"It is the nature of all
greatness not to be exact."
*Edmund Burke, speech on
American taxation, 1774*

10 GRAVITATIONAL RADIATION

We have seen a great many similarities between gravitation and electromagnetism. It should therefore come as no surprise that Einstein's equations, like Maxwell's equations, have radiative solutions.

No one has yet certainly detected gravitational radiation, but the reason for this is not hard to find; Einstein's theory predicts that gravitational radiation is produced in extremely small quantities in ordinary atomic processes. For instance, the probability that a transition between two atomic states will proceed by emission of gravitational, rather than electromagnetic, radiation is typically of order GE^2/e^2, where E is the energy released and e is the electronic charge. For $E \simeq 1$ eV, this probability is about 3×10^{-54}.

Why then study gravitational radiation? One reason is of course that some day we may find a strong source of gravitational radiation. Such a source may indeed already have been detected. (See Section 10.7.) However, gravitational radiation would be interesting even if there were no chance of ever detecting any, for the theory of gravitational radiation provides a crucial link between general relativity and the microscopic frontier of physics.

We have learned in recent years to describe the fundamental observables of microscopic phenomena in terms of elementary particles and their collisions. In classical electrodynamics it is the plane-wave solutions of Maxwell's equations that lead most naturally to an interpretation in terms of a particle, the photon. Similarly, it is the radiative solutions of Einstein's equations that will lead here to the concept of a particle of gravitational radiation, the *graviton*.

Unfortunately, the theory of gravitational radiation is complicated by the nonlinearity of Einstein's equations. In the spirit of Section 7.6, we may say that any gravitational wave is itself a distribution of energy and momentum that contributes to the gravitational field of the wave. This complication prevents our being able to find general radiative solutions of the exact Einstein equations.

There are two approaches to this difficulty. One is to study only the weak-field radiative solutions of Einstein equations, which describe waves carrying not enough energy and momentum to affect their own propagation. The other approach is to look long and hard for special solutions of the exact Einstein field equations. A great deal of mathematical ingenuity has gone into the second approach, with results of some elegance. However, this chapter deals with only the first, weak-field, approach to gravitational radiation. One reason is that any observable gravitational radiation is likely to be of very low intensity. A second, deeper, reason is that it is only possible to attach a precise meaning to the concept of an elementary particle when it is far away from all other particles, and for gravitons this corresponds to a weak-field solution of the field equations.

The reader should not conclude that there is any fundamental gap in our understanding of gravitation because we cannot find general exact solutions of the nonlinear field equations. Indeed, similar problems arise in electrodynamics: The problem of computing the exact electromagnetic field produced by a decaying current in an electrical oscillator is highly nonlinear, because the field acts back on the current that produces it. Even though this problem was not solved for many years after Maxwell's theory, still there was no doubt that electrical oscillators would produce the electromagnetic waves studied by Maxwell. Gravitational waves are more complicated than electromagnetic waves because they contribute to their own source outside the material gravitational antenna. However, the simple properties of both electromagnetic and gravitational waves emerge when we look far out into the wave zone, where the fields are weak.

1 The Weak-Field Approximation

We suppose the metric to be close to the Minkowski metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{10.1.1}$$

where $|h_{\mu\nu}| \ll 1$. To first order in h, the Ricci tensor is then

$$R_{\mu\nu} \simeq \frac{\partial}{\partial x^\nu}\Gamma^\lambda_{\lambda\mu} - \frac{\partial}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu} + O(h^2) \tag{10.1.2}$$

and the affine connection is

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}\eta^{\lambda\rho}\left[\frac{\partial}{\partial x^\mu}h_{\rho\nu} + \frac{\partial}{\partial x^\nu}h_{\rho\mu} - \frac{\partial}{\partial x^\rho}h_{\mu\nu}\right] + O(h^2) \tag{10.1.3}$$

As long as we restrict ourselves to first order in h, we must raise and lower *all* indices using $\eta^{\mu\nu}$, not $g^{\mu\nu}$; that is,

$$\eta^{\lambda\rho}h_{\rho\nu} \equiv h^{\lambda}{}_{\nu} \qquad \eta^{\lambda\rho}\frac{\partial}{\partial x^{\rho}} \equiv \frac{\partial}{\partial x_{\lambda}}, \qquad \text{etc.}$$

