigma_a{}^{AA'}$  are given by:

$$\sigma_0 = -\frac{I}{\sqrt{2}}, \quad \sigma_i = \frac{\Sigma_i}{\sqrt{2}}, \tag{3.3}$$

where  $\Sigma_i$  are the Pauli matrices. We are now going to focus our attention on some more general aspects, following [23].

In defining spinors at a point of space-time, we may start by addressing the question of how an array of complex numbers  $\mu_{CD}^{AB}$  gets transformed in going from a tetrad  $v$  at  $p$  to a tetrad  $w$  at  $p$ . The mapping  $L: v \rightarrow w$  between  $v$  and  $w$  is realized by an element  $L$  of the restricted Lorentz group  $L_0$  (so that it preserves temporal direction and spatial parity). Now, to each  $L$  there correspond two elements  $\pm U^A{}_B$  of  $SL(2, C)$ . Thus the transformation law contains a sign ambiguity:

$$\mu_{CD}^{AB}(w) = \pm U^A{}_E \bar{U}^{B'}{}_{F'} (U^{-1})^G{}_C \bar{U}^{-1H'}{}_D \mu_{GH'}^{EF'}(v).$$

So as to remove this sign ambiguity, let us consider the six-dimensional space:  $\psi \equiv \{\text{set of all tetrads at } p\}$ . We then move to the universal covering manifold  $\tilde{\psi}$  of  $\psi$ :

$$\tilde{\psi} \equiv \{(v, \alpha) : v \in \psi, \alpha = \text{path in } \psi \text{ from } v \text{ to } w\}.$$

*Definition 3.1.*  $(v, \alpha)$  is equivalent to  $(u, \beta)$  if  $u = v$  and if we can continuously deform  $\alpha$  into  $\beta$  keeping fixed the terminal points.

An important property (usually described by the Dirac scissors argument) is that the tetrad at  $p$  changes after a  $2\pi$  rotation, but gets unchanged after a  $4\pi$  rotation. The advantage of considering  $\tilde{\psi}$  is that in so doing,  $\forall v, w \in \tilde{\psi}$ , there is a unique element  $U^A{}_B$  of  $SL(2, C)$  which transforms  $v$  into  $w$ . Thus we give [23]:

*Definition 3.2.* A spinor at  $p$  is a rule which assigns to each  $v \in \tilde{\psi}$  an array  $\mu_{CD}^{AB}$  of complex numbers such that, given  $v, w \in \tilde{\psi}$  related by  $U^A_B \in SL(2, C)$ , then:

$$\mu_{CD}^{AB}(w) = U^A_E \bar{U}^{B'}_{F'} (U^{-1})^G_C \bar{U}^{-1H'}_{D'} \mu_{GH}^{EF}(v). \quad (3.4)$$

In defining spinor structures on  $M$ , we start by considering:  $B =$  principal fibre bundle of oriented orthonormal tetrads on  $M$ . The structure group of  $B$  is the restricted Lorentz group, and the fibre at  $p \in M$  is the collection  $\psi$  of tetrads at  $p$  with given temporal and spatial orientation. The sign ambiguity is corrected taking a fibre bundle whose fibre is the universal covering space  $\tilde{\psi}$  [23].

*Definition 3.3.* A spinor structure on  $M$  is a principal fibre bundle  $\tilde{B}$  on  $M$  with group  $SL(2, C)$ , together with a  $2 - 1$  application  $\phi: \tilde{B} \rightarrow B$  such that:

- (1)  $\phi$  realizes the mapping of each fibre of  $\tilde{B}$  into a single fibre of  $B$ ;
- (2)  $\phi$  commutes with the group operations. Namely,  $\forall U \in SL(2, C)$  we have:  $\phi U = E(U) \phi$  where  $E: SL(2, C) \rightarrow L_0$  is the covering group of  $L_0$ .

*Definition 3.4.* A spinor field on  $M$  is a mapping  $\mu$  of  $\tilde{B}$  into arrays of complex numbers such that (3.4) holds.

The basic theorems about spinor structures are the following [23]:

*Theorem 3.1.* If a space-time  $(M, g)$  has a spinor structure, for this structure to be unique  $M$  must be simply connected.

*Theorem 3.2.* A space-time  $(M, g)$  oriented in space and time has spinor structure if and only if the second Stiefel-Whitney class vanishes.

*Remark:* Stiefel-Whitney classes  $w_i$  can be defined for each vector bundle  $\xi$  by means of a sequence of cohomology classes  $w_i(\xi) \in H^i(B(\xi); Z_2)$ . In so doing, we denote by  $H^i(B(\xi); Z_2)$  the  $i$ -th singular cohomology group of  $B(\xi)$  with coefficients in  $Z_2$ , the group of integers modulo 2 [24]. If  $w_2 \neq 0$ , one cannot define parallel transport of spinors on  $M$ . The orientability of space-time assumed in theorem 3.2. and in section 2.1. implies that also the first Stiefel-Whitney class must vanish.

*Theorem 3.3.*  $(M, g)$  has a spinor structure if and only if the fundamental homotopy groups of  $B$  and  $M$  are related by:

$$\pi_1(B) \approx \pi_1(M) \oplus \pi_1(\psi) = \pi_1(M) \oplus Z_2. \quad (3.5)$$

*Theorem 3.4.* A space-time  $(M, g)$  space and time-oriented has spinor structure if and only if each of its covering manifolds has spinor structure.

*Theorem 3.5.* Let  $M$  be noncompact. Then  $(M, g)$  has spinor structure if and only if a global system of orthonormal tetrads exists on  $M$  [23].

When we unwrap  $\psi$ , we annihilate  $\pi_1(\psi)$ . The existence of a spinor structure implies we can unwrap all fibres on  $B$ . Spinor structures are related to the second homotopy group of  $M$ , whereas covering spaces are related to the first homotopy group. However, it is wrong to think that a spinor structure can be created simply by taking a covering manifold. In a space-time  $(M, g)$  which does not have spinor structure, there must be some closed curve  $\gamma$  which lies in the fibre over  $p \in M$  such that [23]:

- (a)  $\gamma$  is not homotopically zero in the fibre;
- (b)  $\gamma$  can be contracted to a point in the whole bundle of frames.

A very important application of the spinorial formalism in general relativity will be studied in section 6, where we define Ashtekar's spinorial variables for canonical gravity.

#### 4. Causal Structure

Let  $(M, g)$  be a space-time, and let  $p \in M$ . The chronological future of  $p$  is defined as [1, 12]:

$$I^+(p) \equiv \{q \in M : p \ll q\}, \quad (4.1)$$

namely  $I^+(p)$  is the set of all points  $q$  of  $M$  such that there is a future-directed timelike curve from  $p$  to  $q$ . Similarly, we define the chronological past of  $p$ :

$$I^-(p) \equiv \{q \in M : q \ll p\}. \quad (4.2)$$

The causal future of  $p$  is then defined by:

$$J^+(p) \equiv \{q \in M : p \leq q\}, \quad (4.3)$$

and similarly for the causal past:

$$J^-(p) \equiv \{q \in M : q \leq p\}, \quad (4.4)$$

where  $a \leq b$  means there is a future-directed nonspacelike curve from  $a$  to  $b$ . The causal structure of  $(M, g)$  is the collection of past and future sets at all points of  $M$  together with their properties. Following [19] and [25], we shall here recall the following definitions, which will then be useful in section 4.3. and for further readings.

*Definition 4.1.* A set  $\Sigma$  is achronal if no two points of  $\Sigma$  can be joined by a timelike curve.

*Definition 4.2.* A point  $p$  is an endpoint of the curve  $\lambda$  if  $\lambda$  enters and remains in any neighbourhood of  $p$ .

*Definition 4.3.* Let  $\Sigma$  be a spacelike or null achronal three-surface in  $M$ . The future Cauchy development (or future domain of dependence)  $D^+(\Sigma)$  of  $\Sigma$  is the set of points  $p \in M$  such that every past-directed timelike curve from  $p$  without past endpoint intersects  $\Sigma$ .

*Definition 4.4.* The past Cauchy development  $D^-(\Sigma)$  of  $\Sigma$  is defined interchanging future and past in definition 4.3. The total Cauchy development of  $\Sigma$  is then given by  $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$ .

*Definition 4.5.* The future Cauchy horizon  $H^+(\Sigma)$  of  $\Sigma$  is given by:

$$H^+(\Sigma) \equiv \{X : X \in D^+(\Sigma), I^+(X) \cap D^+(\Sigma) = \emptyset\}. \quad (4.5)$$

Similarly, the past Cauchy horizon  $H^-(\Sigma)$  is defined as:

$$H^-(\Sigma) \equiv \{X : X \in D^-(\Sigma), I^-(X) \cap D^-(\Sigma) = \emptyset\}. \quad (4.6)$$

*Definition 4.6.* The edge of an achronal set  $\Sigma$  is given by all points  $p \in \bar{\Sigma}$  such that any neighbourhood  $U$  of  $p$  contains a timelike curve from  $I^-(p, U)$  to  $I^+(p, U)$  that does not meet  $\Sigma$  [18].

Our definitions of Cauchy developments differ indeed from the ones in [12], in that Hawking and Ellis look at past-inextendible curves which are timelike or null, whereas we agree with Penrose and Geroch in not including null curves in the definition. We are now going to discuss three fundamental causality conditions: strong causality, stable causality and global hyperbolicity.

