AN INTRODUCTION TO QUANTUM GRAVITY

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Summary

Quantum gravity was born as that branch of modern theoretical physics that tries to unify its guiding principles, i.e., quantum mechanics and general relativity. Nowadays it is providing new insight into the unification of all fundamental interactions, while giving rise to new developments in mathematics. The various competing theories, e.g. string theory and loop quantum gravity, have still to be checked against observations. We review the classical and quantum foundations necessary to

study field-theory approaches to quantum gravity, the passage from old to new unification in quantum field theory, canonical quantum gravity, the use of functional integrals, the properties of gravitational instantons, the use of spectral zeta-functions in the quantum theory of the universe, Hawking radiation, some theoretical achievements and some key experimental issues.

1. Introduction

The aim of theoretical physics is to provide a clear conceptual framework for the wide variety of natural phenomena, so that not only are we able to make accurate predictions to be checked against observations, but the underlying mathematical structures of the world we live in can also become sufficiently well understood by the scientific community. What are therefore the key elements of a mathematical description of the physical world? Can we derive all basic equations of theoretical physics from a set of symmetry principles? What do they tell us about the origin and evolution of the universe? Why is gravitation so peculiar with respect to all other fundamental interactions?

The above questions have received careful consideration and have led, in particular, to several approaches to a theory aimed at achieving a synthesis of quantum physics on the one hand, and general relativity on the other. This remains, possibly, the most important task of theoretical physics. In the early work of the 1930s, Rosenfeld [131,132] computed the gravitational self-energy of a photon in the lowest order of perturbation theory, and obtained a quadratically divergent result. With hindsight, one can say that Rosenfeld's result implies merely a renormalization of charge rather than a non-vanishing photon mass [40]. A few years after Rosenfeld's papers [131,132], Bronstein realized that the limitation posed by general relativity on the mass density radically distinguishes the theory from quantum electrodynamics and would ultimately lead to the need to reject Riemannian geometry and perhaps also to reject our ordinary concepts of space and time [20,135].

Indeed, since the merging of quantum theory and special relativity has given rise to quantum field theory in Minkowski spacetime, while quantum field theory and classical general relativity, taken without modifications, have given rise to an incomplete scheme such as quantum field theory in curved spacetime [65], which however predicts substantially novel features like Hawking radiation [87,88], here outlined in Section 7, one is led to ask what would result from the "unification" of quantum field theory and gravitation, despite the lack of a quantum gravity phenomenology in earth-based laboratories. The resulting theory is expected to suffer from ultraviolet divergences [157], and the 1-loop [94] and 2-loop [74] calculations for pure gravity which are outstanding pieces of work. As is well described in Ref. [157], if the coupling constant of a field theory has dimension mass^d in $\hbar = c = 1$ units, then the integral for a Feynman diagram of order N behaves at large momenta like $\int p^{A-Nd} dp$, where A depends on the physical process considered but not on the order N. Thus, the "harmful" interactions are those having negative values of d, which is precisely the case for Newton's constant G, where $d = -2$, since $G = 6.67 \times 10^{-39} \mathrm{GeV}^{-2}$ in $\hbar = c = 1$ units. More precisely, since the scalar curvature contains second derivatives of the metric, the corresponding momentum-space vertex functions behave like p^2, and the propagator like p^{-2}. In d dimensions each loop integral contributes p^d, so that with L loops, V vertices and P internal lines, the superficial degree D of divergence of a Feynman diagram is given by [53]

$$D = dL + 2V - 2P. \tag{1}$$

Moreover, a topological relation holds:

$$L = 1 - V + P, \tag{2}$$

which leads to [53]

$$D = (d - 2)L + 2. \qquad (3)$$

In other words, D increases with increasing loop order for $d > 2$, so that it clearly leads to a non-renormalizable theory.

A quantum theory of gravity is expected, for example, to shed new light on singularities in classical cosmology. More precisely, the singularity theorems prove that the Einstein theory of general relativity leads to the occurrence of spacetime singularities in a generic way [86]. At first sight one might be tempted to conclude that a breakdown of all physical laws occurred in the past, or that general relativity is severely incomplete, being unable to predict what came out of a singularity. It has been therefore pointed out that all these pathological features result from the attempt of using the Einstein theory well beyond its limit of validity, i.e. at energy scales where the fundamental theory is definitely more involved. General relativity might be therefore viewed as a low-energy limit of a richer theory, which achieves the synthesis of both the **basic principles** of modern physics and the **fundamental interactions** in the form currently known.

So far, no less than 16 major approaches to quantum gravity have been proposed in the literature. Some of them make a direct or indirect use of the action functional to develop a Lagrangian or Hamiltonian framework. They are as follows.

1. Canonical quantum gravity [16,17,43,44,32,99,100,6,54,144].
2. Manifestly covariant quantization [116, 33, 94, 74, 7, 152, 21, 103].
3. Euclidean quantum gravity [68, 90].
4. R-squared gravity [142].
5. Supergravity [64,148].
6. String and brane theory [162, 98, 10].
7. Renormalization group and Weinberg’s asymptotic safety [129,106].
8. Non-commutative geometry [26, 75].

   Among these 8 approaches, string theory is peculiar because it is not field-theoretic, spacetime points being replaced by extended structures such as strings.

   A second set of approaches relies instead upon different mathematical structures with a more substantial (but not complete) departure from conventional pictures, i.e.

9. Twistor theory [122,123].
10. Asymptotic quantization [67, 5].
11. Lattice formulation [114, 22].
12. Loop space representation [133,134,136,145,154].
13. Quantum topology [101], motivated by Wheeler’s quantum geometrodynamics [159].
14. Simplicial quantum gravity [72, 1, 109, 2] and null-strut calculus [102].
15. Condensed-matter view: the universe in a helium droplet [155].
16. Affine quantum gravity [105].

After such a concise list of a broad range of ideas, we hereafter focus on the presentation of some very basic properties which underlie whatever treatment of classical and quantum gravity, and are therefore of interest for the general reader rather than (just) the specialist. He or she should revert to the above list only after having gone through the material in Sections 2–7.

2. Classical and Quantum Foundations

Before any attempt to quantize gravity we should spell out how classical gravity can be described in modern language. This is done in the subsection below.

2.1. Lorentzian Spacetime and Gravity

In modern physics, thanks to the work of Einstein [51], space and time are unified into the spacetime manifold (M,g), where the metric g is a real-valued symmetric bilinear map

$$g : T_p(M) \times T_p(M) \to \mathbf{R}$$

of Lorentzian signature. The latter feature gives rise to the light-cone structure of spacetime, with vectors being divided into timelike, null or spacelike depending on whether $g(X,X)$ is negative, vanishing or positive, respectively. The classical laws of nature are written in tensor language, and *gravity is the curvature of spacetime*. In the theory of general relativity, gravity couples to the energy-momentum tensor of matter through the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{4}$$

The Einstein–Hilbert action functional for gravity, giving rise to Eq. (4), is diffeomorphism-invariant, and hence general relativity belongs actually to the general set of theories ruled by an infinite-dimensional [31] invariance group (or pseudo-group). With hindsight, following DeWitt [39], one can say that general relativity was actually the first example of a non-Abelian gauge theory (about 38 years before Yang–Mills theory [164]).

Note that the spacetime manifold is actually an equivalence class of pairs (M,g), where two metrics are viewed as equivalent if one can be obtained from the other through the action of the diffeomorphism group $\mathrm{Diff}(M)$. The metric is an additional geometric structure that does not necessarily solve any field equation.

2.2. From Schrödinger to Feynman

Quantum mechanics deals instead, mainly, with a probabilistic description of the world on atomic or sub-atomic scale. It tells us that, on such scales, the world can be described by a Hilbert space structure, or suitable generalizations. Even in the relatively simple case of the hydrogen atom, the appropriate Hilbert space is infinite-dimensional, but finite-dimensional Hilbert spaces play a role as well. For example, the space of spin-states of a spin-s particle is $\mathbf{C}^{2s+1}$ and is therefore finite-dimensional. Various pictures or formulations of quantum mechanics have been developed over the years, and their key elements can be summarized as follows:

(i) In the *Schrödinger picture*, one deals with wave functions evolving in time according to a first-order equation. More precisely, in an abstract Hilbert space $\mathcal{H}$, one studies the Schrödinger equation

$$\mathrm{i}\hbar \frac{d\psi}{dt} = \hat{H}\psi, \tag{5}$$

where the state vector ψ belongs to $\mathcal{H}$, while $\hat{H}$ is the Hamiltonian operator. In wave mechanics, the emphasis is more immediately put on partial differential equations, with the wave function viewed as a complex-valued map $\psi : (x,t) \to \mathbf{C}$ obeying the equation

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m} \Delta + V \right) \psi, \tag{6}$$

where $-\Delta$ is the Laplacian in Cartesian coordinates on $\mathbf{R}^3$ (with this sign convention, its symbol is positive-definite).

(ii) In the *Heisenberg picture*, what evolves in time is instead the operators, according to the first-order equation

$$\mathrm{i}\hbar \frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}]. \tag{7}$$

Heisenberg performed a quantum mechanical re-interpretation of kinematic and mechanical relations [93] because he wanted to formulate quantum theory in terms of observables only.

(iii) In the *Dirac quantization*, from an assessment of the Heisenberg approach and of Poisson brackets [41], one discovers that quantum mechanics can be made to rely upon the basic commutation relations involving position and momentum operators:

$$[\hat{q}^j, \hat{q}^k] = [\hat{p}_j, \hat{p}_k] = 0, \tag{8}$$

$$[\hat{q}^j, \hat{p}_k] = \mathrm{i}\hbar \delta^j_k. \tag{9}$$

For generic operators depending on $\hat{q}, \hat{p}$ variables, their formal Taylor series, jointly with application of (8) and (9), should yield their commutator.

(iv) *Weyl quantization*. The operators satisfying the canonical commutation relations (9) cannot be both bounded [57], whereas it would be nice to have quantization rules not involving unbounded operators and domain problems. For this purpose, one can consider the strongly continuous 1-parameter unitary groups having position and momentum as their infinitesimal generators. These read as $V(t) \equiv \mathrm{e}^{\mathrm{i}t\hat{q}}$, $U(s) \equiv \mathrm{e}^{\mathrm{i}s\hat{p}}$, and satisfy the Weyl form of canonical commutation relations, which is given by

$$U(s)V(t) = \mathrm{e}^{\mathrm{i}st\hbar} V(t)U(s). \tag{10}$$

Here the emphasis was, for the first time, on group-theoretical methods, with a substantial departure from the historical development, that relied instead heavily on quantum commutators and their relation with classical Poisson brackets.

(v) *Feynman quantization* (i.e., Lagrangian approach). The Weyl approach is very elegant and far-sighted, with several modern applications [57], but still has to do with a more rigorous way of doing canonical quantization, which is not suitable for an inclusion of relativity. A spacetime approach to ordinary quantum mechanics was instead devised by Feynman [62] (and partly Dirac himself [42]), who proposed to express the Green kernel of the Schrödinger equation in the form

$$G[x_f, t_f; x_i, t_i] = \int_{\text{all paths}} \mathrm{e}^{\mathrm{i}S/\hbar} d\mu, \tag{11}$$

where $d\mu$ is a suitable (putative) measure on the set of all spacetime paths (including continuous, piecewise continuous, or even discontinuous paths) matching the initial and final conditions. This point of view has enormous potentialities in the quantization of field theories, since it preserves manifest covariance and the full symmetry group, being derived from a Lagrangian.

It should be stressed that quantum mechanics regards wave functions only as a technical tool to study bound states (corresponding to the discrete spectrum of the Hamiltonian operator $\hat{H}$), scattering states (corresponding instead to the continuous spectrum of $\hat{H}$), and to evaluate probabilities (of finding the values taken by the observables of the theory). Moreover, it is meaningless to talk about an elementary phenomenon on atomic (or sub-atomic) scale unless it is registered [160], and quantum mechanics in the laboratory needs also an external observer and assumes the so-called reduction of the wave packet (see [57] and references therein). There exist indeed different interpretations of quantum mechanics, e.g. Copenhagen [160], hidden variables [15], many worlds [60, 35].