#### 4.1. Strong causality

The underlying idea for the definition of strong causality is that there should be no point  $p$  such that every small neighbourhood of  $p$  intersects some timelike curve more than once [26]. Namely, the space-time  $(M, g)$  does not “almost contain” closed timelike curves. In rigorous terms, strong causality is defined as follows [19]:

*Definition 4.7.* Strong causality holds at  $p \in M$  if arbitrarily small neighbourhoods of  $p$  exist which each intersect no timelike curve in a disconnected set.

A very important characterization of strong causality can be given by defining at first the Alexandrov topology [12].

*Definition 4.8.* In the Alexandrov topology, a set is open if and only if it is the union of one or more sets of the form:  $I^+(p) \cap I^-(q)$ ,  $p, q \in M$ .

Thus any open set in the Alexandrov topology will be open in the manifold topology. Now, the following fundamental result holds [14]:

*Theorem 4.1.* The following three requirements on a space-time  $(M, g)$  are equivalent:

- (1)  $(M, g)$  is strongly causal;
- (2) the Alexandrov topology agrees with the manifold topology;
- (3) the Alexandrov topology is Hausdorff.

#### 4.2. Stable causality

Strong causality is not enough to ensure that space-time is not just about to violate causality [26]. The situation can be considerably improved if stable causality holds. For us to be able to properly define this concept, we must discuss the problem of putting a topology on the space of all Lorentz metrics on a four-manifold  $M$ . Essentially three possible topologies seem to be of major interest [26]: compact-open topology, open topology, fine topology.

##### 4.2.1. Compact-open topology

$\forall i = 0, 1, \dots, r$ , let  $\varepsilon_i$  be a set of continuous positive functions on  $M$ ,  $U$  be a compact set  $\subset M$  and  $g$  the Lorentz metric under study. We then define:  $G(U, \varepsilon_i, g) =$  set of all Lorentz metrics  $\tilde{g}$  such that:

$$\left| \frac{\partial^i \tilde{g}}{\partial x^i} - \frac{\partial^i g}{\partial x^i} \right| < \varepsilon_i \quad \text{on } U \quad \forall i.$$

In the compact-open topology, open sets are obtained from the  $G(U, \varepsilon_i, g)$  through the operations of arbitrary union and finite intersection.

##### 4.2.2. Open topology

We no longer require  $U$  to be compact, and we take  $U = M$  in section 4.2.1.



### 4.2.3. Fine topology

We define:  $H(U, \varepsilon_i, g) =$  set of all Lorentz metrics  $\tilde{g}$  such that:

$$\left| \frac{\partial^i \tilde{g}}{\partial x^i} - \frac{\partial^i g}{\partial x^i} \right| < \varepsilon_i,$$

and  $\tilde{g} = g$  out of the compact set  $U$ . Moreover, we set:  $G'(\varepsilon_i, g) = \cup H(U, \varepsilon_i, g)$ . A sub-basis for the fine topology is then given by the neighbourhoods  $G'(\varepsilon_i, g)$  [26].

Now, the underlying idea for stable causality is that space-time must not contain closed timelike curves, and we still fail to find closed timelike curves if we open out the null cones. In view of the former definitions, this idea can be formulated as follows:

*Definition 4.9.* A metric  $g$  satisfies the stable causality condition if, in the  $C^0$  open topology (see section 4.2.2.), an open neighbourhood of  $g$  exists no metric of which has closed timelike curves.

The Minkowski, FRW, Schwarzschild and Reissner-Nordstrom space-times are all stably causal. If stable causality holds, the differentiable and conformal structure can be determined from the causal structure, and space-time cannot be compact (because in a compact space-time there are closed timelike curves). A very important characterization of stable causality is given by the following theorem [12]:

*Theorem 4.2.* A space-time  $(M, g)$  is stably causal if and only if a cosmic time function exists on  $M$ , namely a function whose gradient is everywhere timelike.

### 4.3. Global hyperbolicity

Global hyperbolicity plays a key role in developing a rigorous theory of geodesics in Lorentzian geometry and in proving singularity theorems. Its ultimate meaning can be seen as requiring the existence of Cauchy surfaces, namely spacelike hypersurfaces which each nonspacelike curve intersects exactly once. In fact some authors [27] take this property as the starting point in discussing global hyperbolicity. Indeed, LERAY's original idea was that the set of nonspacelike curves from  $p$  to  $q$  must be compact in a suitable topology [28]. We shall here follow [12], [25] and [27] defining and proving in part what follows.

*Definition 4.10.* A space-time  $(M, g)$  is globally hyperbolic if:

- (a) strong causality holds;
- (b)  $J^+(p) \cap J^-(q)$  is compact  $\forall p, q \in M$ .

*Theorem 4.3.* In a globally hyperbolic space-time, the following properties hold:

- (1)  $J^+(p)$  and  $J^-(p)$  are closed  $\forall p$ ;
- (2)  $\forall p, q$ , the space  $C(p, q)$  of all nonspacelike curves from  $p$  to  $q$  is compact in a suitable topology;
- (3) there are Cauchy surfaces.

*Proof of (1).* It is well-known that, if  $(X, F)$  is a Hausdorff space and  $A \subset X$  is compact, then  $A$  is closed. In our case, this implies that  $J^+(p) \cap J^-(q)$  is closed. Moreover, it is not difficult to see that  $J^+(p)$  itself must be closed. In fact, otherwise we could find a point  $r \in \overline{J^+(p)}$  such that  $r \notin J^+(p)$ . Let us now choose  $q \in I^+(r)$ . We would then have:  $r \in \overline{J^+(p) \cap J^-(q)}$  but  $r \notin J^+(p) \cap J^-(q)$ , which implies that  $J^+(p) \cap J^-(q)$  is not closed, not in agreement with what we found before. Similarly we also prove that  $J^-(p)$  is closed.

*Remark:* A stronger result can also be proved. Namely, if  $(M, g)$  is globally hyperbolic and  $K \subset M$  is compact, then  $J^+(K)$  is closed [27].

*Proof of (3).* The proof will use the following ideas:

*Step 1.* We define a function  $f^+$ , and we prove that global hyperbolicity implies continuity of  $f^+$  on  $M$  [12].

*Step 2.* We consider the function:

$$f: p \in M \rightarrow f(p) \equiv \frac{f^-(p)}{f^+(p)}, \quad (4.7)$$

and we prove that the  $f = \text{constant}$  surfaces are Cauchy surfaces [25].

*Step 1*

The function  $f^+$  we are looking for is given by  $f^+: p \in M \rightarrow \text{volume of } J^+(p, M)$ . This can only be done with a suitable choice of measure. The measure is chosen in such a way that the total volume of  $M$  is equal to 1. For  $f^+$  to be continuous on  $M$ , it is sufficient to show that  $f^+$  is continuous on any nonspacelike curve  $\gamma$ . In fact, let  $r \in \gamma$ , and let  $\{x_n\}$  be a sequence of points on  $\gamma$  in the past of  $r$ . We now define:

$$T \equiv \cap J^+(x_n, M). \quad (4.8)$$

If  $f^+$  were not upper semi-continuous on  $\gamma$  in  $r$ , there would be a point  $q \in T - J^+(r, M)$ , with  $r \notin J^-(q, M)$ . But on the other hand, the fact that  $x_n \in J^-(q, M)$  implies that  $r \in J^-(q, M)$ , which is impossible in view of global hyperbolicity. The absurd proves that  $f^+$  is upper semi-continuous. In the same way (exchanging the role of past and future) we can prove lower semi-continuity, and thus continuity. It becomes then trivial to prove the continuity of the function  $f^+: p \in M \rightarrow \text{volume of } I^+(p, M)$ . From now on, we shall mean by  $f^+$  the volume function of  $I^+(p, M)$ .

*Step 2*

Let  $\Sigma$  be the set of points where  $f = 1$ , and let  $p \in M$  be such that  $f(p) > 1$ . The idea is to prove that every past-directed timelike curve from  $p$  intersects  $\Sigma$ , so that  $p \in D^+(\Sigma)$ . In a similar way, if  $f(p) < 1$ , one can then prove that  $p \in D^-(\Sigma)$  (which finally implies that  $\Sigma$  is indeed a Cauchy surface). The former result can be proved as follows [25].

*Step 2a*

We consider any past-directed timelike curve  $\mu$  without past endpoint from  $p$ . In view of the continuity of  $f$  proved in step 1, such a curve  $\mu$  must intersect  $\Sigma$ , provided one can show that there is  $\varepsilon \rightarrow 0^+$ :  $f_{\text{on } \mu} = \varepsilon$ , where  $\varepsilon$  is arbitrary.

*Step 2b*

Given  $q \in M$ , we denote by  $U$  a subset of  $M$  such that  $U \subset I^+(q)$ . The subsets  $U$  of this form cover  $M$ . Moreover, any  $U$  cannot be in  $I^-(r) \forall r \in \mu$ . This is forbidden by global hyperbolicity. In fact, suppose for absurd that  $q \in \bigcap_{r \in \mu} I^-(r)$ . We then choose a sequence  $\{t_i\}$  of points on  $\mu$  such that:

$$t_{i+1} \in I^-(t_i) \exists i: z \in I^-(t_i) \forall z \in \mu.$$

$\forall i$ , we also consider a timelike curve  $\mu'$  such that:

- (1)  $\mu'$  begins at  $p$ ;
- (2)  $\mu' = \mu$  to  $t_i$ ;
- (3)  $\mu'$  continues to  $q$ .