2.3. Spacetime Singularities

Now we revert to the geometric side. In Riemannian or pseudo-Riemannian geometry, geodesics are curves whose tangent vector X moves by parallel transport [85], so that eventually

$$\frac{dX^{\lambda}}{ds}+\Gamma_{\mu\nu}^{\lambda}X^{\mu}X^{\nu}=0, \tag{12}$$

where s is the affine parameter and $\Gamma_{\mu\nu}^{\lambda}$ are the connection coefficients. In general relativity, timelike geodesics correspond to the trajectories of freely moving observers, while null geodesics describe the trajectories of zero-rest-mass particles (Section 8.1 of Ref. [85]). Moreover, a spacetime (M,g) is said to be singularity-free if *all timelike and null geodesics can be extended to arbitrary values of their affine parameter*. At a spacetime singularity in general relativity, all laws of classical physics would break down, because one would witness very pathological events such as the sudden disappearance of freely moving observers, and one would be completely unable to predict what came out of the singularity. In the 1960s, Penrose [121] proved first an important theorem on the occurrence of singularities in gravitational collapse (e.g. formation of black holes). Subsequent work by Hawking [79, 80, 81, 82, 83], Geroch [66], Ellis and Hawking [84, 52], Hawking and Penrose [86] proved that spacetime singularities are generic properties of general relativity, provided that physically realistic energy conditions hold. Very little analytic use of the Einstein equations is made, whereas the key role emerges of topological and global methods in general relativity.

On the side of singularity theory in classical cosmology, explicit mention should be made of the work in Ref. [14], since it has led to significant progress by Damour et al. [27], despite having failed to prove singularity avoidance in classical cosmology. As pointed out in Ref. [27], the work by Belinsky et al. is remarkable because it gives a description of the generic asymptotic behaviour of the gravitational field in 4-dimensional spacetime in the vicinity of a spacelike singularity. Interestingly, near the singularity the spatial points essentially decouple, i.e. the evolution of the spatial metric at each spatial point is asymptotically governed by a set of second-order, non-linear ordinary differential equations in the time variable [14]. Moreover, the use of qualitative Hamiltonian methods leads naturally to a billiard description of the asymptotic evolution, where the logarithms of spatial scale factors define a geodesic motion in a region of the Lobachevskii plane, interrupted by geometric reflections against the walls bounding this region. Chaos follows because the Bianchi IX billiard has finite volume [27]. A self-contained derivation of the billiard picture for inhomogeneous solutions in D dimensions, with dilaton and p-form gauge fields, has been obtained in Ref. [27].

2.4. Unification of All Fundamental Interactions

The fully established unifications of modern physics are as follows.

(i) *Maxwell*: electricity and magnetism are unified into electromagnetism. All related phenomena can be described by an antisymmetric rank-two tensor field, and derived from a 1-form, called the potential.

(ii) *Einstein*: space and time are unified into the spacetime manifold. Moreover, inertial and gravitational mass, conceptually different, are actually unified as well.

(iii) *Standard model of particle physics*: electromagnetic, weak and strong forces are unified by a non-Abelian gauge theory, normally considered in Minkowski spacetime (this being the base space in fibre-bundle language).

The physics community is now familiar with a picture relying upon four fundamental interactions: electromagnetic, weak, strong and gravitational. The large-scale structure of the universe, however, is ruled by gravity only. All unifications beyond Maxwell involve non-Abelian gauge groups (either Yang–Mills or Diffeomorphism group). At least three extreme views have been developed along the years, i.e.,

(i) Gravity arose first, temporally, in the very early Universe, then all other fundamental interactions.

(ii) Gravity might result from Quantum Field Theory (this was the Sakharov idea [139]).

(iii) The vacuum of particle physics is regarded as a cold quantum liquid in equilibrium. Protons, gravitons and gluons are viewed as collective excitations of this liquid [155].

3. Canonical Quantum Gravity

Although Hamiltonian methods differ substantially from the Lagrangian approach used in the construction of the functional integral (see the following sections), they remain nevertheless of great importance both in cosmology and in light of modern developments in canonical quantum gravity [6,136,144], which is here presented within the original framework of quantum geometrodynamics. For this purpose, it may be useful to describe the main ideas of the Arnowitt-Deser-Misner (hereafter referred to as ADM) formalism. This is a canonical formalism for general relativity that enables one to re-write Einstein's field equations in first-order form and explicitly solved with respect to a time variable. For this purpose, one assumes that 4-dimensional spacetime (M,g) can be foliated by a family of $t = \text{constant}$ spacelike surfaces S_t, giving rise to a $3+1$ decomposition of the original 4-geometry. The basic geometric data of this decomposition are as follows [53].

(1) The induced 3-metric h of the three-dimensional spacelike surfaces S_t. This yields the intrinsic geometry of the 3-space. h is also called the first fundamental form of S_t, and is positive-definite with our conventions.

(2) The way each S_t is imbedded in (M,g). This is known once we are able to compute the spatial part of the covariant derivative of the normal n to S_t. On denoting by ∇ the 4-connection of (M,g), one is thus led to define the tensor

$$K_{ij} \equiv -\nabla_j n_i. \tag{13}$$

Note that K_{ij} is symmetric if and only if ∇ is symmetric. In general relativity, an equivalent definition of K_{ij} is $K_{ij} \equiv -\frac{1}{2}(L_n h)_{ij}$, where L_n denotes the Lie derivative along the normal to S_t. The tensor K is called extrinsic-curvature tensor, or second fundamental form of S_t.

(3) How the coordinates are propagated off the surface S_t. For this purpose one defines the vector $(N, N^1, N^2, N^3)dt$ connecting the point (t, x^i) with the point $(t+dt, x^i)$. Thus, given the surface $x^0 = t$ and the surface $x^0 = t + dt$, $Ndt \equiv d\tau$ specifies a displacement normal to the surface $x^0 = t$. Moreover, $N^i dt$ yields the displacement from the point (t, x^i) to the foot of the normal to $x^0 = t$ through $(t+dt, x^i)$. In other words, the N^i arise since the $x^i = \text{constant}$ lines do not coincide in general with the normals to the $t = \text{constant}$ surfaces. According to a well-established terminology, N is the lapse function, and the N^i are the shift functions. They are the tool needed to achieve the desired space-time foliation.

In the light of points (1)–(3) above, the 4-metric g can be locally cast in the form

$$g = h_{ij}(dx^i + N^i dt) \otimes (dx^j + N^j dt) - N^2 dt \otimes dt. \tag{14}$$

This implies that

$$g_{00} = -(N^2 - N_i N^i), \tag{15}$$

$$g_{i0} = g_{0i} = N_i, \tag{16}$$

$$g_{ij} = h_{ij}, \tag{17}$$

whereas, using the property $g^{\lambda\nu} g_{\nu\mu} = \delta^\lambda{}_\mu$, one finds

$$g^{00} = -\frac{1}{N^2}, \tag{18}$$

$$g^{i0} = g^{0i} = \frac{N^i}{N^2}, \tag{19}$$

$$g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}. \tag{20}$$

Interestingly, the covariant g_{ij} and h_{ij} coincide, whereas the contravariant g^{ij} and h^{ij} differ as shown in (20). In terms of N, N^i and h, the extrinsic-curvature tensor defined in (13) takes the form

$$K_{ij} \equiv \frac{1}{2N}\left(-\frac{\partial h_{ij}}{\partial t} + N_{i|j} + N_{j|i}\right), \tag{21}$$

where the stroke $|$ denotes covariant differentiation on the spacelike 3-surface S_t, and indices of K_{ij} are raised using h^{il}. Equation (21) can be also written as

$$\frac{\partial h_{ij}}{\partial t} = N_{i|j} + N_{j|i} - 2NK_{ij}. \tag{22}$$

Equation (22) should be supplemented by another first-order equation expressing the time evolution of K_{ij} (recall that π^{ij} is related to K^{ij}), i.e.

$$\frac{\partial K_{ij}}{\partial t} = -N_{|ij} + N\left[{}^{(3)}R_{ij} + K_{ij}(\mathrm{tr}K) - 2K_{im}K_j^{\ m}\right] + [N^m K_{ij|m} + N^m_{|i}K_{jm} + N^m_{|j}K_{im}]. \tag{23}$$

On using the ADM variables described so far, the form of the action integral I for pure gravity that is stationary under variations of the metric vanishing on the boundary is (in $c=1$ units)

$$I \equiv \frac{1}{16\pi G}\int_M {}^{(4)}R\sqrt{-g}\, d^4x + \frac{1}{8\pi G}\int_{\partial M} K_i^i \sqrt{h}\, d^3x = \frac{1}{16\pi G}\int_M [{}^{(3)}R + K_{ij}K^{ij} - (K_i^i)^2]N\sqrt{h}\, d^3x\, dt. \tag{24}$$

The boundary term appearing in (24) is necessary since ${}^{(4)}R$ contains second derivatives of the metric, and integration by parts in the Einstein–Hilbert part

$$I_H \equiv \frac{1}{16\pi G}\int_M {}^{(4)}R\sqrt{-g}\, d^4x$$

of the action also leads to a boundary term equal to $-\frac{1}{8\pi G}\int_{\partial M} K_i^i \sqrt{h}\, d^3x$. On denoting by $G_{\mu\nu}$ the Einstein tensor $G_{\mu\nu} \equiv {}^{(4)}R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, {}^{(4)}R$, and defining

$$\delta\Gamma^\rho_{\mu\nu} \equiv \frac{1}{2}g^{\rho\lambda}\left[\nabla_\mu(\delta g_{\lambda\nu}) + \nabla_\nu(\delta g_{\lambda\mu}) - \nabla_\lambda(\delta g_{\mu\nu})\right], \tag{25}$$

one then finds [165]

$$(16\pi G)\delta I_H = -\int_M \sqrt{-g}\, G^{\mu\nu}\, \delta g_{\mu\nu}\, d^4x + \int_{\partial M} \sqrt{-g}(g^{\mu\nu}\delta^\sigma_\rho - g^{\mu\sigma}\delta^\nu_\rho)\delta\Gamma^\rho_{\mu\nu}(d^3x)_\sigma, \tag{26}$$

which clearly shows that I_H is stationary if the Einstein equations hold, and the normal derivatives of the variations of the metric vanish on the boundary ∂M. In other words, I_H is not stationary under *arbitrary* variations of the metric, and stationarity is only achieved after adding to I_H the boundary term appearing in (24), if $\delta g_{\mu\nu}$ is set to zero on ∂M. Other useful forms of the boundary term can be found in [68,165]. Note also that, strictly, in writing down (24) one should also take into account a term arising from I_H [32]:

$$I_t \equiv \frac{1}{8\pi G}\int dt \int_{\partial M} d^3x\, \partial_i\left[\sqrt{h}\left(K_l^l\, N^i - h^{ij}\, N_{|j}\right)\right]. \tag{27}$$

However, we have not explicitly included I_t since it does not modify the results derived or described hereafter.

We are now ready to apply Dirac's technique to the Hamiltonian quantization of general relativity. This requires that all classical constraints which are first-class are turned into operators that annihilate the wave functional [53]. Hereafter, we assume that this step has already been performed. As we know, consistency of the quantum constraints is proved if one can show that their commutators lead to no new constraints [53]. For this purpose, it may be useful to recall the equal-time commutation relations of the canonical variables, i.e.

$$\left[N(x), \pi(x')\right] = \mathrm{i}\delta(x, x'), \tag{28}$$

$$\left[N_j(x), \pi^k(x')\right] = \mathrm{i}\delta_j^{k'}, \tag{29}$$

$$\left[h_{jk}, \pi^{l'm'}\right] = \mathrm{i}\delta_{jk}{}^{l'm'}. \tag{30}$$

Note that, following [32], primes have been used, either on indices or on the variables themselves, to distinguish different points of 3-space. In other words, one defines

$$\delta_i^{j'} \equiv \delta_i^j \; \delta(x, x'), \tag{31}$$

$$\delta_{ij}{}^{k'l'} \equiv \delta_{ij}{}^{kl} \; \delta(x, x'), \tag{32}$$

$$\delta_{ij}{}^{kl} \equiv \frac{1}{2}(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k). \tag{33}$$

The reader can check that, since [32]

$$\mathcal{H} \equiv \sqrt{h}(K_{ij}K^{ij} - K^2 - {}^{(3)}R), \tag{34}$$

$$\mathcal{H}^i \equiv -2\pi^{ij}_{,j} - h^{il}(2h_{jl,k} - h_{jk,l})\pi^{jk}, \tag{35}$$

one has

$$[\pi(x), \pi^i(x')] = [\pi(x), \mathcal{H}^i(x')] = [\pi(x), \mathcal{H}(x')] = [\pi^i(x), \mathcal{H}^j(x')] = [\pi^i(x), \mathcal{H}(x')] = 0. \tag{36}$$