Global hyperbolicity plays a role in ensuring that the sequence  $\{t_i\}$  has a limit curve  $\Omega$ , which by construction contains  $\mu$ . On the other hand, we know this is impossible. In fact, if  $\mu$  were contained in a causal curve from  $p$  to  $q$ , it should have a past endpoint, which is not in agreement with the hypothesis. Thus, having proved that  $\exists r \in \mu: U \not\subset I^-(r)$ , we find that  $f^-(r) \rightarrow 0$  when  $r$  continues into the past on  $\mu$ , which in turn implies that  $\mu$  intersects  $\Sigma$  as we said in step 2a [25].

The proof of (2) is not given here, and can be found in [12]. Global hyperbolicity plays a key role in proving singularity theorems because, if  $p$  and  $q$  lie in a globally hyperbolic set and  $q \in J^+(p)$ , there is a nonspacelike geodesic from  $p$  to  $q$  whose length is greater than or equal to that of any other nonspacelike curve from  $p$  to  $q$ . The proof that arbitrary, sufficiently small variations in the metric do not destroy global hyperbolicity can be found for example in [25]. Globally hyperbolic space-times are also peculiar in that for them the Lorentzian distance function defined in section 2.2.2. is finite and continuous as the Riemannian distance function (see [1], p. 86). The relation between strong causality, finite distance function and global hyperbolicity is proved on p. 107 of [1]. More recent work on causal structure of Lorentzian manifolds can be found in [29] and references therein.

## 5. Asymptotic Structure

Under this name one can discuss black holes theory, gravitational radiation, positive mass theorems (for the ADM and Bondi's mass), the singularity problem. Here we choose to focus on two topics: the asymptotic symmetry group of space-time and the definition of boundary of space-time.

### 5.1. The Bondi-Metzner-Sachs group

For a generic space-time, the isometry group is simply the identity, and thus does not provide relevant information. But isometry groups play a very important role in physics. The most important example is given by the Poincaré group, which is the group of all real transformations of Minkowski space-time:

$$x' = Ax + a, \tag{5.1}$$

which leave invariant the length  $(x - y)^2$ . Namely, the Poincaré group is given by the semidirect product  $SO(3, 1) \otimes T_4$  of the Lorentz group  $O(3, 1)$  and of translations  $T_4$  in Minkowski space-time.

It is therefore very important to generalize the concept of isometry group to a suitably regular curved space-time [3]. The diffeomorphism group is not really useful because it is "too large" and it only preserves the differentiable structure of space-time. The concept of asymptotic symmetry group makes sense for any space-time  $(M, g)$  which tends to infinity either to Minkowski or to a Friedmann-Robertson-Walker model. The goal is achieved adding to  $(M, g)$  a boundary given by future null infinity, past null infinity or the whole of null infinity (hereafter referred to as "scri"). We are now going to formulate in a precise way this idea. For this purpose let us begin by recalling that the cuts of scri are spacelike two-surfaces in scri orthogonal to the generators of scri. Each cut has  $S^2$  topology. They

can be regarded as Riemann spheres with coordinates  $(\zeta, \zeta^*)$ , where  $\zeta = x + iy$  and  $\zeta^*$  is the complex conjugate of  $\zeta$ , so that locally the metric is given by:  $ds^2 = -d\zeta d\zeta^*$ . Thus, defining [20]:

$$\zeta \equiv e^{i\phi} \cot \frac{\theta}{2}, \quad (5.2)$$

we find:

$$ds^2 = -\frac{1}{4}(1 + \zeta\zeta^*)^2 d\Sigma^2, \quad d\Sigma^2 = d\theta^2 + (\sin \theta)^2 d\phi^2. \quad (5.3)$$

Thus, if we choose a conformal factor  $\Omega = \frac{2}{(1 + \zeta\zeta^*)}$ , each cut becomes the unit two-sphere.

The choice of a chart can then be used to define an asymptotic symmetry group. Indeed, the following simple but fundamental result holds [20]:

*Theorem 5.1.* All holomorphic bijections  $f$  of the Riemann sphere are of the form:

$$\hat{\zeta} = f(\zeta) = \frac{a\zeta + b}{c\zeta + d}, \quad (5.4)$$

where  $ad - bc = 1$ .

The transformations (5.4) are called fractional linear transformations (FLT). Now, if a cut is to remain a unit sphere under (5.4), we must perform another conformal transformation:  $d\hat{\Sigma}^2 = K^2 d\Sigma^2$ , where [20]:

$$K = \frac{1 + \zeta\zeta^*}{(a\zeta + b)(c^*\zeta^* + b^*) + (c\zeta + d)(a^*\zeta^* + d^*)}. \quad (5.5)$$

Finally, for the theory to remain invariant under (5.5), the lengths along the generators of scri must change according to:  $d\hat{u} = K du$ , which implies:

$$\hat{u} = K[u + \alpha(\zeta, \zeta^*)]. \quad (5.6)$$

The transformations (5.4–6) form the Bondi-Metzner-Sachs (BMS) asymptotic symmetry group of space-time. The subgroups of BMS are:

#### 5.1.1. Supertranslations

This is the subgroup  $S$  defined by:

$$\hat{u} = u + \alpha(\zeta, \zeta^*), \quad \hat{\zeta} = \zeta. \quad (5.7)$$

The quotient group  $\frac{\text{(BMS)}}{S}$  represents the orthochronous proper Lorentz group.

#### 5.1.2. Translations

This four-parameter subgroup  $T$  is given by (5.7) plus the following relation:

$$\alpha = \frac{A + B\zeta + B^*\zeta^* + C\zeta\zeta^*}{1 + \zeta\zeta^*}. \quad (5.8)$$

The name is due to the fact that a translation in Minkowski space-time generates a member of  $T$ . In fact, denoting by  $(t, x, y, z)$  cartesian coordinates in Minkowski space-time, if we set:

$$u = t - r, \quad r^2 = x^2 + y^2 + z^2, \quad \zeta = e^{i\phi} \cot \frac{\theta}{2}, \quad Z = \frac{1}{1 + \zeta\zeta^*}, \quad (5.9)$$

we find that [20]:

$$Z^2\zeta = (x + iy) \frac{1 - \frac{z}{r}}{4r}, \quad (5.10)$$

$$x = r(\zeta + \zeta^*), \quad y = -ir(\zeta - \zeta^*)Z, \quad z = r(\zeta\zeta^* - 1)Z. \quad (5.11)$$

Thus the translation:

$$t' = t + a, \quad x' = x + b, \quad y' = y + c, \quad z' = z + d, \quad (5.12)$$

implies that:

$$u' = u + Z(A + B\zeta + B^*\zeta^* + C\zeta\zeta^*) + O\left(\frac{1}{r}\right), \quad (5.13)$$

which agrees with (5.7–8).

### 5.1.3. Poincaré

A BMS transformation is obtained from a Lorentz transformation and a supertranslation. This is why there are several Poincaré groups at scri, one for each supertranslation which is not a translation, and no one of them is preferred. This implies there is not yet agreement about how to define angular momentum in an asymptotically flat space-time (because this is related to the Lorentz group which is a part of the Poincaré group as explained before). Still, the energy-momentum tensor is well-defined, because it is only related to the translations.

The Poincaré group can be defined as the subgroup of BMS which maps good cuts into good cuts [30]. Namely, there is a four-parameter collection of cuts, called good cuts, whose asymptotic shear vanishes. These good cuts provide the structure needed so as to reduce BMS to the Poincaré group. In fact, the asymptotic shear  $\sigma^0(u, \zeta, \zeta^*)$  of the  $u = \text{constant}$  null surfaces is related to the  $(\sigma')^0(u', \zeta', (\zeta'^*))$  of the  $u' = \text{constant}$  null surfaces through the relation:

$$(\sigma')^0(u', \zeta', (\zeta'^*)) = K^{-1}[\sigma^0(u, \zeta, \zeta^*) + (\text{edth})^2 \alpha(\zeta, \zeta^*)], \quad (5.14)$$

where  $\zeta'$  is the one given in (5.4), and the operator edth is defined on page 8 of [30]. In view of (5.7), for the supertranslations the relation (5.14) assumes the form:

$$(\sigma')^0(u, \zeta, \zeta^*) = \sigma^0(u' - \alpha, \zeta, \zeta^*) + (\text{edth})^2 \alpha. \quad (5.15)$$

For stationary space-times (which have a timelike Killing vector field), the Bondi system exists where  $\sigma^0 = 0$ . Therefore, a supertranslation between two Bondi systems both having  $\sigma^0 = 0$  leads to the equation:  $(\text{edth})^2 \alpha = 0$ , which is solved by the translation group. This proves in turn that there is indeed a collection of good cuts as defined before. As explained in [31] (see also [3]), in geometrical terms the main ideas can be summarized as follows. The generators of scri are the integral curves of a null vector field  $N$ . A vector field  $X$  is

called an (asymptotic) symmetry if it generates a diffeomorphism which leaves invariant the integral curves of  $N$ . Denoting by  $h$  the intrinsic metric on  $\text{scri}$ , one then has [31]:  $L_X N = -\varrho N$ ,  $L_X h = 2\varrho h$ ,  $L_N \varrho = 0$  where  $\varrho$  is a smooth function. Any linear combination and any Lie bracket of symmetries is still a symmetry, so that they form a Lie algebra denoted by  $B$ , say. Given the vector field  $X = \beta N$ , one finds that  $X$  is a symmetry if and only if [31]:  $L_N \beta = 0$ . The symmetries of this form are the supertranslations  $ST \subset B$ . As clarified in [31],  $ST$  is the Abelian infinite-dimensional ideal of  $B$ , and the quotient  $B/ST$  is found to be the Lie algebra of the Lorentz group. As remarked in [32], it should be emphasized that the basic problem in asymptotics, namely the existence of solutions to Einstein's equations whose asymptotic properties are described by the  $\text{scri}$  formalism, is still unsolved. We refer the reader to [32] for a detailed study of this problem.