It now remains to compute the three commutators $[\mathcal{H}_i, \mathcal{H}_{j'}]$, $[\mathcal{H}_i, \mathcal{H}']$, $[\mathcal{H}, \mathcal{H}']$. The first two commutators are obtained by using Eq. (35) and defining $\mathcal{H}_i \equiv h_{ij}\mathcal{H}^j$. Interestingly, $\mathcal{H}_i$ is homogeneous bilinear in the h_{ij} and π^{ij}, with the momenta always to the right. As we said before, following Dirac, the operator version of constraints should annihilate the wave function since the classical constraints are first-class (i.e. their Poisson brackets are linear combinations of the constraints themselves). This condition reads as

$$\int_{S_t} \mathcal{H}\xi \; d^3x \, \psi = 0 \quad \forall \xi, \tag{37}$$

$$\int_{S_t} \mathcal{H}_i \xi^i \; d^3x \, \psi = 0 \quad \forall \xi^i . \tag{38}$$

In the applications to cosmology, Eq. (37) is known as the Wheeler–DeWitt equation, and the functional ψ is then called the *wave function of the universe* [78].
We begin by computing [32]

$$\left[h_{jk}, \mathrm{i} \int_{S_t} \mathcal{H}_{k'} \delta\xi^{k'} \; d^3x' \right] = -h_{jk,l} \; \delta\xi^l - h_{lk} \; \delta\xi^l{}_{,j} - h_{jl} \; \delta\xi^l{}_{,k} , \tag{39}$$

$$\left[\pi^{jk}, \mathrm{i} \int_{S_t} \mathcal{H}_{k'} \delta\xi^{k'} \; d^3x' \right] = -(\pi^{jk} \delta\xi^l)_{,l} + \pi^{lk} \; \delta\xi^j{}_{,l} + \pi^{jl} \; \delta\xi^k{}_{,l} . \tag{40}$$

This calculation shows that the $\mathcal{H}_i$ are generators of 3-dimensional coordinate transformations $\overline{x}^i = x^i + \delta\xi^i$. Thus, by using the definition of structure constants of the general coordinate-transformation group [32], i.e.

$$c^{k''}{}_{ij'} \equiv \delta^{k''}{}_{i,l''} \; \delta^{l''}_{j'} - \delta^{k''}{}_{j',l''} \; \delta^{l''}_{i} , \tag{41}$$

the results (39)–(40) may be used to show that

$$[\mathcal{H}_j(x), \mathcal{H}_k(x')] = -\mathrm{i} \int_{S_t} \mathcal{H}_{l''} \; c^{l''}{}_{jk'} \; d^3x'' , \tag{42}$$

$$[\mathcal{H}_j(x), \mathcal{H}(x')] = \mathrm{i} \mathcal{H} \; \delta_{,j}(x, x'). \tag{43}$$

Note that the only term of $\mathcal{H}$ which might lead to difficulties is the one quadratic in the momenta. However, all factors appearing in this term have homogeneous linear transformation laws under the 3-dimensional coordinate-transformation group. They thus remain undisturbed in position when commuted with $\mathcal{H}_j$ [32].

Last, we have to study the commutator $[\mathcal{H}(x), \mathcal{H}(x')]$. The following remarks are in order:
(i) Terms quadratic in momenta contain no derivatives of h_{ij} or π^{ij} with respect to 3-space coordinates. Hence they commute;
(ii) The terms $\sqrt{h(x)}({}^{(3)}R(x))$ and $\sqrt{h(x')}({}^{(3)}R(x'))$ contain no momenta, so that they also commute;
(iii) The only commutators we are left with are the cross-commutators, and they can be evaluated by using the variational formula [32]

$$\delta\left(\sqrt{h} \; {}^{(3)}R\right) = \sqrt{h} \; h^{ij} h^{kl} \left(\delta h_{ik,jl} - \delta h_{ij,kl} \right) - \sqrt{h} \left[{}^{(3)}R^{ij} - \frac{1}{2} h^{ij} \left({}^{(3)}R \right) \right] \delta h_{ij} , \tag{44}$$

which leads to

$$\left[\int_{S_t} \mathcal{H} \; \xi_1 \; d^3x, \int_{S_t} \mathcal{H} \; \xi_2 \; d^3x \right] = \mathrm{i} \int_{S_t} \mathcal{H}^l \, (\xi_1 \; \xi_{2,l} - \xi_{1,l} \; \xi_2) \; d^3x. \tag{45}$$

The commutators (42)–(43) and (45) clearly show that the constraint equations of canonical quantum gravity are first-class. The Wheeler-DeWitt equation (37) is an equation on the superspace (here Σ is a Riemannian 3-manifold diffeomorphic to S_t)

$$S(\Sigma) \equiv \text{Riem}(\Sigma)/\text{Diff}(\Sigma).$$

In this quotient space, two Riemannian metrics on Σ are identified if they are related through the action of the diffeomorphism group $\text{Diff}(\Sigma)$.

Two very useful classical formulae frequently used in Lorentzian canonical gravity are

$$\mathcal{H} \equiv (16\pi G) G_{ijkl} p^{ij} p^{kl} - \frac{\sqrt{h}}{16\pi G} ({}^{(3)}R), \tag{46}$$

$$\mathcal{H} \equiv (16\pi G)^{-1} [G^{ijml} K_{ij} K_{ml} - \sqrt{h} ({}^{(3)}R)], \tag{47}$$

where the rank-4 tensor density is the DeWitt supermetric on superspace, with covariant and contravariant forms

$$G_{ijkl} \equiv \frac{1}{2\sqrt{h}} \left(h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl} \right), \tag{48}$$

$$G^{ijkl} \equiv \frac{\sqrt{h}}{2} \left(h^{ik} h^{jl} + h^{il} h^{jk} - 2 h^{ij} h^{kl} \right), \tag{49}$$

and p^{ij} is here defined as $-\frac{\sqrt{h}}{16\pi G}(K^{ij} - h^{ij} K)$. Note that the factor -2 multiplying $h^{ij} h^{kl}$ in (49) is needed so as to obtain the identity

$$G_{ijmn} G^{mnkl} = \frac{1}{2} \left(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k \right). \tag{50}$$

Equation (46) clearly shows that H contains a part quadratic in the momenta and a part proportional to ${}^{(3)}R$ (cf. (34)). On quantization, it is then hard to give a well-defined meaning to the second functional derivative $\frac{\delta^2}{\delta h_{ij} \delta h_{kl}}$, whereas the occurrence of ${}^{(3)}R$ makes it even more difficult to solve exactly the Wheeler-DeWitt equation.

It should be stressed that wave functions built from the functional integral (see the following sections) which generalizes the path integral of ordinary quantum mechanics (see (11)) do not solve the Wheeler–DeWitt equation (37) unless some suitable assumptions are made [77], and counterexamples have been built, i.e. a functional integral for the wave function of the universe which does not solve the Wheeler–DeWitt equation [37].

4. From Old to New Unification

Here we outline how the space-of-histories formulation provides a common ground for describing the ‘old’ and ‘new’ unifications of fundamental theories.

4.1. Old Unification

Quantum field theory begins once an action functional S is given, since *the first and most fundamental assumption of quantum theory is that every isolated dynamical system can be described by a characteristic action functional* [31]. The Feynman approach makes it necessary to consider an infinite-dimensional manifold such as the space Φ of all field histories φ^i. On this space there exist (in the case of gauge theories) vector fields

$$Q_\alpha = Q^i_{\ \alpha} \frac{\delta}{\delta\varphi^i} \tag{51}$$

that leave the action invariant, i.e. [39]

$$Q_\alpha S = 0. \tag{52}$$

The Lie brackets of these vector fields lead to a classification of all gauge theories known so far.

4.2. Type-I Gauge Theories

The peculiar property of type-I gauge theories is that these Lie brackets are equal to linear combinations of the vector fields themselves, with structure constants, i.e. [38]

$$[Q_\alpha, Q_\beta] = C^\gamma_{\ \alpha\beta}\, Q_\gamma, \tag{53}$$

where $\frac{\delta C^\gamma_{\ \alpha\beta}}{\delta\varphi^i} = 0$. The Maxwell, Yang–Mills, Einstein theories are all example of type-I theories (this is the 'unifying feature'). All of them, being gauge theories, need supplementary conditions, since the second functional derivative of S is not an invertible operator. After imposing such conditions, the theories are ruled by an invertible differential operator of D'Alembert type (or Laplace type, if one deals instead with Euclidean field theory), or a non-minimal operator at the very worst (for arbitrary choices of gauge parameters). For example, when Maxwell theory is quantized via functional integrals in the Lorenz [110] gauge, one deals with a gauge-fixing functional

$$\Phi(A) = \nabla^\mu A_\mu, \tag{54}$$

and the second-order differential operator acting on the potential in the gauge-fixed action functional reads as

$$P_\mu^{\ \nu} = -\delta_\mu^{\ \nu} \Box + R_\mu^{\ \nu} + \left(1 - \frac{1}{\alpha}\right) \nabla_\mu \nabla^\nu, \tag{55}$$

where α is an arbitrary gauge parameter. The Feynman choice $\alpha = 1$ leads to the minimal operator

$$\widetilde{P}_\mu^{\ \nu} = -\delta_\mu^{\ \nu} \Box + R_\mu^{\ \nu},$$

which is the standard wave operator on vectors in curved spacetime. Such operators play a leading role in the 1-loop expansion of the Euclidean effective action, i.e. the quadratic order in $\hbar$ in the asymptotic expansion of the functional ruling the quantum theory with positive-definite metrics.

The closure property expressed by Eq. (53) implies that the gauge group decomposes the space of histories Φ into sub-spaces to which the Q_α are tangent. These sub-spaces are known as orbits, and

Φ may be viewed as a principal fibre bundle of which the orbits are the fibres. The space of orbits is, strictly, the quotient space $\Phi / \mathcal{G}$, where $\mathcal{G}$ is the proper gauge group, i.e. the set of transformations of Φ into itself obtained by exponentiating the infinitesimal gauge transformation

$$\delta\varphi^i = Q_\alpha^i \, \delta\xi^\alpha, \tag{56}$$

and taking products of the resulting exponential maps. Suppose one performs the transformation [39]

$$\varphi^i \rightarrow I^A, K^\alpha \tag{57}$$

from the field variables φ^i to a set of fibre-adapted coordinates I^A and K^α. With this notation, the I's label the fibres, i.e. the points in $\Phi / \mathcal{G}$, and are gauge invariant because

$$Q_\alpha I^A = 0. \tag{58}$$

The K's label the points within each fibre, and one often makes specific choices for the K's, corresponding to the choice of supplementary condition [31], more frequently called gauge condition. One normally picks out a base point φ_* in Φ and chooses the $K's$ to be local functionals of the φ's in such a way that the formula

$$\widehat{\mathcal{F}}_\beta^\alpha \equiv Q_\beta K^\alpha = K_{,i}^\alpha \, Q_\beta^i \tag{59}$$

defines a non-singular differential operator, called the *ghost operator*, at and in a neighbourhood of φ_*. Thus, what is often called choosing a gauge amounts to choosing a hypersurface $K^\alpha = \text{constant}$ in a fibre-adapted coordinate patch. The fields acted upon by the ghost operator are called *ghost fields*, and have opposite statistics with respect to the fields occurring in the gauge-invariant action functional (see Refs. [63,61,33] for the first time that ghost fields were considered in quantized gauge theories). The gauge-fixed action in the functional integral reads as [39]

$$S_{\text{g.f.}} = S + \frac{1}{2} K^\alpha \omega_{\alpha\beta'} K^{\beta'}, \tag{60}$$

where $\omega_{\alpha\beta'}$ is a non-singular matrix of gauge parameters (strictly, it is written with matrix notation, but it contains Dirac's delta, i.e. $\omega_{\alpha\beta'} \equiv \omega_{\alpha\beta}\delta(x, x')$).

4.3. Type-II Gauge Theories

For type-II gauge theories, Lie brackets of vector fields Q_α are as in Eq. (53) for type-I theories, but the structure constants are promoted to structure functions. An example is given by simple supergravity (a supersymmetric [73,158] gauge theory of gravity, with a symmetry relating bosonic and fermionic fields) in four spacetime dimensions, with auxiliary fields [148].

4.4. Type-III Gauge Theories

In this case, the Lie bracket (53) is generalized by

$$[Q_\alpha, Q_\beta] = C^\gamma_{\alpha\beta}\, Q_\gamma + U^i_{\ \alpha\beta}\, S_{,i}, \qquad (61)$$

and it therefore reduces to (53) only on the *mass-shell*, i.e. for those field configurations satisfying the Euler–Lagrange equations. An example is given by theories with gravitons and gravitinos such as Bose–Fermi supermultiplets of both simple and extended supergravity in any number of spacetime dimensions, without auxiliary fields [148].

4.5. From General Relativity to Supergravity

It should be stressed that general relativity is naturally related to supersymmetry, since the requirement of gauge-invariant Rarita–Schwinger equations [128] in curved spacetime implies Ricci-flatness in four dimensions [29], which is then equivalent to vacuum Einstein equations. Of course, despite such a relation does exist, general relativity can be (and is) formulated without any use of supersymmetry.

The Dirac operator [56] is more fundamental in this framework, since the m-dimensional spacetime metric is entirely re-constructed from the γ-matrices, in that

$$g^{\mu\nu} = 2^{-[m/2]-1}\mathrm{tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu). \qquad (62)$$

In 4-dimensional spacetime, one can use the tetrad formalism, with Latin indices (a,b) corresponding to tensors in flat space (the tangent frames, the freely falling lifts) while Greek indices (μ,ν) correspond to coordinates in curved space. The contravariant form of the spacetime metric g is then given by

$$g^{\mu\nu} = \eta^{ab} e_a^\mu e_b^\nu, \qquad (63)$$

where η^{ab} is the Minkowski metric and e_a^μ are the tetrad vectors. The curved-space γ-matrices γ^μ are then obtained from the flat-space γ-matrices γ^a and from the tetrad according to

$$\gamma^\mu = \gamma^a e_a^\mu. \qquad (64)$$

In Ref. [64], the authors assumed that the action functional describing the interaction of tetrad fields and Rarita–Schwinger fields in curved spacetime, subject to the Majorana constraint

$\psi_\rho(x) = C\overline{\psi}_\rho(x)^T$, reads as

$$I = \int d^4x \left[\frac{1}{4}\kappa^{-2}\sqrt{-g}R - \frac{1}{2}\varepsilon^{\lambda\rho\mu\nu}\overline{\psi}_\lambda(x)\gamma_5\gamma_\mu D_\nu\psi_\rho(x) \right], \qquad (65)$$

where the covariant derivative of Rarita–Schwinger fields is defined by

$$D_\nu\psi_\rho(x) \equiv \partial_\nu\psi_\rho(x) - \Gamma^\sigma_{\nu\rho}\psi_\sigma + \frac{1}{2}\omega_{\nu ab}\sigma^{ab}\psi_\rho. \qquad (66)$$

With a standard notation, $\Gamma^\sigma_{\nu\rho}$ are the Christoffel symbols built from the curved spacetime metric $g^{\mu\nu}$, $\omega_{\nu ab}$ is the spin-connection (the gauge field associated to the generators of the Lorentz algebra)

$$\omega_{\nu ab} = \frac{1}{2}\left[e_a^{\mu} (\partial_\nu e_{b\mu} - \partial_\mu e_{b\nu}) + e_a^{\rho} e_b^{\sigma} (\partial_\sigma e_{c\rho}) e_\nu^c \right] - (a \leftrightarrow b), \tag{67}$$

while σ_{ab} is proportional to the commutator of γ-matrices in Minkowski spacetime, i.e.

$$\sigma_{ab} \equiv \frac{1}{4}[\gamma_a, \gamma_b]. \tag{68}$$

To investigate the possible supersymmetry possessed by the above action functional, the authors of Ref. [64] considered the transformation laws

$$\delta\psi_\mu(x) = \kappa^{-1} D_\mu \varepsilon(x), \tag{69}$$

$$\delta e_\mu^a(x) = \mathrm{i}\kappa\bar{\varepsilon}(x)\gamma^a \psi_\mu(x), \tag{70}$$

$$\delta g_{\mu\nu}(x) = \mathrm{i}\kappa\bar{\varepsilon}(x)\left[\gamma_\mu \psi_\nu(x) + \gamma_\nu \psi_\mu(x) \right], \tag{71}$$

where the supersymmetry parameter is taken to be an arbitrary Majorana spinor field $\bar{\varepsilon}(x)$ of dimension square root of length. The assumption of local supersymmetry was non-trivial, and was made necessary by the coordinate-invariant Lagrangian (i.e. at that stage one had to avoid, for consistency, the coordinate-dependent notion of constant, space-time-independent spinor). After a lengthy calculation the authors of Ref. [64] managed to prove full gauge invariance of the supergravity action. With geometrical hindsight, one can prove it in a quicker and more elegant way by looking at a formulation of supergravity as a Yang–Mills Theory [149].

4.6. New Unification

In modern high energy physics, the emphasis is no longer on fields (sections of vector bundles in classical field theory [156], operator-valued distributions in quantum field theory [161]), but rather on extended objects such as strings [28]. In string theory, particles are not described as points, but instead as strings, i.e., 1-dimensional extended objects. While a point particle sweeps out a 1-dimensional worldline, the string sweeps out a worldsheet, i.e., a 2-dimensional real surface. For a free string, the topology of the worldsheet is a cylinder in the case of a closed string, or a sheet for an open string. It is assumed that different elementary particles correspond to different vibration modes of the string, in much the same way as different minimal notes correspond to different vibrational modes of musical string instruments [28]. The five different string theories [4] are different aspects of a more fundamental unified theory, called M-theory [13].

In the latest developments, one deals with 'branes', which are classical solutions of the equations of motion of the low-energy string effective action, that correspond to new non-perturbative states of string theory, break half of the supersymmetry, and are required by duality arguments in theories with open strings. They have the peculiar property that open strings have their end-points attached to them [45,46]. Branes have made it possible not only to arrive at the formulation of M theory, but also to study perturbative and non-perturbative properties of the gauge theories living on the world-volume [47]. The so-called Dirichlet branes [124], or Dp branes, admit indeed two distinct descriptions. On the one hand, they are classical solutions of the low-energy string effective action (as we said before) and may be therefore described in terms of closed strings. On the other hand, their dynamics is determined by the degrees of freedom of the open strings with endpoints attached to their world-volume, satisfying Dirichlet boundary conditions along the directions transverse to the branes. They may be thus described in terms of open strings as well. Such a twofold description of Dp branes laid the foundations of the Maldacena conjecture [112] providing the equivalence between a closed string theory, as the IIB theory on 5-dimensional anti-de Sitter space times the 5-sphere, and $N=4$ super Yang–Mills with degrees of freedom corresponding to the massless excitations of the open strings having their endpoints attached to a $D3$ brane.

For the impact of braneworld picture on phenomenology and unification, we refer the reader to the seminal work in Refs. [126,127], while for the role of extra dimensions in cosmology we should mention also the work in Refs. [137,138]. With the language of pseudo-Riemannian geometry, branes are timelike surfaces embedded into bulk spacetime [11, 10]. According to this picture, gravity lives on the bulk, while standard-model gauge fields are confined on the brane [12]. For branes, the normal vector N is spacelike with respect to the bulk metric G_{AB}, i.e.,

$$G_{AB}N^A N^B = N_C N^C > 0. \qquad (72)$$

For a wide class of brane models, the action functional S pertaining to the combined effect of bulk and brane geometry can be taken to split into the sum [10] ($g_{\alpha\beta}(x)$ being the brane metric)

$$S = S_4[g_{\alpha\beta}(x)] + S_5[G_{AB}(X)], \qquad (73)$$

while the effective action [38] Γ is formally given by

$$\mathrm{e}^{\mathrm{i}\Gamma} = \int DG_{AB}(X)\, \mathrm{e}^{\mathrm{i}S} \times \text{gauge} - \text{fixing term.} \qquad (74)$$

In the functional integral, the gauge-fixed action reads as (here there is summation as well as integration over repeated indices [31, 38, 10])

$$S_{\text{g.f.}} = S_4 + S_5 + \frac{1}{2}F^A \Omega_{AB} F^B + \frac{1}{2}\chi^\mu \omega_{\mu\nu} \chi^\nu, \qquad (75)$$

where F^A and χ^μ are bulk and brane gauge-fixing functionals, respectively, while Ω_{AB} and $\omega_{\mu\nu}$ are non-singular matrices of gauge parameters, similarly to the end of Section 4.2. The gauge-invariance properties of bulk and brane action functionals can be expressed by saying that there exist vector fields on the space of histories such that (cf. Eq. (52))

$$R_B S_5 = 0, R_\nu S_4 = 0, \qquad (76)$$

whose Lie brackets obey a relation formally analogous to Eq. (53) for ordinary type-I theories, i.e.

$$[R_B, R_D] = C^A_{\ BD}\ R_A, \qquad (77)$$

$$[R_\mu, R_\nu] = C^\lambda_{\ \mu\nu}\ R_\lambda. \qquad (78)$$

Equations (77) and (78) refer to the sharply different Lie algebras of diffeomorphisms on the bulk and the brane, respectively. The bulk and brane ghost operators are therefore

$$Q^A_{\ B} \equiv R_B F^A = F^A_{\ ,a}\ R^a_{\ B}, \qquad (79)$$

$$J^{\mu}_{\ \nu} \equiv R_{\nu} \chi^{\mu} = \chi^{\mu}_{\ ,i} \, R^{i}_{\ \nu}, \tag{80}$$

respectively, where the commas denote functional differentiation with respect to the field variables. The full bulk integration means integrating first with respect to all bulk metrics G_{AB} inducing on the boundary ∂M the given brane metric $g_{\alpha\beta}(x)$, and then integrating with respect to all brane metrics. Thus, one first evaluates the cosmological wave function [10] of the bulk spacetime (which generalizes the wave function of the universe encountered in canonical quantum gravity), i.e.

$$\psi_{\text{Bulk}} = \int_{G_{AB}[\partial M]=g_{\alpha\beta}} \mu\left(G_{AB}, S_C, T^D\right) \mathrm{e}^{\mathrm{i}\tilde{S}_5}, \tag{81}$$

where μ is taken to be a suitable measure, the S_C, T^D are ghost fields, and

$$\tilde{S}_5 \equiv S_5[G_{AB}] + \frac{1}{2} F^A \Omega_{AB} F^B + S_A Q^A_{\ B} T^B. \tag{82}$$

Eventually, the effective action results from

$$\mathrm{e}^{\mathrm{i}\Gamma} = \int \tilde{\mu}\left(g_{\alpha\beta}, \rho_{\gamma}, \sigma^{\delta}\right) \mathrm{e}^{\mathrm{i}\tilde{S}_4} \psi_{\text{Bulk}}, \tag{83}$$

where $\tilde{\mu}$ is another putative measure, ρ_{γ} and σ^{δ} are brane ghost fields, and

$$\tilde{S}_4 \equiv S_4 + \frac{1}{2} \chi^{\mu} \omega_{\mu\nu} \chi^{\nu} + \rho_{\mu} J^{\mu}_{\ \nu} \sigma^{\nu}. \tag{84}$$

We would like to stress here that infinite-dimensional manifolds are the natural arena for studying the quantization of the gravitational field, even prior to considering a space-of-histories formulation. There are, indeed, at least three sources of infinite-dimensionality in quantum gravity:

(i) The infinite-dimensional Lie group (or pseudo-group) of spacetime diffeomorphisms, which is the invariance group of general relativity in the first place [31], [143].
(ii) The infinite-dimensional space of histories in a functional-integral quantization [38, 39].
(iii) The infinite-dimensional Geroch space of asymptotically simple spacetimes [67].

5. Functional Integrals and Background Fields

We now study in greater detail some aspects of the use of functional integrals in quantum gravity, after the previous (formal) applications to a space of histories formulation.

5.1. The 1-Loop Approximation

In the 1-loop approximation (also called stationary phase or JWKB method) one first expands *both* the metric g and the fields ϕ coupled to it about a metric g_0 and a field ϕ_0 which are solutions of the classical field equations:

$$g = g_0 + \overline{g}, \tag{85}$$

$$\phi = \phi_0 + \overline{\phi}. \tag{86}$$

One then assumes that the fluctuations $\bar{g}$ and $\bar{\phi}$ are so small that the dominant contribution to the functional integral for the in-out amplitude comes from the quadratic order in the Taylor-series expansion of the action about the background fields g_0 and ϕ_0 [90]:

$$I_E[g,\phi] = I_E[g_0,\phi_0] + I_2[\bar{g},\bar{\phi}] + \text{higher} - \text{order terms}, \qquad (87)$$

so that the logarithm of the quantum-gravity amplitude $\tilde{A}$ can be expressed as

$$\log(\tilde{A}) \sim -I_E[g_0,\phi_0] + \log\int D[\bar{g},\bar{\phi}]\mathrm{e}^{-I_2[\bar{g},\bar{\phi}]}. \qquad (88)$$

It should be stressed that *background fields need not be a solution of any field equation* [36], but this possibility will not be exploited in our presentation. For our purposes we are interested in the second term appearing on the right-hand side of (88). An useful factorization is obtained if ϕ_0 can be set to zero. One then finds that $I_2[\bar{g},\bar{\phi}] = I_2[\bar{g}] + I_2[\bar{\phi}]$, which implies [91]

$$\log(\tilde{A}) \sim -I_E[g_0] + \log D[\phi]\mathrm{e}^{-I_2[\phi]} + \log\int D[\bar{g}]\mathrm{e}^{-I_2[\bar{g}]}. \qquad (89)$$

The 1-loop term for matter fields with various spins (and boundary conditions) is extensively studied in the literature. We here recall some basic results, following again Ref. [91].