## 5.2. The boundary of space-time

The singularity theorems in general relativity [12] were proved using a definition of singularities based on the  $g$ -boundary. Namely, one defines a topological space, the  $g$ -boundary, whose points are equivalence classes of incomplete nonspacelike geodesics. The points of the  $g$ -boundary are then the singular points of space-time. As emphasized for example in [8], this definition has two basic drawbacks:

- (1) it is based on geodesics, whereas in [33] it was proved there are geodesically complete space-times with curves of finite length and bounded acceleration;
- (2) there are several alternative ways of forming equivalence classes and defining the topology.

Schmidt's method is along the following lines:

*Step 1.* Connections are known to provide a parallelization of the bundle  $L(M)$  of linear frames.

*Step 2.* This parallelization can be used to define a Riemannian metric.

*Step 3.* This Riemannian metric has the effect of making a connected component of  $L(M)$  into a metric space. This connected component  $L'(M)$  is dense in a complete metric space  $L'_c(M)$ .

*Step 4.* One defines  $\bar{M}$  as the set of orbits of the transformation group on  $L'_c(M)$ .

*Step 5.* The  $b$ -boundary  $\partial M$  of  $M$  is then defined as:  $\partial M \equiv \bar{M} - M$ .

*Step 6.* Singularities of  $M$  are defined as points of the  $b$ -boundary  $\partial M$  which are contained in the  $b$ -boundary of any extension of  $M$ .

A few more details about this construction can now be given.

### *Step 1*

The parallelization of  $L(M)$  is obtained defining horizontal and vertical vector fields. For this purpose, we denote at first by  $\pi: L(M) \rightarrow M$  the mapping of the frame at  $x$  into  $x$ .

*Definition 5.1.* The curve  $\gamma$  in  $L(M)$  is horizontal if the frames  $Y_1(t), \dots, Y_n(t)$  are parallel along  $\pi(\gamma(t))$ .

*Definition 5.2.* The horizontal vector fields  $B_i$  are the unique vector fields such that:

$$\pi_*((B_i)_\gamma) = Y_i, \quad \pi_*((B(\xi))_\gamma) = \xi^i Y_i, \quad (5.16)$$

if  $\gamma = Y_1, \dots, Y_n$ , where  $\pi_*$  denotes as usual the pull-back of  $\pi$ .

*Definition 5.3.* Vertical vector fields are given by:

$$(E^*) = \left( \frac{d}{dt} R_{a(t)\gamma} \right)_{t=0}, \quad (5.17)$$

where  $R_a$  is the action of the general linear group  $GL(n, R)$  on  $L(M)$ . The parallelization of  $L(M)$  is then given by  $(E_k^{*i}, B_i)$ .

*Step 2*

*Definition 5.4.* Denoting by  $gl(n, R)$  the Lie algebra of  $GL(n, R)$ , a  $gl(n, R)$ -valued one-form  $\omega$  is expressed as:

$$\omega(Y) = \omega_k^i(Y) E_i^k. \quad (5.18)$$

*Definition 5.5.* The canonical 1-forms  $\theta^i$  are given by:

$$\pi_*(Y_\gamma) = \theta^i(Y_\gamma) Y_i, \quad (5.19)$$

if  $\gamma = Y_1, \dots, Y_n$ .

*Definition 5.6.* The Riemannian metric  $g$  is then [8]:

$$g(X, Y) = \sum_i \theta^i(X) \theta^i(Y) + \sum_{i,k} \omega_k^i(X) \omega_k^i(Y). \quad (5.20)$$

*Step 3*

The Riemannian metric  $g$  defines a distance function according to (2.2). Thus the connected component  $L'(M)$  of  $L(M)$  is a metric space, and it uniquely determines a complete metric space,  $L'_C(M)$ . Moreover,  $L'(M)$  is dense in  $L'_C(M)$ .

*Step 4*

One proves [8] that  $GL(n, R)$  is a topological transformation group on  $L'_C(M)$ , in that the transformations  $R_a$  are uniformly continuous and can be extended in a uniformly continuous way on the closure of  $L'(M)$  in  $L'_C(M)$ .

However, also Schmidt's definition has some drawbacks. In fact:

- (1) in a closed FRW universe the initial and final singularities form the same single point of the  $b$ -boundary [34];
- (2) in the FRW and Schwarzwild solutions the  $b$ -boundary points are not Hausdorff separated from the corresponding space-time [35].

A fully satisfactory improvement of Schmidt's definition is still an open problem. Unfortunately, a recent attempt appeared in [9] was not correct.

## 6. Hamiltonian Structure

Dirac's theory of constrained Hamiltonian systems [36–37] has been successfully applied to general relativity, though many unsolved problems remain on quantization [10]. The ADM formalism for general relativity is discussed in [37–39]. The derivation of boundary terms in the action integral can be found in [40–42], whereas a modern treatment of the ADM phase space for general relativity in the asymptotically flat case is in [10]. More recently, ASHTEKAR's spinorial variables have given rise to a renewed interest in canonical gravity [10]. We are now going to analyze the "new" phase space of general relativity, only paying attention to the classical theory.

The basic postulate of canonical gravity is that space-time is topologically  $\Sigma \times R$ , and it admits a foliation in spacelike three-manifolds  $\Sigma_t$ , which are all diffeomorphic to  $\Sigma$ . Ashtekar's variables for canonical gravity are very important at least for the following reasons:

- (1) they are one of the most striking applications of the spinorial formalism to general relativity;
- (2) the constraint equations assume a polynomial form, which is not achieved using the old variables;
- (3) they realize a formal analogy between gravity and Yang-Mills theory;
- (4) they could lead to an exact solution of the constraint equations of the quantum theory.

The basic ideas of the formalism of  $SU(2)$  spinors in Euclidean three-space are the following [10]. We consider  $(V, \varepsilon_{AB}, G_{A'B})$ , where  $V$  is a complex two-dimensional vector space with a nondegenerate symplectic form  $\varepsilon_{AB}$  and a positive-definite Hermitian scalar product  $G_{A'B}$ . Then, given a real three-manifold  $\Sigma$ , we take the vector bundle  $B$  over  $\Sigma$  whose fibres are isomorphic to  $(V, \varepsilon_{AB}, G_{A'B})$ . The  $SU(2)$  spinor fields on  $\Sigma$  are thus the cross-sections of  $B$ . The isomorphism between the space of symmetric, second-rank Hermitian spinors  $\lambda^{AB}$  and the tangent space to  $\Sigma$  is realized by the soldering form  $\sigma^a{}_{AB}$ , and the metric  $h$  on  $\Sigma$  is given by:

$$h^{ab} \equiv \sigma^a{}_{AB} \sigma^b{}_{CD} \varepsilon^{AC} \varepsilon^{BD} = -\text{Tr}(\sigma^a \sigma^b). \quad (6.1)$$

The conjugation of  $SU(2)$  spinors obeys the rules:

$$(\psi_A + \lambda \phi_A)^+ = \psi_A^+ + \lambda^* \phi_A^+, \quad (\psi_A^+)^+ = -\psi_A, \quad (6.2)$$

$$\varepsilon_{AB}^+ = \varepsilon_{AB}, \quad (\psi_A \phi_B)^+ = \psi_A^+ \phi_B^+, \quad (6.3)$$

$$(\psi^A)^+ \psi_A > 0, \quad \forall \psi_A \neq 0. \quad (6.4)$$

We now consider a new configuration space  $C$  in the asymptotically flat case, defined as the space of all  $\sigma^a{}_{A^B}$  such that [10]:

$$\sigma^a{}_{A^B} = \left[ 1 + \frac{M(\theta, \phi)}{r} \right]^2 (\sigma^0)^a{}_{A^B} + O\left(\frac{1}{r^2}\right). \quad (6.5)$$

The momentum conjugate to  $\sigma^a{}_{A^B}$ , following [10], is denoted by  $M_{aA^B}$  and it obeys the relations:

$$\text{Tr}(M_a \sigma^a) = O\left(\frac{1}{r^3}\right), \quad (6.6)$$

$$M_{aA^B} + \frac{1}{3} \text{Tr}(M_l \sigma^l) \sigma_{aA^B} = O\left(\frac{1}{r^2}\right). \quad (6.7)$$

The extended phase space  $\Gamma$  is the space whose points are the pairs  $(\sigma^a{}_{A^B}, M_{aA^B})$  obeying (6.5–7), and the Poisson brackets among observables are defined by:

$$\{u, v\} \equiv \int_{\Sigma} \text{Tr} \left( \frac{\delta u}{\delta M_a} \frac{\delta v}{\delta \sigma^a} - \frac{\delta u}{\delta \sigma^a} \frac{\delta v}{\delta M_a} \right) d^3x. \quad (6.8)$$