A familiar form of $I_2[\phi]$ is

$$I_2[\phi] = \frac{1}{2}\int \phi\mathcal{B}\phi\sqrt{g_0}\, d^4x, \qquad (90)$$

where the elliptic differential operator $\mathcal{B}$ depends on the background metric g_0. Note that $\mathcal{B}$ is a second-order operator for bosonic fields, whereas it is first-order for fermionic fields. In light of (90) it is clear that we are interested in the eigenvalues $\{\lambda_n\}$ of $\mathcal{B}$, with corresponding eigenfunctions $\{\phi_n\}$. If boundaries are absent, it is sometimes possible to know explicitly the eigenvalues with their degeneracies. This is what happens for example in de Sitter space. If boundaries are present, however, very little is known about the detailed form of the eigenvalues, once boundary conditions have been imposed.

We here assume for simplicity to deal with bosonic fields subject to (homogeneous) Dirichlet conditions on the boundary surface: $\phi = 0$ on ∂M, and $\phi_n = 0$ on ∂M, $\forall n$. It is in fact well-known that the Laplace operator subject to Dirichlet conditions has a positive-definite spectrum [23]. The field ϕ can then be expanded in terms of the eigenfunctions ϕ_n of $\mathcal{B}$ as

$$\phi = \sum_{n=n_0}^{\infty} y_n\phi_n, \qquad (91)$$

where the eigenfunctions ϕ_n are normalized so that

$$\int \phi_n\phi_m\sqrt{g_0}\, d^4x = \delta_{nm}. \qquad (92)$$

Another formula we need is the one expressing the measure on the space of all fields ϕ as

$$D[\phi] = \prod_{n=n_0}^{\infty} \mu \, dy_n , \tag{93}$$

where the normalization parameter μ has dimensions of mass or $(\text{length})^{-1}$. Note that, if gauge fields appear in the calculation, the choice of gauge-fixing and the form of the measure in the functional integral are not a trivial problem.

On using well-known results about Gaussian integrals, the 1-loop amplitudes $\widetilde{A}_\phi^{(1)}$ can be now obtained as

$$\widetilde{A}_\phi^{(1)} \equiv \int D[\phi] \mathrm{e}^{-I_2[\phi]} = \prod_{n=n_0}^{\infty} \int \mu \, dy_n \, \mathrm{e}^{-\frac{\lambda_n}{2} y_n^2} = \prod_{n=n_0}^{\infty} (2\pi\mu^2 \lambda_n^{-1})^{\frac{1}{2}} = \frac{1}{\sqrt{\det(\frac{1}{2}\pi^{-1}\mu^{-2}\mathcal{B})}} . \tag{94}$$

When fermionic fields appear in the functional integral for the in-out amplitude, one deals with a first-order elliptic operator, the Dirac operator, acting on independent spinor fields ψ and $\widetilde{\psi}$. These are anticommuting Grassmann variables obeying the Berezin integration rules

$$\int dw = 0, \int w \, dw = 1. \tag{95}$$

The formulae (95) are all what we need, since powers of w greater than or equal to 2 vanish in light of the anticommuting property. The reader can then check that the 1-loop amplitude for fermionic fields is

$$\widetilde{A}_\psi^{(1)} = \det\left(\frac{1}{2}\mu^{-2}\mathcal{B}\right). \tag{96}$$

The main difference with respect to bosonic fields is the direct proportionality to the determinant. The following comments can be useful in understanding the meaning of (96).

Let us denote again by γ^μ the curved-space γ-matrices, and by λ_i the eigenvalues of the Dirac operator $\gamma^\mu D_\mu$, and suppose that no zero-modes exist. More precisely, the eigenvalues of $\gamma^\mu D_\mu$ occur in equal and opposite pairs: $\pm\lambda_1, \pm\lambda_2$, ..., whereas the eigenvalues of the Laplace operator on spinors occur as $(\lambda_1)^2$ twice, $(\lambda_2)^2$ twice, and so on. For Dirac fermions (D) one thus finds

$$\det{}_D\left(\gamma^\mu D_\mu\right) = \left(\prod_{i=1}^{\infty} |\lambda_i|\right)\left(\prod_{i=1}^{\infty} |\lambda_i|\right) = \prod_{i=1}^{\infty} |\lambda_i|^2 , \tag{97}$$

whereas in the case of Majorana spinors (M), for which the number of degrees of freedom is halved, one finds

$$\det{}_M(\gamma^\mu D_\mu) = \prod_{i=1}^{\infty} |\lambda_i| = \sqrt{\det{}_D(\gamma^\mu D_\mu)} . \tag{98}$$

5.2. Zeta-Function Regularization of Functional Integrals

The formal expression (94) for the 1-loop quantum amplitude clearly diverges since the eigenvalues λ_n increase without bound, and a regularization is thus necessary. For this purpose, the following technique has been described and applied by many authors [48,89,91].
Bearing in mind that Riemann's zeta-function $\zeta_R(s)$ is defined as

$$\zeta_R(s) \equiv \sum_{n=1}^{\infty} n^{-s}, \qquad (99)$$

one first defines a generalized (also called spectral) zeta-function $\zeta(s)$ obtained from the (positive) eigenvalues of the second-order, self-adjoint operator $\mathcal{B}$. Such a $\zeta(s)$ can be defined as (cf. [141])

$$\zeta(s) \equiv \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} d_m(n) \lambda_{n,m}^{-s}. \qquad (100)$$

This means that all the eigenvalues are completely characterized by two integer labels n and m, while their degeneracy d_m only depends on n. Note that formal differentiation of (100) at the origin yields

$$\det(\mathcal{B}) = \mathrm{e}^{-\zeta'(0)}. \qquad (101)$$

This result can be given a sensible meaning since, in four dimensions, $\zeta(s)$ converges for $Re(s) > 2$, and one can perform its analytic extension to a meromorphic function of s which only has poles at $s = \frac{1}{2}, 1, \frac{3}{2}, 2$. Since $\det(\mu\mathcal{B}) = \mu^{\zeta(0)} \det(\mathcal{B})$, one finds the useful formula

$$\log\left(\tilde{A}_\phi\right) = \frac{1}{2}\zeta'(0) + \frac{1}{2}\log(2\pi\mu^2)\zeta(0). \qquad (102)$$

As we said following (90), it may happen quite often that the eigenvalues appearing in (100) are unknown, since the eigenvalue condition, i.e. the equation leading to the eigenvalues by virtue of the boundary conditions, is a complicated equation which cannot be solved exactly for the eigenvalues. However, since the scaling properties of the 1-loop amplitude are still given by $\zeta(0)$ (and $\zeta'(0)$) as shown in (102), efforts have been made to compute $\zeta(0)$ also in this case. The various steps of this program are as follows [89].

(1) One first studies the heat equation for the operator $\mathcal{B}$, i.e.

$$\frac{\partial}{\partial \tau} F(x, y, \tau) + \mathcal{B} F(x, y, \tau) = 0, \qquad (103)$$

where the Green's function F satisfies the initial condition $F(x, y, 0) = \delta(x, y)$.

(2) Assuming completeness of the set $\{\phi_n\}$ of eigenfunctions of $\mathcal{B}$, the field ϕ can be expanded as

$$\phi = \sum_{n=n_i}^{\infty} a_n \phi_n .$$

(3) The Green's function $F(x,y,\tau)$ is then given by

$$F(x,y,\tau) = \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} \mathrm{e}^{-\lambda_{n,m}\tau} \phi_{n,m}(x) \otimes \phi_{n,m}(y). \qquad (104)$$

(4) The corresponding integrated heat kernel is then

$$G(\tau) = \int_M \mathrm{Tr}\; F(x,x,\tau)\sqrt{g}\; d^4x = \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} \mathrm{e}^{-\lambda_{n,m}\tau}. \qquad (105)$$

(5) In light of (100) and (105), the generalized zeta-function can be also obtained as an integral transform (also called inverse Mellin transform) of the integrated heat kernel, i.e. [89,71]

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \tau^{s-1} G(\tau)\, d\tau. \qquad (106)$$

(6) The hard part of the analysis is now to prove that $G(\tau)$ has an asymptotic expansion as $\tau \to 0^+$ [76]. This property has been proved for all boundary conditions such that the Laplace operator is self-adjoint and the boundary-value problem is strongly elliptic [71,8]. The corresponding asymptotic expansion of $G(\tau)$ can be written as

$$G(\tau) \sim A_0 \tau^{-2} + A_{\frac{1}{2}} \tau^{-\frac{3}{2}} + A_1 \tau^{-1} + A_{\frac{3}{2}} \tau^{-\frac{1}{2}} + A_2 + \mathrm{O}\left(\sqrt{\tau}\right), \qquad (107)$$

which implies

$$\zeta(0) = A_2. \qquad (108)$$

The result (108) is proved by splitting the integral in (106) into an integral from 0 to 1 and an integral from 1 to ∞. The asymptotic expansion of $\int_0^1 \tau^{s-1} G(\tau)\, d\tau$ is then obtained by using (107).

In other words, for a given second-order self-adjoint elliptic operator, we study the corresponding heat equation, and the integrated heat kernel $G(\tau)$. The $\zeta(0)$ value is then given by the constant term appearing in the asymptotic expansion of $G(\tau)$ as $\tau \to 0^+$. The $\zeta(0)$ value also yields the 1-loop divergences of the theory for bosonic and fermionic fields [55].

5.3. Gravitational Instantons

This section is devoted to the study of the background gravitational fields. These gravitational instantons are complete 4-geometries solving the Einstein equations $R(X,Y) - \Lambda g(X,Y) = 0$ when the 4-metric g has signature $+4$ (i.e. it is positive-definite, and thus called Riemannian). They are of interest because they occur in the tree-level approximation of the partition function, and in light of their role in studying tunnelling phenomena. Moreover, they can be interpreted as the stationary phase metrics in the path integrals for the partition functions, Z, of the thermal canonical ensemble and the volume canonical ensemble. In these cases the action of the instanton gives the dominant contribution to $-\log Z$. Following [125], essentially three cases can be studied.

5.3.1. Asymptotically Locally Euclidean Instantons

Even though it might seem natural to define first the asymptotically Euclidean instantons, it turns out that there is not much choice in this case, since the only asymptotically Euclidean instanton is flat space. It is in fact well-known that the action of an asymptotically Euclidean metric with vanishing scalar curvature is ≥ 0, and it vanishes if and only if the metric is flat. Suppose now that such a metric is a solution of the Einstein equations $R(X,Y)=0$. Its action should be thus stationary also under constant conformal rescalings $g \to k^2 g$ of the metric. However, the whole action rescales then as $I_E \to k^2 I_E$, so that it can only be stationary and finite if $I_E = 0$. By virtue of the theorem previously mentioned, the metric g must then be flat [70,108].

In the asymptotically locally Euclidean case, however, the boundary at infinity has topology S^3/Γ rather than S^3, where Γ is a discrete subgroup of the group $SO(4)$. Many examples can then be found. The simplest was discovered by Eguchi and Hanson [49,50], and corresponds to $\Gamma = Z_2$ and $\partial M = RP^3$. This instanton is conveniently described using three left-invariant 1-forms $\{\omega_i\}$ on the 3-sphere, satisfying the $SU(2)$ algebra $d\omega_i = -\frac{1}{2}\varepsilon_i^{\ jk}\, \omega_j \wedge \omega_k$, and parametrized by Euler angles as follows:

$$\omega_1 = (\cos\psi)d\theta + (\sin\psi)(\sin\theta)d\phi, \tag{109}$$

$$\omega_2 = -(\sin\psi)d\theta + (\cos\psi)(\sin\theta)d\phi, \tag{110}$$

$$\omega_3 = d\psi + (\cos\theta)d\phi, \tag{111}$$

where $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$. The metric of the Eguchi–Hanson instanton may be thus written in the Bianchi-IX form [125]

$$g_1 = \left(1-\frac{a^4}{r^4}\right)^{-1} dr \otimes dr + \frac{r^2}{4}\left[(\omega_1)^2 + (\omega_2)^2 + \left(1-\frac{a^4}{r^4}\right)(\omega_3)^2\right], \tag{112}$$

where $r \in [a,\infty[$. The singularity of g_1 at $r=a$ is only a coordinate singularity. We may get rid of it by defining $4\frac{\rho^2}{a^2} \equiv 1-\frac{a^4}{r^4}$, so that, as $r \to a$, the metric g_1 is approximated by the metric

$$g_2 = d\rho \otimes d\rho + \rho^2\left[d\psi + (\cos\theta)d\phi\right]^2 + \frac{a^2}{4}\left[d\theta \otimes d\theta + (\sin\theta)^2 d\phi \otimes d\phi\right]. \tag{113}$$

Regularity of g_2 at $\rho = 0$ is then guaranteed provided that one identifies ψ with period 2π. This implies in turn that the local surfaces $r = \text{constant}$ have topology RP^3 rather than S^3, as we claimed. Note that at $r = a \Rightarrow \rho = 0$ the metric becomes that of a 2-sphere of radius $\frac{a}{2}$. Following Ref. [69], we say that $r=a$ is a bolt, where the action of the Killing vector $\frac{\partial}{\partial\psi}$ has a 2-dimensional fixed-point set [125].

A whole family of multi-instanton solutions is obtained by taking the group $\Gamma = Z_k$. They all have a self-dual Riemann-curvature tensor, and their metric takes the form

$$g = V^{-1}(d\tau + \underline{\gamma} \cdot \underline{dx})^2 + V\ \underline{dx} \cdot \underline{dx}. \qquad (114)$$

Following [125], $V = V(\underline{x})$ and $\underline{\gamma} = \underline{\gamma}(\underline{x})$ on an auxiliary flat 3-space with metric $\underline{dx} \cdot \underline{dx}$. This metric g solves the Einstein vacuum equations provided that $\mathrm{grad}\, V = \mathrm{curl}\underline{\gamma}$, which implies $\Delta V = 0$. If one takes

$$V = \sum_{i=1}^{n} \frac{1}{|\underline{x} - \underline{x}_i|}, \qquad (115)$$

one obtains the desired asymptotically locally Euclidean multi-instantons. In particular, if $n = 1$ in (115), g describes flat space, whereas $n = 2$ leads to the Eguchi–Hanson instanton. If $n > 2$, there are $(3n - 6)$ arbitrary parameters, related to the freedom to choose the positions $\underline{x}_i$ of the singularities in V. These singularities correspond actually to coordinate singularities in (113), and can be removed by using suitable coordinate transformations [125].

5.3.2. Asymptotically Flat Instantons

This name is chosen since the underlying idea is to deal with metrics in the functional integral which tend to the flat metric in three directions but are periodic in the Euclidean-time dimension. The basic example is provided by the Riemannian version $g_R^{(1)}$ (also called Euclidean) of the Schwarzschild solution, i.e.

$$g_R^{(1)} = \left(1 - 2\frac{M}{r}\right) d\tau \otimes d\tau + \left(1 - 2\frac{M}{r}\right)^{-1} dr \otimes dr + r^2 \Omega_2, \qquad (116)$$

where $\Omega_2 = d\theta \otimes d\theta + (\sin\theta)^2 d\phi \otimes d\phi$ is the metric on a unit 2-sphere. It is indeed well-known that, in the Lorentzian case, the metric g_L is more conveniently written by using Kruskal–Szekeres coordinates

$$g_L = 32M^3 r^{-1} \mathrm{e}^{-\frac{r}{2M}} (-dz \otimes dz + dy \otimes dy) + r^2 \Omega_2, \qquad (117)$$

where z and y obey the relations

$$-z^2 + y^2 = \left(\frac{r}{2M} - 1\right) \mathrm{e}^{\frac{r}{2M}}, \qquad (118)$$

$$\frac{(y+z)}{(y-z)} = \mathrm{e}^{\frac{t}{2M}}. \qquad (119)$$

In the Lorentzian case, the coordinate singularity at $r = 2M$ can be thus avoided, whereas the curvature singularity at $r = 0$ remains and is described by the surface $z^2 - y^2 = 1$. However, if we set $\zeta = \mathrm{i}z$, the analytic continuation to the section of the complexified space-time where ζ is real yields the positive-definite (i.e. Riemannian) metric

$$g_R^{(2)} = 32M^3 r^{-1} e^{-\frac{r}{2M}} \left(d\zeta \otimes d\zeta + dy \otimes dy \right) + r^2 \Omega_2, \qquad (120)$$

where

$$\zeta^2 + y^2 = \left(\frac{r}{2M} - 1 \right) e^{\frac{r}{2M}}. \qquad (121)$$

It is now clear that also the curvature singularity at $r = 0$ has disappeared, since the left-hand side of (121) is ≥ 0, whereas the right-hand side of (121) would be equal to -1 at $r = 0$. Note also that, by setting $z = -i\zeta$ and $t = -i\tau$ in (119), and writing $\zeta^2 + y^2$ as $(y + i\zeta)(y - i\zeta)$ in (121), one finds

$$y + i\zeta = e^{\frac{i\tau}{4M}} \sqrt{\frac{r}{2M} - 1}\ e^{\frac{r}{4M}}, \qquad (122)$$

$$y = \cos\left(\frac{\tau}{4M} \right) \sqrt{\frac{r}{2M} - 1}\ e^{\frac{r}{4M}}, \qquad (123)$$

which imply that the Euclidean time τ is periodic with period $8\pi M$. This periodicity on the Euclidean section leads to the interpretation of the Riemannian Schwarzschild solution as describing a black hole in thermal equilibrium with gravitons at a temperature $(8\pi M)^{-1}$ [125]. Moreover, the fact that any matter-field Green's function on this Schwarzschild background is also periodic in imaginary time leads to some of the thermal-emission properties of black holes. This is one of the greatest conceptual revolutions in modern gravitational physics.

Interestingly, a new asymptotically flat gravitational instanton has been found by Chen and Teo in Ref. [24]. It has an $U(1) \times U(1)$ isometry group and some novel global features with respect to the other two asymptotically flat instantons, i.e. Euclidean Schwarzschild and Euclidean Kerr.

There is also a local version of the asymptotically flat boundary condition in which ∂M has the topology of a non-trivial S^1-bundle over S^2, i.e. S^3/Γ, where Γ is a discrete subgroup of $SO(4)$. However, unlike the asymptotically Euclidean boundary condition, the S^3 is distorted and expands with increasing radius in only two directions rather than three [125]. The simplest example of an asymptotically locally flat instanton is the self-dual (i.e. with self-dual curvature 2-form) Taub-NUT solution, which can be regarded as a special case of the 2-parameter Taub-NUT metrics

$$g = \frac{(r+M)}{(r-M)} dr \otimes dr + 4M^2 \frac{(r-M)}{(r+M)} (\omega_3)^2 + (r^2 - M^2) \left[(\omega_1)^2 + (\omega_2)^2 \right], \qquad (124)$$

where the $\{\omega_i\}$ have been defined in (109)–(111). The main properties of the metric (124) are

(I) $r \in [M, \infty[$, and $r = M$ is a removable coordinate singularity provided that ψ is identified modulo 4π;

(II) the $r = \text{constant}$ surfaces have S^3 topology;

(III) $r = M$ is a point at which the isometry generated by the Killing vector $\frac{\partial}{\partial \psi}$ has a zero-dimensional fixed-point set.

In other words, $r = M$ is a nut, using the terminology in Ref. [69].

There is also a family of asymptotically locally flat multi-Taub-NUT instantons. Their metric takes the form (114), but one should bear in mind that the formula (115) is replaced by

$$V = 1 + \sum_{i=1}^{n} \frac{2M}{\left|\underline{x} - \underline{x}_i\right|}. \qquad (125)$$

Again, the singularities at $\underline{x} = \underline{x}_i$ can be removed, and the instantons are all self-dual.

5.3.3. Compact Instantons

Compact gravitational instantons occur in the course of studying the topological structure of the gravitational vacuum. This can be done by first of all normalizing all metrics in the functional integral to have a given 4-volume V, and then evaluating the instanton contributions to the partition function as a function of their topological complexity. One then sends the volume V to infinity at the end of the calculation. If one wants to constrain the metrics in the functional integral to have a volume V, this can be obtained by adding a term $\frac{\Lambda}{8\pi}V$ to the action. The stationary points of the modified action are solutions of the Einstein equations with cosmological constant Λ, i.e. $R(X,Y) - \Lambda g(X,Y) = 0$.

The few compact instantons that are known can be described as follows [125].

(1) The 4-sphere S^4, i.e. the Riemannian version of de Sitter space obtained by analytic continuation to positive-definite metrics. Setting to 3 for convenience the cosmological constant, the metric on S^4 takes the form [125]

$$g_I = d\beta \otimes d\beta + \frac{1}{4}(\sin\beta)^2[(\omega_1)^2 + (\omega_2)^2 + (\omega_3)^2], \qquad (126)$$

where $\beta \in [0,\pi]$. The apparent singularities at $\beta = 0,\pi$ can be made into regular nuts, provided that the Euler angle ψ is identified modulo 4π. The $\beta = \text{constant}$ surfaces are topologically S^3, and the isometry group of the metric (126) is $SO(5)$.

(2) If in C^3 we identify (z_1, z_2, z_3) and $(\lambda z_1, \lambda z_2, \lambda z_3)$, $\forall \lambda \in C - \{0\}$, we obtain, by definition, the complex projective space CP^2. For this 2-dimensional complex space one can find a real 4-dimensional metric, which solves the Einstein equations with cosmological constant Λ. If we set Λ to 6 for convenience, the metric of CP^2 takes the form [125]

$$g_{II} = d\beta \otimes d\beta + \frac{1}{4}(\sin\beta)^2[(\omega_1)^2 + (\omega_2)^2 + (\cos\beta)^2(\omega_3)^2], \qquad (127)$$

where $\beta \in [0,\frac{\pi}{2}]$. A bolt exists at $\beta = \frac{\pi}{2}$, where $\frac{\partial}{\partial\psi}$ has a 2-dimensional fixed-point set. The isometry group of g_{II} is locally $SU(3)$, which has a $U(2)$ subgroup acting on the 3-spheres $\beta = \text{constant}$.

(3) The Einstein metric on the product manifold $S^2 \times S^2$ is obtained as the direct sum of the metrics on two 2-spheres, i.e.

$$g = \frac{1}{\Lambda}\sum_{i=1}^{2}(d\theta_i \otimes d\theta_i + (\sin\theta_i)^2 d\phi_i \otimes d\phi_i). \qquad (128)$$

The metric (128) is invariant under the $SO(3)\times SO(3)$ isometry group of $S^2\times S^2$, but is not of Bianchi-IX type as (126)-(127). This can be achieved by a coordinate transformation leading to [125]

$$g_{III}=d\beta\otimes d\beta+(\cos\beta)^2(\omega_1)^2+(\sin\beta)^2(\omega_2)^2+(\omega_3)^2, \qquad (129)$$

where $\Lambda=2$ and $\beta\in[0,\frac{\pi}{2}]$. Regularity at $\beta=0,\frac{\pi}{2}$ is obtained provided that ψ is identified modulo 2π (cf. (126)). Remarkably, this is a regular Bianchi-IX Einstein metric in which the coefficients of ω_1,ω_2 and ω_3 are all different.

(4) The nontrivial S^2-bundle over S^2 has a metric which, by setting $\Lambda=3$, may be cast in the form [118,125]

$$g_{IV}=(1+\nu^2)[f_1(x)dx\otimes dx+f_2(x)((\omega_1)^2+(\omega_2)^2)+f_3(x)(\omega_3)^2], \qquad (130)$$

where $x\in[0,1]$, ν is the positive root of

$$w^4+4w^3-6w^2+12w-3=0, \qquad (131)$$

and the functions f_1, f_2, f_3 are defined by

$$f_1(x)\equiv\frac{(1-\nu^2x^2)}{(3-\nu^2-\nu^2(1+\nu^2)x^2)(1-x^2)}, \qquad (132)$$

$$f_2(x)\equiv\frac{(1-\nu^2x^2)}{(3+6\nu^2-\nu^4)}, \qquad (133)$$

$$f_3(x)\equiv\frac{(3-\nu^2-\nu^2(1+\nu^2)x^2)(1-x^2)}{(3-\nu^2)(1-\nu^2x^2)}. \qquad (134)$$

The isometry group corresponding to (130) may be shown to be $U(2)$.