In going to the new phase space we have added three degrees of freedom, which leads to three new constraints:

$$C_{ab} = -\text{Tr}(M_{[a} \sigma_{b]}) = M_{[ab]}, \quad (6.9)$$



in addition to the Hamiltonian and momentum constraints. We are now going to consider the phase space:

$$\Gamma' \equiv \{(\tilde{\sigma}^a_{J^L}, A_{aJ^L})\}, \quad (6.10)$$

where the spinorial variables  $\tilde{\sigma}^a_{J^L}$  and  $A_{aJ^L}$  are obtained from  $\sigma^a_{J^L}$  and  $M_{aJ^L}$  as follows. The variable  $\tilde{\sigma}^a_{J^L}$  is defined by:

$$\tilde{\sigma}^a_{J^L} \equiv \sqrt{h} \sigma^a_{J^L}. \quad (6.11)$$

The step leading to  $A_{aJ^L}$  is simple but not trivial (it can be more thoroughly understood recalling the definition of Sen connection as done in [10]). At first we define a new momentum variable:

$$\pi_{aJ^L} \equiv \frac{1}{\sqrt{h}} \left[ M_{aJ^L} + \frac{1}{2} \text{Tr}(M_b \sigma^b) \sigma_{aJ^L} \right]. \quad (6.12)$$

Now, denoting by  $D$  the connection on the real three-manifold  $\Sigma$ , we define a new connection  $\tilde{D}$  by [10]:

$$\tilde{D}_a \lambda_M \equiv \partial_a \lambda_M + A_{aM}{}^C \lambda_C. \quad (6.13)$$

The spinorial variable  $A_{aJ^L}$  in (6.13) is obtained from (6.12) and from the spin-connection 1-form  $\Gamma_{aJ^L}$  of  $D$  by:

$$A_{aJ^L} \equiv \Gamma_{aJ^L} + \frac{i}{\sqrt{2}} \pi_{aJ^L}, \quad (6.14)$$

where the spin connection is known to be the unique connection which annihilates the soldering form  $\sigma^a_{JL}$ , and is given by:

$$\Gamma_a{}^{JL} \equiv -\frac{1}{2} \sigma_J{}^{EL} [\partial_a \sigma^J{}_E + \Gamma_{ba}^f \sigma^{bj}{}_E], \quad (6.15)$$

and  $\Gamma_{ba}^f$  are the Christoffel symbols involving the three-metric  $h$  on  $\Sigma$ . The new variables defined in (6.11) and (6.14) obey the Poisson bracket relations [10]:

$$\{\tilde{\sigma}^a_{J^L}(x), \tilde{\sigma}^b_{M^N}(y)\} = 0, \quad (6.16)$$

$$\{A_a{}^{JL}(x), \tilde{\sigma}^m_{MN}(y)\} = \frac{i}{\sqrt{2}} \delta(x, y) \delta_M{}^{(J} \delta_N{}^{L)}, \quad (6.17)$$

$$\{A_a{}^{J^L}(x), A_{bM}{}^N(y)\} = 0. \quad (6.18)$$

Finally, denoting by  $F_{abM}{}^N$  the curvature of  $\tilde{D}$ :

$$F_{abM}{}^C \lambda_C = 2\tilde{D}_{[a} \tilde{D}_{b]} \lambda_M, \quad (6.19)$$

the constraints of the theory assume the form [10]:

$$\tilde{D}_a(\tilde{\sigma}^a_{J^L}) = 0, \quad (6.20)$$

$$\text{Tr}(\tilde{\sigma}^a F_{ab}) = 0, \quad (6.21)$$

$$\text{Tr}(\tilde{\sigma}^a \tilde{\sigma}^b F_{ab}) = 0, \quad (6.22)$$

where the Gauss-law constraint (6.20) is due to (6.9), and (6.21–22) are respectively the momentum and Hamiltonian constraints. Setting:

$$E^a \equiv \tilde{\sigma}^a, \quad B^a \equiv \frac{1}{2} \varepsilon^{abc} F_{bc}, \quad (6.23)$$

we see that the new phase space (6.10) can be thought as a submanifold of the constrained phase space of a complexified  $SU(2)$  Yang-Mills theory (see remark (3) at the beginning of this section), and the constraints are indeed polynomial as we anticipated. One can also reverse things, and regard the  $A_{aJ}{}^L$  as configuration variables, so that their momentum conjugate becomes  $\tilde{\sigma}^a{}_J{}^L$ . In so doing the momentum constraints remain linear and the Hamiltonian constraint remains quadratic in the momenta [10].

Also, it should be emphasized that  $\tilde{\sigma}^a{}_J{}^L$  is real whereas  $A_{aJ}{}^L$  is complex, so that they are not conjugate variables in the usual sense. We can overcome this difficulty going to the complex regime. Namely, we consider a complex phase space  $\Gamma_C$  whose points are defined on a real three-manifold  $\Sigma$ . The real section  $\Gamma$  of  $\Gamma_C$  is then defined by [10]:

$$(\tilde{\sigma}^a)^+ = -\tilde{\sigma}^a, \quad (6.24)$$

$$(A_{aJ}{}^L - \Gamma_{aJ}{}^L)^+ = -(A_{aJ}{}^L - \Gamma_{aJ}{}^L). \quad (6.25)$$

In so doing we get back to (real Lorentzian) general relativity, whereas real Euclidean general relativity is defined by the conditions where (6.24) gets unchanged, whereas in (6.25) the spin-connection 1-form  $\Gamma_{aJ}{}^L$  does not appear.

At the classical level, important work has been done in [43]. In that paper, the author has shown that the trace of the extrinsic curvature tensor of the boundary of space-time is the generating function for the canonical transformation of the phase space of general relativity introduced by ASHTEKAR. When torsion is nonvanishing, an additional boundary term is present in the generating function, which has the effect of making the action complex.

## 7. Singularities for Theories with Torsion

The singularity theorems of PENROSE, HAWKING and GEROCH [12, 44–49] show that Einstein's general relativity leads to the occurrence of singularities in cosmology in a rather generic way. On the other hand, much work has also been done on alternative theories of gravitation [50]. It is by now well-known that when we describe gravity as the gauge theory of the Poincaré group, this naturally leads to theories with torsion [4–7]. The basic ideas can be summarized as follows [7, 51–52]. The holonomy theorems imply that torsion and curvature are related respectively to the groups of translations and of homogeneous transformations in the tangent vector spaces to a manifold. The introduction of torsion related to spin gives rise to a strong link between gravitation and particle physics, because it extends the holonomy group to the translations. An enlightening discussion of gauge translations can be found for example in [53–54]. In particular, the introduction of [54] clarifies from the very beginning the main geometric role played by the translations in the gauge group: they change a principal fibre bundle having no special relationship between the points on the fibres and the base manifold into the bundle of linear frames of the base manifold. When we consider the gauge theory of the Poincaré group, we discover that the gauge fields for the translation invariance are the orthonormal frames, and the gauge field for Lorentz transformations is the part of the full connection called contorsion [7]. From the point of view of fibre-bundle theory, the possibility of defining torsion is a peculiarity of relativistic theories of gravitation. Namely, the bundle  $L(M)$  of linear frames is soldered to the base  $B = M$ , whereas for gauge theories other than gravitation the bundle  $L(M)$  is loosely connected to  $M$  [5]. Denoting by  $\theta: TL(M) \rightarrow R^4$  the soldering form and by  $\omega$  a connection 1-form on  $L(M)$ , the torsion 2-form  $T$  is defined by [5]:  $T \equiv d\theta + \omega \wedge \theta$ . The Poincaré group deserves special consideration because it corresponds to an external symmetry, it yields momentum and angular momentum conservation, and its translational part can be seen as carrying matter through space-time [6].

At the very high densities present in the early universe, the effects of spin can no longer be neglected [52]. Thus it is natural to address the question: is there a rigorous theory of singularities in a space-time with torsion? The answer can only be found discussing at first the properties of geodesics in a space-time with torsion, and trying to define what is a singularity in such a theory.

### 7.1. Space-times with torsion and their geodesics

A space-time with torsion (hereafter referred to as  $U_4$  space-time) is defined adding the following fourth requirement to the ones in section 2.1.:

(4) Given a linear  $C^r$  connection  $\tilde{\nabla}$  which obeys the metricity condition, a nonvanishing tensor:

$$S(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y], \tag{7.1}$$

where  $X$  and  $Y$  are arbitrary  $C^r$  vector fields and the square bracket denotes their Lie bracket. The tensor  $\frac{S}{2}$  is then called the torsion tensor (compare with [12]).

Now, it is well-known that the curve  $\gamma$  is defined to be a geodesic curve if its tangent vector moves by parallel transport, so that  $\nabla_X X$  is parallel to  $\left(\frac{\partial}{\partial t}\right)_\gamma$  (see, however, comment before definition 7.1.). A new parameter  $s(t)$ , called affine parameter, can always be found such that, in local coordinates, this condition is finally expressed by the equation:

$$\frac{d^2 x^a}{ds^2} + \Gamma_{bc}^a \frac{dx^b}{ds} \frac{dx^c}{ds} = 0. \tag{7.2}$$

The geodesic equation (7.2) will now contain the effect of torsion through the symmetric part  $S^a_{(bc)}$  (not to be confused with the vanishing  $S_{(bc)^a}$ ). It is very useful to study this equation in a case of cosmological interest. For example, in a closed FRW universe the only nonvanishing components of the torsion tensor are the ones given in [52], so that, setting  $m = 1, 2, 3$  (7.2) yields [11]:

$$\frac{d^2 x^0}{ds^2} + a \frac{da}{ds} \frac{ds}{dt} c_{ii} \left(\frac{dx^i}{ds}\right)^2 = 0, \tag{7.3}$$

$$\frac{d^2 x^m}{ds^2} + \Gamma_{ij}^m \frac{dx^i}{ds} \frac{dx^j}{ds} + 2 \left(\frac{1}{a} \frac{da}{ds} \frac{ds}{dt} - Q\right) \frac{dx^0}{ds} \frac{dx^m}{ds} = 0. \tag{7.4}$$

In (7.3),  $c_{ii}$  are the diagonal components of the unit three-sphere metric, and we are summing over all  $i = 1, 2, 3$ . In (7.4), we used the result of [52] according to which:

$$\Gamma_{0m}^m = \frac{\dot{a}}{a} - 2Q, \quad \Gamma_{m0}^m = \frac{\dot{a}}{a} \quad \forall m = 1, 2, 3. \tag{7.5}$$

Of course,  $\dot{a}$  denotes  $\frac{da}{ds} \frac{ds}{dt}$ . Now, if the field equations are such that both  $\frac{1}{a} \frac{da}{ds} \frac{ds}{dt}$  and  $Q$  remain finite for all values of  $s$ , the model will be nonspacelike geodesically complete. If a torsion singularity is thought as a point where torsion is infinite, we are ruling out this possibility with our criterion, in addition to the requirement that the scale factor never shrinks to zero. Thus it seems that, whatever the physical source of torsion is (spin or

theories with quadratic Lagrangians etc.), nonspacelike geodesic completeness is a concept of physical relevance even though test particles do not move along geodesics [13].

An important comment is now in order. We have defined geodesics exactly as important treatises do in general relativity (see [1], p. 403; [12], p. 33) for reasons which will become even more clear studying maximal timelike geodesics in section 7.2. However, our definition differs from the one adopted in [13]. In that paper, our geodesics are just called autoparallel curves, whereas the authors interpret as geodesics the curves of extremal length whose tangent vector is parallelly transported according to the Christoffel connection. In other words, if test particles were moving along extremal curves in theories with torsion, there would be a strong reason for defining geodesics and studying singularities only as done in [13]. But, as explained in [13], the trajectories of particles differ from extremal curves and from the curves we call geodesics. Thus it appears important to improve our understanding, studying the mathematical properties of a singularity theory based on the definition of geodesics as autoparallel curves. This definition, involving the properties of the full connection, may be expected to have physical relevance, imposing regularity conditions on the geometry and on the torsion of the cosmological model which is studied. Moreover, in view of the fact that the definition of timelike, null and spacelike vectors is not affected by the presence of torsion, the whole theory of causal structure outlined in section 4 remains unchanged. Combining this remark (also made in [13]) with the qualitative argument concerning the geodesic equation, we here give the following preliminary definition [11]:

*Definition 7.1.* A  $U_4$  space-time is singularity-free if it is timelike and null geodesically complete, where geodesics are defined as curves whose tangent vector moves by parallel transport with respect to the full  $U_4$  connection.

This definition differs from the one given in [13] because we rely on a different definition of geodesics, and it has the drawbacks already illustrated at the beginning of section 5.2. However, definition 7.1. is a preliminary definition which allows a direct comparison with the corresponding situation in general relativity, is generic in that it does not depend on the specific physical theory which is the source of torsion and it has physical relevance not only for a closed FRW model but also for completely arbitrary models as we said before. Thus we can now try to make the same (and eventually additional) assumptions which lead to singularity theorems in general relativity, and check whether one gets timelike and or null geodesic incompleteness. Indeed, the extrinsic curvature tensor and the vorticity which appears in the Raychaudhuri equation will now explicitly contain the effects of torsion, and it is not a priori clear what is going to happen. Namely, if one adopts the definition 7.1. as a preliminary definition of singularities in a  $U_4$  space-time, the main issues to be studied seem to be [11]:

- (1) How can we explain from first principles that a space-time which is nonspacelike geodesically incomplete may become nonspacelike geodesically complete in the presence of torsion? And is the converse possible?
- (2) What happens in a  $U_4$  space-time [13] under the assumptions which lead to the theorems of PENROSE, HAWKING and GEROCH?

We shall now partially study question (2) in the next sub-section.

## 7.2. A singularity theorem without causality assumptions for $U_4$ space-times

In this section we shall denote by  $R(X, Y)$  the four-dimensional Ricci tensor with scalar curvature  $R$ , and by  $K(X, Y)$  the extrinsic curvature tensor of a spacelike three-surface. The energy-momentum tensor will be written as  $T(X, Y)$ , so that the Einstein equations are:

$$R(X, Y) - \frac{1}{2} g(X, Y) R = T(X, Y). \quad (7.6)$$

In so doing, we are absorbing the  $8\pi G$  factor into the definition of  $T(X, Y)$ . For the case of general relativity, it was proved in [46] that singularities must occur under certain assumptions, even though no causality requirements are made. In fact, Hawking's result [12, 46] states that space-time cannot be timelike geodesically complete if:

- (1)  $R(X, X) \geq 0$  for any nonspacelike vector  $X$   
 (which can also be written in the form:  $T(X, X) \geq g(X, X) \frac{T}{2}$ );
- (2) there exists a compact spacelike three-surface  $\Sigma$  without edge;
- (3) the trace  $K$  of the extrinsic curvature tensor  $K(X, Y)$  of  $\Sigma$  is either everywhere positive or everywhere negative.

We are now going to study the following problem: is there a suitable generalization of this theorem in the case of a  $U_4$  space-time? Indeed, a careful examination of Hawking's proof (see [12], p. 273) shows that the arguments which are to be modified or adapted in a  $U_4$  space-time are the ones involving the Raychauduri equation and the results which prove the existence or the nonexistence of conjugate points. We are now going to examine them in detail.

### 7.2.1. Raychauduri equation

The generalized Raychauduri equation in the ECSK theory of gravity has been derived in [55–56] (see also [57–58]). It turns out that, denoting by  $\tilde{\omega}_{ab}$  and  $\sigma_{ab}$  respectively the vorticity and the shear tensors, the expansion  $\theta$  for a timelike congruence of curves obeys the equation:

$$\frac{d\theta}{ds} = -(R(U, U) + 2\sigma^2 - 2\tilde{\omega}^2) - \frac{\theta^2}{3} + \tilde{\nabla}_a(\dot{U})^a. \tag{7.7}$$

In (7.7),  $U$  is the unit timelike tangent vector, and we have set:

$$2\sigma^2 \equiv \sigma_{ab}\sigma^{ab}, \quad 2\tilde{\omega}^2 \equiv (\omega_{ab} + \frac{1}{2}\tilde{S}_{ab})(\omega^{ab} + \frac{1}{2}\tilde{S}^{ab}), \tag{7.8}$$

where  $\omega_{ab}$  is the vorticity tensor obtained from the Christoffel symbols, and  $\tilde{S}_{bc}$  is obtained from the spin tensor  $\sigma_{bc}{}^a$  through a relation usually assumed to be of the form [56, 59]:

$$\sigma_{bc}{}^a = \tilde{S}_{bc}U^a. \tag{7.9}$$

### 7.2.2. Existence of conjugate points

Conjugate points are defined as in general relativity [12], but bearing in mind that now the Riemann tensor is the one obtained from the connection  $\tilde{\nabla}$  appearing in (7.1):

$$R(X, Y, Z, W) = [\tilde{\nabla}_X\tilde{\nabla}_Yg(W) - \tilde{\nabla}_Y\tilde{\nabla}_Xg(W) - \tilde{\nabla}_{[X,Y]}g(W)](Z). \tag{7.10}$$

In general relativity, if one assumes that at  $s_0$  one has  $\theta(s_0) = \theta_0 < 0$ , and  $R(U, U) \geq 0$  everywhere, then one can prove there is a point conjugate to  $q$  along  $\gamma(s)$  between  $\gamma(s_0)$  and  $\gamma\left(s_0 - \frac{3}{\theta_0}\right)$ , provided  $\gamma(s)$  can be extended to  $\gamma\left(s_0 - \frac{3}{\theta_0}\right)$ . This result is then extended to prove the existence of points conjugate to a three-surface  $\Sigma$  along  $\gamma(s)$  within a distance  $\frac{3}{\theta'}$  from  $\Sigma$ , where  $\theta'$  is the initial value of  $\theta$  given by the trace  $K$  of  $K(X, Y)$ , provided  $K < 0$

and  $\gamma(s)$  can be extended to that distance (see propositions 4.4.1 and 4.4.3 from [12]). This is achieved studying an equation of the kind (7.7) where  $\tilde{\omega}^2 = \omega^2$  is vanishing because  $\omega_{ab}$  is constant and initially vanishing, and the last term on the right-hand side vanishes as well. However, in the ECSK theory,  $\tilde{\omega}^2$  will still contribute in view of (7.8). Thus the inequality:

$$\frac{d\theta}{ds} \leq -\frac{\theta^2}{3},$$

can only make sense if we assume that:

$$R(U, U) - 2\tilde{\omega}^2 \geq 0, \quad (7.11)$$

where we do not strictly need to include  $2\sigma^2$  on the left-hand side of (7.11) because  $\sigma^2$  is positive [12, 58]. If (7.11) holds, we can write (see (7.7) and set there  $\tilde{\nabla}_a(U^a) = 0$ ):

$$\int_{\theta_0}^{\theta} y^{-2} dy \leq -\frac{1}{3} \int_{s_0}^s dx, \quad (7.12)$$

which implies:

$$\theta \leq \frac{3}{s - \left(s_0 - \frac{3}{\theta_0}\right)}, \quad (7.13)$$

where  $\theta_0 < 0$ . Thus  $\theta$  becomes infinite and there are conjugate points for some  $s \in \left[s_0, s_0 - \frac{3}{\theta_0}\right]$ . However, (7.11) is a restriction on the torsion tensor. In fact, the equations of the ECSK theory are given by (7.6) plus another one more suitably written in the form used in [13] (compare with [11] and [59]):

$$S_{bc}{}^a - \delta_b^a S_{dc}{}^d - \delta_c^a S_{bd}{}^d = \sigma_{bc}{}^a. \quad (7.14)$$