(5) Another compact instanton of fundamental importance is the $K3$ surface, whose explicit metric has not yet been found. $K3$ is defined as the compact complex surface whose first Betti number and first Chern class are vanishing. A physical picture of the K3 gravitational instanton has been obtained by Page [119].

Two topological invariants exist which may be used to characterize the various gravitational instantons studied so far. These invariants are the Euler number χ and the Hirzebruch signature τ. The Euler number can be defined as an alternating sum of Betti numbers, i.e.

$$\chi\equiv B_0-B_1+B_2-B_3+B_4. \qquad (135)$$

The Hirzebruch signature can be defined as

$$\tau\equiv B_2^+-B_2^-, \qquad (136)$$

where B_2^+ is the number of self-dual harmonic 2-forms, and B_2^- is the number of anti-self-dual harmonic 2-forms [in terms of the Hodge-star operator ${}^*F_{ab} \equiv \frac{1}{2}\varepsilon_{abcd}F^{cd}$, self-duality of a 2-form F is expressed as ${}^*F = F$, and anti-self-duality as ${}^*F = -F$]. In the case of compact 4-dimensional manifolds without boundary, χ and τ can be expressed as integrals of the curvature [91]

$$\chi = \frac{1}{128\pi^2}\int_M R_{\lambda\mu\nu\rho}\, R_{\alpha\beta\gamma\delta}\, \varepsilon^{\lambda\mu\alpha\beta}\, \varepsilon^{\nu\rho\gamma\delta}\, \sqrt{g}\, d^4x, \qquad (137)$$

$$\tau = \frac{1}{96\pi^2}\int_M R_{\lambda\mu\nu\rho}\, R^{\lambda\mu}{}_{\alpha\beta}\, \varepsilon^{\nu\rho\alpha\beta}\, \sqrt{g}\, d^4x. \qquad (138)$$

For the instantons previously listed one finds [125]

Eguchi–Hanson: $\chi = 2$, $\tau = 1$.
Asymptotically locally Euclidean multi-instantons: $\chi = n$, $\tau = n-1$.

Schwarzschild: $\chi = 2$, $\tau = 0$.
Taub-NUT: $\chi = 1$, $\tau = 0$.
Asymptotically locally flat multi-Taub-NUT instantons: $\chi = n$, $\tau = n-1$.
S^4: $\chi = 2$, $\tau = 0$.
CP^2: $\chi = 3$, $\tau = 1$.
$S^2 \times S^2$: $\chi = 4$, $\tau = 0$.
S^2-bundle over S^2: $\chi = 4$, $\tau = 0$.
$K3$: $\chi = 24$, $\tau = 16$.

6. Spectral Zeta-Functions in 1-loop Quantum Cosmology

In the late nineties a systematic investigation of boundary conditions in quantum field theory and quantum gravity has been performed (see Refs. [111, 150, 55, 117, 8] and the many references therein). It is now clear that the set of fully gauge-invariant boundary conditions in quantum field theory, providing a unified scheme for Maxwell, Yang–Mills and General Relativity is as follows:

$$[\pi\mathcal{A}]_{\partial M} = 0, \qquad (139)$$

$$[\Phi(\mathcal{A})]_{\partial M} = 0, \qquad (140)$$

$$[\varphi]_{\partial M} = 0, \qquad (141)$$

where π is a projection operator, $\mathcal{A}$ is the Maxwell potential, or the Yang–Mills potential, or the metric (more precisely, their perturbation about a background value which can be set to zero for Maxwell or Yang–Mills), Φ is the gauge-fixing functional, φ denotes the set of ghost fields for these bosonic theories. Equations (139) and (140) are both preserved under infinitesimal gauge transformations provided that the ghost obeys homogeneous Dirichlet conditions as in Eq. (141). For gravity, it may be convenient to choose Φ so as to have an operator P of Laplace type in the Euclidean theory.

6.1. Eigenvalue Condition for Scalar Modes

In a quantum theory of the early universe via functional integrals, the semiclassical analysis remains a valuable tool, but the tree-level approximation might be an oversimplification. Thus, it seems appropriate to consider at least the 1-loop approximation. On the portion of flat Euclidean 4-space bounded by a 3-sphere, called Euclidean 4-ball and relevant for 1-loop quantum cosmology [78, 92, 55] when a portion of 4-sphere bounded by a 3-sphere is studied in the limit of small 3-geometry [140], the metric perturbations $h_{\mu\nu}$ can be expanded in terms of scalar, transverse vector, transverse-traceless tensor harmonics on the 3-sphere S^3 of radius a. For vector, tensor and ghost modes, boundary conditions reduce to Dirichlet or Robin. For scalar modes, one finds eventually the eigenvalues $E = X^2$ from the roots X of [58, 59]

$$J_n'(x) \pm \frac{n}{x} J_n(x) = 0, \tag{142}$$

$$J_n'(x) + \left(-\frac{x}{2} \pm \frac{n}{x} \right) J_n(x) = 0, \tag{143}$$

where J_n are the Bessel functions of first kind. Note that both x and $-x$ solve the same equation.

6.2. Four Spectral Zeta-Functions for Scalar Modes

By virtue of the Cauchy theorem and of suitable rotations of integration contours in the complex plane [19], the eigenvalue conditions (142) and (143) give rise to the following four spectral zeta-functions [58,59]:

$$\zeta_{A,B}^{\pm}(s) \equiv \frac{\sin(\pi s)}{\pi} \sum_{n=3}^{\infty} n^{-(2s-2)} \int_0^{\infty} dz \frac{\frac{\partial}{\partial z} \log F_{A,B}^{\pm}(zn)}{z^{2s}}, \tag{144}$$

where, denoting by I_n the modified Bessel functions of first kind (here $\beta_+ \equiv n, \beta_- \equiv n+2$),

$$F_A^{\pm}(zn) \equiv z^{-\beta_\pm} (znI_n'(zn) \pm nI_n(zn)), \tag{145}$$

$$F_B^{\pm}(zn) \equiv z^{-\beta_\pm} \left(znI_n'(zn) + \left(\frac{z^2 n^2}{2} \pm n \right) I_n(zn) \right). \tag{146}$$

Regularity at the origin is easily proved in the elliptic sectors, corresponding to $\zeta_A^{\pm}(s)$ and $\zeta_B^{-}(s)$ [58,59].

6.3. Regularity at the Origin Of ζ_B^+

With the notation in Refs. [58,59], if one defines the variable $\tau \equiv (1+z^2)^{-\frac{1}{2}}$, one can write the uniform asymptotic expansion of F_B^+ in the form [58,59]

$$F_B^+ \sim \frac{e^{n\eta(\tau)}}{h(n)\sqrt{\tau}} \frac{(1-\tau^2)}{\tau} \left(1 + \sum_{j=1}^{\infty} \frac{r_{j,+}(\tau)}{n^j} \right). \tag{147}$$

On splitting the integral $\int_0^1 d\tau = \int_0^{\mu} d\tau + \int_{\mu}^1 d\tau$ with μ small, one gets an asymptotic expansion of the left-hand side of Eq. (144) by writing, in the first interval on the right-hand side,

$$\log\left(1+\sum_{j=1}^{\infty}\frac{r_{j,+}(\tau)}{n^j}\right)\sim\sum_{j=1}^{\infty}\frac{R_{j,+}(\tau)}{n^j}, \tag{148}$$

and then computing [58,59]

$$C_j(\tau)\equiv\frac{\partial R_{j,+}}{\partial\tau}=(1-\tau)^{-j-1}\sum_{a=j-1}^{4j}K_a^{(j)}\tau^a. \tag{149}$$

Remarkably, by virtue of the identity obeyed by the spectral coefficients $K_a^{(j)}$ on the 4-ball, i.e.

$$g(j)\equiv\sum_{a=j}^{4j}\frac{\Gamma(a+1)}{\Gamma(a-j+1)}K_a^{(j)}=0, \tag{150}$$

which holds $\forall j=1,...,\infty$, one finds [58,59]

$$\lim_{s\to 0}s\zeta_B^+(s)=\frac{1}{6}\sum_{a=3}^{12}a(a-1)(a-2)K_a^{(3)}=0, \tag{151}$$

and [58,59]

$$\zeta_B^+(0)=\frac{5}{4}+\frac{1079}{240}-\frac{1}{2}\sum_{a=2}^{12}\omega(a)K_a^{(3)}+\sum_{j=1}^{\infty}f(j)g(j)=\frac{296}{45}, \tag{152}$$

where, on denoting here by ψ the logarithmic derivative of the Γ-function [58,59],

$$\omega(a)\equiv\frac{1}{6}\frac{\Gamma(a+1)}{\Gamma(a-2)}\left[-\log(2)-\frac{(6a^2-9a+1)}{4}\frac{\Gamma(a-2)}{\Gamma(a+1)}+2\psi(a+1)-\psi(a-2)-\psi(4)\right], \tag{153}$$

$$f(j)\equiv\frac{(-1)^j}{j!}[-1-2^{2-j}+\zeta_R(j-2)(1-\delta_{j,3})+\gamma\delta_{j,3}]. \tag{154}$$

Equation (150) achieves three goals:
(i) Vanishing of $\log(2)$ coefficient in (152);
(ii) Vanishing of $\sum_{j=1}^{\infty}f(j)g(j)$ in (152);
(iii) Regularity at the origin of ζ_B^+.

6.4. Interpretation of the Result

Since all other $\zeta(0)$ values for pure gravity obtained in the literature are negative, the analysis here briefly outlined shows that only fully diffeomorphism-invariant boundary conditions lead to a positive $\zeta(0)$ value for pure gravity on the 4-ball, and hence *only fully diffeomorphism-invariant boundary conditions lead to a vanishing cosmological wave function for vanishing 3-geometries at*

1-loop level, at least on the Euclidean 4-ball. If the probabilistic interpretation is tenable for the whole universe, this means that *the universe has vanishing probability of reaching the initial singularity* at $a=0$, which is therefore avoided by virtue of quantum effects [58,59], since the 1-loop wave function is proportional to $a^{\zeta(0)}$ [140].

Interestingly, quantum cosmology can have observational consequences as well. For example, the work in Ref. [104] has derived the primordial power spectrum of density fluctuations in the framework of quantum cosmology, by performing a Born–Oppenheimer approximation of the Wheeler–DeWitt equation for an inflationary universe with a scalar field. In this way one first recovers the scale-invariant power spectrum that is found as an approximation in the simplest inflationary models. One then obtains quantum gravitational corrections to this spectrum, discussing whether they lead to measurable signatures in the Cosmic Microwave Background anisotropy spectrum [104].

7. Hawking's Radiation

Hawking's theoretical discovery of particle creation by black holes [87,88] has led, along the years, to many important developments in quantum field theory in curved spacetime, quantum gravity and string theory. Thus, we devote this section to a brief review of such an effect, relying upon the DAMTP lecture notes by Townsend [147]. For this purpose, we consider a massless scalar field Φ in a Schwarzschild black hole spacetime. The positive-frequency outgoing modes of Φ are known to behave, near future null infinity $\mathcal{F}^+$, as

$$\Phi_\omega \sim \mathrm{e}^{-\mathrm{i}\omega u}. \tag{155}$$

According to a geometric optics approximation, a particle's worldline is a null ray γ of constant phase u, and we trace this ray backwards in time from $\mathcal{F}^+$. The later it reaches $\mathcal{F}^+$, the closer it must approach the future event horizon $\mathcal{H}^+$ in the exterior spacetime before entering the star. The ray γ belongs to a family of rays whose limit as $t \to \infty$ is a null geodesic generator, denoted by γ_H, of $\mathcal{H}^+$. One can specify γ by giving its affine distance from γ_H along an ingoing null geodesic passing through $\mathcal{H}^+$. The affine parameter on this ingoing null geodesic is U, so $U=-\varepsilon$. One can thus write, on γ near $\mathcal{H}^+$ (κ being the surface gravity),

$$u = -\frac{1}{\kappa}\log \varepsilon, \tag{156}$$

so that positive-frequency outgoing modes have, near $\mathcal{H}^+$, the approximate form

$$\Phi_\omega \sim \mathrm{e}^{\frac{\mathrm{i}\omega}{\kappa}\log\varepsilon}. \tag{157}$$

This describes increasingly rapid oscillations as $\varepsilon \to 0$, and hence the geometric optics approximation is indeed justified at late times.