In (7.14) we have absorbed the  $8\pi G$  factor into the definition of  $\sigma_{bc}{}^a$ , whereas this is not done in (7.9). Setting  $\varepsilon = g(U, U) = -1$ ,  $\varrho = 8\pi G$ , the insertion of (7.9) into (7.14) and the multiplication by  $U_a$  yields:

$$\tilde{S}_{bc} = \frac{1}{\varrho\varepsilon} (U_a S_{bc}{}^a - U_b S_{dc}{}^d - U_c S_{bd}{}^d), \quad (7.15)$$

which implies, defining:

$$\begin{aligned} f(\omega, \omega S) &\equiv \omega_{ab} \omega^{ab} + \frac{1}{2} \omega_{ab} \tilde{S}^{ab} + \frac{1}{2} \tilde{S}_{ab} \omega^{ab} \\ &= \omega_{ab} \omega^{ab} + \frac{\omega_{ab}}{2\varrho\varepsilon} (U_h S^{abh} - U^a S_h{}^{bh} - U^b S_h{}^a{}^h) \\ &\quad + \frac{\omega^{ab}}{2\varrho\varepsilon} (U_h S_{ab}{}^h - U_a S_{hb}{}^h - U_b S_{ah}{}^h), \end{aligned} \quad (7.16)$$

and using (7.8) and (7.11), that:

$$\begin{aligned} 8\tilde{\omega}^2 &= 4f(\omega, \omega S) + \tilde{S}_{bc} \tilde{S}^{bc} \\ &= 4f(\omega, \omega S) + \varrho^{-2} (U_h S_{bc}{}^h - U_b S_{hc}{}^h - U_c S_{bh}{}^h) (U_f S^{bcf} - U^b S_f{}^{cf} - U^c S^b{}_f{}^f) \\ &\leq 4R(U, U). \end{aligned} \quad (7.17)$$

Indeed some cases have been studied [55] where  $\omega_{ab}$  is vanishing. However we here prefer to write the equations in general form. Moreover, in extending (7.13) so as to prove the existence of conjugate points to spacelike three-surfaces, the assumption  $K < 0$  on the trace  $K$  of  $K(X, Y)$  also implies another condition on the torsion tensor. In fact, denoting by  $\chi(X, Y)$  the tensor obtained from the metric and from the lapse and shift functions as the extrinsic curvature in general relativity, in a  $U_4$  space-time one has:

$$K(X, Y) = \chi(X, Y) + \lambda(X, Y), \tag{7.18}$$

where the symmetric part of  $\lambda(X, Y)$  (the only one which contributes to  $K$ ) is given by:

$$\lambda_{(ab)} = -2n^\mu S_{(a\mu b)}. \tag{7.19}$$

In (7.19) we have changed sign with respect to PILATI [60] because his convention for  $K(X, Y)$  is opposite to Hawking's convention, and we are here following Hawking so as to avoid confusion in comparing theorems. Thus the condition  $K < 0$  implies the following restriction on torsion:

$$\lambda = -2g^{ab}n^\mu S_{(a\mu b)} < -\chi. \tag{7.20}$$

When (7.11) and (7.20) hold, one follows exactly the same technique which leads to (7.13) in proving there are points conjugate to a spacelike three-surface.

### 7.2.3. Maximal timelike geodesics

In general relativity, it is known (proposition 4.5.8 of [12]) that a timelike geodesic curve  $\gamma$  from  $q$  to  $p$  is maximal if and only if there is no point conjugate to  $q$  along  $\gamma$  in  $(q, p)$ . We are now going to sum up how this result is proved and then extended so as to rule out the existence of points conjugate to three-surfaces. This last step will then be enlightening in understanding what changes in a  $U_4$  space-time [11].

We shall here follow the conventions of section 4.5 of [12], denoting by  $L(Z_1, Z_2)$  the second derivative of the arc length defined in (2.3), by  $V$  the unit tangent vector  $\frac{\partial}{\partial s}$  and by  $T_\gamma$  the vector space consisting of all continuous, piecewise  $C^2$  vector fields along the timelike geodesic  $\gamma$  orthogonal to  $V$  and vanishing at  $q$  and  $p$ . We are here just interested in proving that, if the timelike geodesic  $\gamma$  from  $q$  to  $p$  is maximal, this implies there is no point conjugate to  $q$ . The idea is to suppose for absurd that  $\gamma$  is maximal but there is a point conjugate to  $q$ . One then finds that  $L(Z, Z) > 0$ , which in turn implies that  $\gamma$  is not maximal, against the hypothesis. This is achieved taking a Jacobi field  $W$  along  $\gamma$  vanishing at  $q$  and  $r$ , and extending it to  $p$  putting  $W = 0$  in the interval  $[r, p]$ . Moreover, one considers a vector  $M \in T_\gamma$  so that  $g\left(M, \frac{D}{\partial s} W\right) = -1$  at  $r$ . In what follows, we shall just say that  $M$  is suitably chosen, in a way which will become clear later. One then defines:

$$Z \equiv \varepsilon M + \varepsilon^{-1} W, \tag{7.21}$$

where  $\varepsilon$  is positive and constant. Thus, the general formula for  $L(Z_1, Z_2)$  implies that (see lemma 4.5.6 of [12]):

$$L(Z, Z) = \varepsilon^2 L(M, M) + 2L(W, M) + \varepsilon^{-2} L(W, W) = \varepsilon^2 L(M, M) + 2, \tag{7.22}$$

which implies that  $L(Z, Z)$  is  $> 0$  if  $\varepsilon$  is suitably small, as we anticipated. The same method is also used in proving there cannot be points conjugate to a three-surface  $\Sigma$  if the timelike geodesic  $\gamma$  from  $\Sigma$  to  $p$  is maximal. However, as proved in lemma 4.5.7 of [12], in the case of a three-surface  $\Sigma$ , the formula for  $L(Z_1, Z_2)$  is of the kind:

$$L(Z_1, Z_2) = F(Z_1, Z_2) - \chi(Z_1, Z_2), \quad (7.23)$$

where  $\chi(X, Y)$  is the extrinsic curvature tensor of  $\Sigma$ . But we know that in a  $U_4$  space-time  $\chi(X, Y)$  gets replaced by the nonsymmetric tensor  $K(X, Y)$  defined in (7.18–19), which can be completed with the relation for the antisymmetric part of  $\lambda(X, Y)$ :

$$\lambda_{[ab]} = -n^\mu S_{ba\mu}. \quad (7.24)$$

Thus now the splitting (7.21) leads to a formula of the kind (7.22) where the requirement

$$L(W, M) + L(M, W) = c > 0, \quad (7.25)$$

will involve torsion because (7.23) gets replaced by:

$$L(Z_1, Z_2) = \tilde{F}(Z_1, Z_2) - K(Z_1, Z_2). \quad (7.26)$$

Namely, the left-hand side of (7.25) will contain  $K(W, M) + K(M, W)$ . The condition (7.25) also clarifies how to suitably choose  $M$  in a  $U_4$  space-time. It is worth emphasizing that only  $\lambda_{(ab)}$  contributes to (7.25) because the contributions of  $\lambda_{[ab]}$  coming from  $K(M, W)$  and  $K(W, M)$  add up to zero. In proving (7.26), the first step is the generalization of lemma 4.5.4 of [12] to a  $U_4$  space-time. This is achieved remarking that the relation:

$$\frac{\partial}{\partial u} g \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = 2g \left( \frac{D}{\partial u} \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right), \quad (7.27)$$

is also valid in a  $U_4$  space-time, where now  $\frac{D}{\partial u}$  denotes the covariant derivative along the curve with respect to the full  $U_4$  connection. In fact, denoting by  $X$  the vector  $\frac{\partial}{\partial t}$  and using the definition of covariant derivative along a curve one finds [11]:

$$\frac{\partial}{\partial u} g(X, X) = 2g \left( \frac{D}{\partial u} X, X \right) + X^a X^b \frac{D}{\partial u} g_{ab}, \quad (7.28)$$

where  $\frac{D}{\partial u} g_{ab}$  is vanishing if the connection obeys the metricity condition, which is also assumed in a  $U_4$  space-time (see section 7.1. and [4]). In other words, the key role is played by the connection which obeys the metricity condition, and  $\frac{\partial}{\partial u} g(X, X)$  will implicitly contain the effects of torsion because of the relation:

$$\frac{DX^a}{\partial u} \equiv \frac{\partial X^a}{\partial u} + \Gamma_{bc}^a \frac{dx^b}{du} X^c. \quad (7.29)$$

Although this point seems to be elementary, it plays a vital role in leading to (7.26). This is why we chose to emphasize it [11].



## 7.2.4. The singularity theorem

If we now compare the results discussed or proved in sections 7.2.1–3. with p. 273 of [12], we are led to state the following singularity theorem:

*Theorem 7.1.* The  $U_4$  space-time of the ECSK theory cannot be timelike geodesically complete if:

- (1)  $R(U, U) - 2\tilde{\omega}^2 \geq 0$  for any nonspacelike vector  $U$ ;
- (2) there exists a compact spacelike three-surface  $S$  without edge;
- (3) the trace  $K$  of the extrinsic curvature tensor  $K(X, Y)$  of  $S$  is either everywhere positive or everywhere negative, and  $L(W, M) + L(M, W) = c > 0$  as defined in (7.25–26) and before.

Conditions (1) and (3) will then involve the torsion tensor defined through (7.1). Indeed, the second part of condition (3) can be seen as a prerequisite, but we have chosen to insert it into the statement of the theorem so as to present together all conditions which involve the extrinsic curvature tensor  $K(X, Y)$ . The compatibility of (1) with the field equations of the ECSK theory is expressed by (7.17) whenever (7.9) makes sense. Otherwise, (7.17) is to be replaced by a different relation. It is worth emphasizing that if we switch off torsion, condition (1) becomes the one required in general relativity because, as explained on pp. 96–97 of [12], the vorticity of the torsion-free connection vanishes wherever a  $3 \times 3$  matrix which appears in the Jacobi fields is nonsingular. Finally, if  $\tilde{\nabla}_a(\dot{U})^a$  is not vanishing as we assumed so far (see (7.7) and comment before (7.12)) following [55–56], the condition (1) of our theorem is to be replaced by [11]:

- (1')  $R(U, U) - 2\tilde{\omega}^2 - \tilde{\nabla}_a(\dot{U})^a \geq 0$  for any nonspacelike vector  $U$ .

## 8. Concluding Remarks

At first we have seen our task as presenting an unified description of some aspects of the differentiable, spinor, causal, asymptotic and Hamiltonian structure of space-time. There is a very rich literature on these topics on specialized books [1, 3, 10, 12, 14, 16–18] and on the original papers (see also [61–78]), but we thought it was important to analyze them in a single paper. This choice of arguments helps also nonexpert readers in gaining familiarity with concepts and techniques frequently used in classical gravity and which also find application in quantum gravity, as it happens for the theory of spinors and for Ashtekar's variables. Moreover, causal and asymptotic structure play a key role for singularity theory in cosmology, which is a basic problem of classical theories of gravity and constitutes the main motivation for the study of quantum cosmology (see papers in [79–80]). As a partial completion of what we studied so far, the following remarks and mentions are now in order.

- (a) An alternative description of space-time has been recently proposed in [81]. In that paper, the author has shown that the gravitational field in general relativity has the properties of a parametric manifold, namely a mathematical structure generalizing the concept of gauge fields. The author then explains how space-time can be seen as a parametric three-manifold supplied with a metric and a gravitational potential, and he develops the theory of parametric spinors [81].
- (b) Relativists are quite often interested in four-dimensional space-time models with the associated two-component spinor language [21]. However, from the point of view of theoretical physics, different formalisms are also of interest. Thus the readers can look at [82–83], and also consider a classical paper such as [84]. Other remarkable results on spinors are the ones proved in [85].

- (c) In section 4.2. we briefly motivated and described stable causality. Recent progress on the topology of stable causality is due to [86]. The authors give an enlightening discussion of causally convex and stably causally convex sets and of their topologies. For example, they prove that a point of space-time has arbitrarily small neighbourhoods that are stably causally convex sets if and only if stable causality holds. They also define returning sets, analyzing the structure of subsets that control the fulfillment of strong and stable causality at a point. In [87] volume functions have been used to characterize strong causality, global hyperbolicity and other causality conditions, and in [88] the causal boundary for stably causal space-times has been analyzed. In [89], strong and stable causality have been characterized in terms of causal functions through two important theorems.
- (d) There is a very rich literature on the asymptotic structure of space-time [90–91]. For example, the structure of the gravitational field at spatial infinity is studied through an analysis of asymptotically Euclidean and Minkowskian space-times in [92–94]. More recently, impressive progress has been made in studying the global structure of simple space-times in [95]. In that paper, the author has proved what follows [95]:
- (1) Future null infinity is diffeomorphic to the complement of a point in some contractible open three-manifold;
  - (2) The strongly causal region  $\Sigma_{SC}$  of future null infinity is diffeomorphic to  $S^2 \times \mathbb{R}$ ;
  - (3) Every compact connected spacelike two-surface in future null infinity is contained in  $\Sigma_{SC}$ ;
  - (4) Space-time must be globally hyperbolic.
- (e) In section 6 we only focused on some classical aspects of Ashtekar’s formalism for canonical gravity. However its main motivation is the development of a nonperturbative approach to quantum gravity. At present, the main interest is in a representation where quantum states arise as functions of loops on a three-manifold, and in so doing a class of exact solutions to all quantum constraints has been obtained for the first time (see [96] and references therein).
- (f) Very recent progress in singularity theory in cosmology is due to [97–98]. In [97] the author has studied an alternative interpretation according to which gravitational collapse may give rise not to singularities but to chronology violation. He has found an example of a singularity-free chronology violating space-time with a nonachronal closed trapped surface. In [98], a remarkable proof has been given that a nonempty chronology violation set with compact boundary causes singularities. In other words, these papers shed new light on the problem of whether causality violations lead to the occurrence of singularities, and one can now prove that causality violations that do not extend to infinity must cause singularities (see [98] and references therein). In [99] it has been shown that a large class of time-dependent solutions to Einstein’s equations are classical solutions to string theory. Interestingly enough, these include metrics with large curvature and some with space-time singularities.

Finally, in section 7 we have studied other aspects of the singularity problem in cosmology. We have then taken the point of view according to which nonspacelike geodesic incompleteness can be used as a preliminary definition of singularities also in space-times with torsion. We have finally been able to show under which conditions Hawking’s singularity theorem without causality assumptions can be extended to the space-time of the ECSK theory. However, when we assume (7.9) and we require consistency of the additional condition (7.11) with the equations of the ECSK theory, we end up with the relation (7.17) which explicitly involves the torsion tensor on the left-hand side (of course, the torsion tensor is also present in  $R(U, U)$  through the connection coefficients, but this is an implicit appearance of torsion, and it is better not to make this splitting). Also the conditions (7.20) and (7.25) involve the torsion tensor in an explicit way if one uses the

formula (7.18). This is why we interpret our result as an indication of the fact that the presence of singularities in the ECSK theory is less generic than in general relativity. Our result is to be compared with [13]. The relevant differences between our work and their work are:

- (1) We rely on a different definition of geodesics, as explained in section 7.1.;
- (2) We emphasize the role played by the full extrinsic curvature tensor and by the variation formulae in  $U_4$  theory, a remark which is absent in [13];
- (3) We keep the field equations of the ECSK theory in their original form, whereas the authors in [13] cast them in a form analogous to general relativity, but with a modified energy-momentum tensor which contains torsion. We think this technique is not strictly needed [11]. Moreover, from a Hamiltonian point of view, the splitting of the Riemann tensor into the one obtained from the Christoffel symbols plus the one explicitly related to torsion does not seem to be in agreement with the choice of the full connection as a canonical variable. In fact, if we look for example at models with quadratic Lagrangians in  $U_4$  theory, the frame and the full connection are to be regarded as independent variables [52], and this choice of canonical variables has also been made for the ECSK theory [100–102].

Problems to be studied for further research are the generalization to  $U_4$  space-times of the other singularity theorems in [12] using our approach, and of the results in [8] that we outlined in section 5.2. Moreover, the generalization to  $U_4$  space-times of the classification of singularities appeared in [69], and its relation to the preliminary definition of singularities we used in this paper (namely specification of the regularity condition needed for the Riemann tensor and for the full connection) deserves careful consideration.

Recently, the singularity problem for space-times with torsion has also been studied in [103] for the case of classical  $N = 1$  supergravity. In that paper the author has used the modified weak-energy condition for theories with torsion developed in [13]. Pullin has found that spin-spin contact interactions cannot avert singularities in general. Finally, he has presented a singularity-free model for a spatially homogeneous Rarita-Schwinger field in a FRW space-time. Another recent approach to the gauge theory of gravity is the one in [104], where the authors have given a coordinate-free description of the  $SO(3, 2)$  theory of gravity. The groups  $SO(3, 2)$  and  $SO(4, 1)$  are of special interest because they are the only ones reducing to the Poincaré group by a process called Wigner-Inonu contraction [105].  $SO(3, 2)$  is the group which leads to supersymmetry as a natural extension. The geometric analysis in [104] yields a better understanding of the embedding of vierbein and Lorentz connection into the connection of a larger symmetry group.

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