The positive-frequency outgoing modes should be matched onto a solution of the Klein–Gordon equation near past null infinity $\mathcal{F}^-$. When geometric optics holds, one performs parallel transport of the vectors n parallel to $\mathcal{F}^-$ and l orthogonal to n back to $\mathcal{F}^-$ along the continuation of γ_H. Such a continuation can be taken to meet $\mathcal{F}^-$ at $v=0$. The continuation of the null ray γ back to $\mathcal{F}^-$ meets $\mathcal{F}^-$ at an affine distance ε along an outgoing null geodesic on $\mathcal{F}^-$. The affine parameter on outgoing null geodesics in $\mathcal{F}^-$ is v, because the line element takes on $\mathcal{F}^-$ the form

$$ds^2 = du\, dv + r^2 d\Omega^2, \tag{158}$$

$d\Omega^2$ being the line element on a unit 2-sphere, so that $v = -\varepsilon$ and

$$\Phi_\omega \sim \mathrm{e}^{\frac{\mathrm{i}\omega}{\kappa}\log(-v)}. \tag{159}$$

This holds for negative values of v. When v is instead positive, an ingoing null ray from $\mathcal{F}^-$ passes through $\mathcal{H}^+$ and does not reach $\mathcal{F}^+$, hence the positive-frequency outgoing modes depend on v on $\mathcal{F}^-$, where

$$\Phi_\omega(v) = 0 \text{ if } v > 0, \mathrm{e}^{\frac{\mathrm{i}\omega}{\kappa}\log(-v)} \text{ if } v < 0. \tag{160}$$

Consider now the Fourier transform

$$\widetilde{\Phi}_\omega \equiv \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega' v}\Phi_\omega(v)dv = \int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}\omega' v + \frac{\mathrm{i}\omega}{\kappa}\log(-v)}dv. \tag{161}$$

In this integral, let us choose the branch cut in the complex v-plane to lie along the real axis. For positive ω' let us rotate contour to the positive imaginary axis and then set $v = \mathrm{i}x$ to get

$$\widetilde{\Phi}_\omega(\omega') = -\mathrm{i}\int_0^{\infty} \mathrm{e}^{-\omega' x + \frac{\mathrm{i}\omega}{\kappa}\log(x\mathrm{e}^{-\mathrm{i}\pi/2})}dx = -\mathrm{e}^{\frac{\pi\omega}{2\kappa}}\int_0^{\infty} \mathrm{e}^{-\omega' x + \frac{\mathrm{i}\omega}{\kappa}\log(x)}dx. \tag{162}$$

Since ω' is positive the integral converges. When ω' is negative one can rotate the contour to the negative imaginary axis and then set $v = -\mathrm{i}x$ to get

$$\widetilde{\Phi}_\omega(\omega') = -\mathrm{i}\int_0^{\infty} \mathrm{e}^{-\omega' x + \frac{\mathrm{i}\omega}{\kappa}\log(x\mathrm{e}^{-\mathrm{i}\pi/2})}dx = -\mathrm{e}^{\frac{\pi\omega}{2\kappa}}\int_0^{\infty} \mathrm{e}^{-\omega' x + \frac{\mathrm{i}\omega}{\kappa}\log(x)}dx. \tag{163}$$

From the two previous formulae one gets

$$\widetilde{\Phi}_\omega(-\omega') = -\mathrm{e}^{-\frac{\pi\omega}{\kappa}}\widetilde{\Phi}_\omega(\omega') \text{ if } \omega' > 0. \tag{164}$$

Thus, a mode of positive frequency ω on $\mathcal{F}^+$ matches, at late times, onto positive and negative modes on $\mathcal{F}^-$. For positive ω' one can identify

$$A_{\omega\omega'} = \widetilde{\Phi}_\omega(\omega'), \tag{165}$$

$$B_{\omega\omega'} = \widetilde{\Phi}_\omega(-\omega') = -\mathrm{e}^{-\frac{\pi\omega}{\kappa}}\widetilde{\Phi}_\omega(\omega'), \tag{166}$$

as the Bogoliubov coefficients. These formulae imply that

$$B_{ij} = -\mathrm{e}^{-\frac{\pi\omega}{\kappa}}A_{ij}. \tag{167}$$

On the other hand, the matrices A and B should satisfy the Bogoliubov relations, from which

$$\delta_{ij} = (AA^{\dagger} - BB^{\dagger})_{ij} = \sum_{l} (A_{il}A^{*}_{jl} - B_{il}B^{*}_{jl}) = [\mathrm{e}^{\frac{\pi(\omega_i+\omega_j)}{\kappa}} - 1]\sum_{l} B_{il}B^{*}_{jl}, \qquad (168)$$

where we have inserted the formula relating B_{ij} to A_{ij}. Now one can take $i = j$ to get

$$(BB^{\dagger})_{ii} = \frac{1}{\mathrm{e}^{\frac{2\pi\omega_i}{\kappa}} - 1}. \qquad (169)$$

Eventually, one needs the inverse Bogoliubov coefficients corresponding to a positive-frequency mode on $\mathcal{F}^-$ matching onto positive- and negative-frequency modes on $\mathcal{F}^+$. Since the inverse B coefficient is found to be

$$B' = -B^T, \qquad (170)$$

the late-time particle flux through $\mathcal{F}^+$, given a vacuum on $\mathcal{F}^-$, turns out to be

$$\langle N_i \rangle_{\mathcal{F}^+} = ((B')^{\dagger} B')_{ii} = (B^* B^T)_{ii} = (BB^T)^*_{ii} \qquad (171)$$

From reality of $(BB^T)_{ii}$, the previous formulae lead to

$$\langle N_i \rangle_{\mathcal{F}^+} = \frac{1}{\mathrm{e}^{\frac{2\pi\omega_i}{\kappa}} - 1} \qquad (172)$$

Remarkably, this is the Planck distribution for black body radiation from a Schwarzschild black hole at the Hawking temperature

$$T_H = \frac{\hbar\kappa}{2\pi}. \qquad (173)$$

8. Achievements and Open Problems

At this stage, the general reader might well be wondering what has been gained by working on the quantum gravity problem over so many decades. Indeed, at the theoretical level, at least the following achievements can be brought to his (or her) attention:

(i) The ghost fields [63] necessary for the functional-integral quantization of gravity and Yang–Mills theories [33, 61] have been discovered, jointly with a deep perspective on the space of histories formulation.

(ii) The Vilkovisky–DeWitt gauge-invariant effective action [151, 38] has been obtained and thoroughly studied.

(iii) We know that black holes emit a thermal spectrum and have temperature and entropy by virtue of semiclassical quantum effects [87, 88, 40]. A full theory of quantum gravity should account for this and should tell us whether or not the black holes evaporation process comes to an end [153].

(iv) The manifestly covariant theory leads to the detailed calculation of physical quantities such as cross-sections for gravitational scattering of identical scalar particles, scattering of gravitons by scalar particles, scattering of one graviton by another and gravitational bremsstrahlung [34], but no laboratory experiment is in sight for these effects.

(v) *The ultimate laboratory for modern high energy physics is the whole universe*. We have reasons to believe that either we need string and brane theory with all their (extra) ingredients, or we have to resort to radically different approaches such as, for example, loop space or twistors, the latter two living however in isolation with respect to deep ideas such as supersymmetry and supergravity (but we acknowledge that twistor string theory [163] is making encouraging progress [113,146]).

Although string theory may provide a finite theory of quantum gravity that unifies all fundamental interactions at once, its impact on particle physics phenomenology and laboratory experiments remains elusive. Some key issues are therefore in sight:

(i) What is the impact (if any) of Planck-scale physics on cosmological observations [166]?
(ii) Will general relativity retain its role of fundamental theory, or shall we have to accept that it is only the low-energy limit of string or M-theory?
(iii) Are renormalization-group methods a viable way to do non-perturbative quantum gravity [129,18], after the recent discovery of a non-Gaussian ultraviolet fixed point [106,130,107] of the renormalization-group flow?
(iv) Is there truly a singularity avoidance in quantum cosmology [58,59] or string theory [97,95,96]?

8.1. Experimental Side

As is well stressed, for example, in Ref. [75], gravity is so weak that it can only produce measurable effects in the presence of big masses, and this makes it virtually impossible to detect radiative corrections to it. Nevertheless, at least four items can be brought to the attention of the reader within the experimental framework.
(i) Colella et al. [25] have used a neutron interferometer to observe the quantum-mechanical phase shift of neutrons caused by their interaction with the Earth's gravitational field.

(ii) Page and Geilker [120] have considered an experiment that gave results inconsistent with the simplest alternative to quantum gravity, i.e. the semiclassical Einstein equation. This evidence supports, but does not prove, the hypothesis that a consistent theory of gravity coupled to quantized matter should also have the gravitational field quantized [30].

(iii) Balbinot et al. have shown [9] that, in a black hole-like configuration realized in a Bose–Einstein condensate, a particle creation of the Hawking type does indeed take place and can be unambiguously identified via a characteristic pattern in the density-density correlations. This has opened the concrete possibility of the experimental verification of this effect.

(iv) Mercati et al. [115], for the study of the Planck-scale modifications [3] of the energy-momentum dispersion relations, have considered the possible role of experiments involving nonrelativistic particles and particularly atoms. They have extended a recent result, establishing that measurements of atom-recoil frequency can provide insight that is valuable for some theoretical models.

We are already facing unprecedented challenges, where the achievements of spacetime physics and quantum field theory are called into question. The years to come will hopefully tell us whether the many new mathematical concepts considered in theoretical physics lead really to a better understanding of the physical universe and its underlying structures.

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Glossary

ψ : The quantum-mechanical wave function, which obeys the Schrödinger equation written in Eq. (5).

$\mathcal{H}$: The (abstract) Hilbert space of ordinary quantum mechanics

$\hat{H}$: The Hamiltonian operator of quantum mechanics

g : The spacetime metric

(M,g) : The spacetime manifold, where M is taken to be connected, Hausdorff, smooth; the metric g is taken to be Lorentzian, and a globally defined timelike vector field on M is taken to exist. Strictly, one deals with an equivalence class of pairs (M,g), since the pairs (M_1, g_1) and (M_2, g_2) are viewed as equivalent if a diffeomorphism α exists from M_1 to M_2 such that g_2 is obtainable from g_1 through the action of α .

Dirac operator: the most fundamental operator in the theory of elliptic operators on Riemannian manifolds. The square of an operator of Dirac type yields an operator of Laplace type, and this is closely related to the fact that the spacetime metric can be recovered from the Dirac γ -matrices as is shown in equation (62) (see also Eq. (63) for the relation between metric and tetrads).

Wheeler—DeWitt equation: the basic equation of canonical quantum gravity (see Eqs. (37) and (46)). It results from Dirac's prescription, according to which the quantum version of first-class constraints should annihilate the wave functional.

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Biographical Sketch

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His original contributions are mainly devoted to quantum gravity and quantum field theory on manifolds with boundary (1-loop conformal anomalies, mixed and diff-invariant boundary conditions, heat-kernel asymptotics, Casimir effect), spontaneous symmetry breaking in the early universe, scattering from singular potentials in quantum mechanics. He has received 1831 citations for his papers until 8 September 2011 according to the SLAC database criteria, including 7 top-cited papers (2 very well-known and 5 well-known) and 45 known papers. This yields an h-index for his work equal to 23, i.e. well above the minimal h-value of 18 for promotion to full professor position (see Hirsch in quant-ph/0508025).

He is author or co-author of 180 publications, including, in particular, 27 papers in Classical and Quantum Gravity, 19 papers in Physical Review D15, papers in various other journals including Communications in Mathematical Physics, Journal of High Energy Physics, Nuclear Physics B and Physical Review Letters, and 6 monographs: "Quantum Gravity, Quantum Cosmology and Lorentzian Geometries" (Springer-Verlag 1992 and 1994), "Complex General Relativity" (Kluwer 1995), "Euclidean Quantum Gravity on Manifolds with Boundary" with A.Yu. Kamenshchik and G. Pollifrone (Kluwer 1997), "Dirac Operators and Spectral Geometry" (Cambridge University Press 1998), "Quantum Gravity in Four Dimensions" (Nova Science 2001), "From Classical to Quantum Mechanics" with G. Marmo and G. Sudarshan (Cambridge University Press 2004).

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