With this understanding, Eqs. (10.1.2) and (10.1.3) yield the first-order Ricci tensor

$$R_{\mu\nu} \simeq R^{(1)}_{\mu\nu} \equiv \frac{1}{2}\left(\Box^2 h_{\mu\nu} - \frac{\partial}{\partial x^{\lambda}\,\partial x^{\mu}} h^{\lambda}{}_{\nu} - \frac{\partial^2}{\partial x^{\lambda}\,\partial x^{\nu}} h^{\lambda}{}_{\mu} + \frac{\partial^2}{\partial x^{\mu}\,\partial x^{\nu}} h^{\lambda}{}_{\lambda}\right)$$

The Einstein field equations therefore read

$$\Box^2 h_{\mu\nu} - \frac{\partial}{\partial x^{\lambda}\,\partial x^{\mu}} h^{\lambda}{}_{\nu} - \frac{\partial^2}{\partial x^{\lambda}\,\partial x^{\nu}} h^{\lambda}{}_{\mu} + \frac{\partial^2}{\partial x^{\mu}\,\partial x^{\nu}} h^{\lambda}{}_{\lambda} = -16\pi G S_{\mu\nu} \qquad (10.1.4)$$

$$S_{\mu\nu} \equiv T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}T^{\lambda}{}_{\lambda} \qquad (10.1.5)$$

Here $T_{\mu\nu}$ is taken to lowest order in $h_{\mu\nu}$, so it is independent of $h_{\mu\nu}$, and satisfies the ordinary conservation conditions

$$\frac{\partial}{\partial x^{\mu}} T^{\mu}{}_{\nu} = 0 \qquad (10.1.6)$$

(If gravitational forces play an important role in the structure of the radiating system, then $\tau^{\mu\nu}$ should be used in place of $T^{\mu\nu}$; see Section 7.6.) Note that it is this form of the conservation law that is needed for the consistency of (10.1.4), because (10.1.6) implies

$$\frac{\partial}{\partial x^{\mu}} S^{\mu}{}_{\nu} = \frac{1}{2}\frac{\partial}{\partial x^{\nu}} S^{\lambda}{}_{\lambda}$$

whereas the linearized Ricci tensor satisfies Bianchi identities of the form

$$\frac{\partial}{\partial x^{\mu}} R^{(1)\mu}{}_{\nu} = \frac{1}{2}\frac{\partial}{\partial x^{\nu}}\left[\Box^2 h^{\lambda}{}_{\lambda} - \frac{\partial^2 h^{\lambda\nu}}{\partial x^{\lambda}\,\partial x^{\nu}}\right] = \frac{1}{2}\frac{\partial R^{(1)\lambda}{}_{\lambda}}{\partial x^{\nu}}$$

As already discussed in Section 7.4, we cannot expect a field equation such as (10.1.4) to yield unique solutions, because given any solution, we can always generate other solutions by performing coordinate transformations. The most general coordinate transformation that leaves the field weak is of the form

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \qquad (10.1.7)$$

where $\partial\varepsilon^{\mu}/\partial x^{\nu}$ is at most of the same order of magnitude as $h_{\mu\nu}$. The metric in the new coordinate system is given by

$$g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\rho}} g^{\lambda\rho}$$

or, since $g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu}$,

$$h'^{\mu\nu} = h^{\mu\nu} - \frac{\partial \varepsilon^\mu}{\partial x^\lambda}\eta^{\lambda\nu} - \frac{\partial \varepsilon^\nu}{\partial x^\rho}\eta^{\rho\mu}$$

Thus, if $h_{\mu\nu}$ is a solution of (10.1.4), then so will be

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \varepsilon_\mu}{\partial x^\nu} - \frac{\partial \varepsilon_\nu}{\partial x^\mu} \tag{10.1.8}$$

where $\varepsilon_\mu \equiv \varepsilon^\nu \eta_{\mu\nu}$ are four small but otherwise arbitrary functions of x^μ. That this is the case can be verified by direct inspection of Eq. (10.1.4); this property is called the *gauge invariance* of the field equation.

The gauge invariance of Eq. (10.1.4) is a nuisance when it comes to actually solving the field equations. However, the difficulty can be removed by choosing some particular gauge, that is, coordinate system. The most convenient choice is to work in a *harmonic coordinate system*, for which

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$$

Using (10.1.3), this gives to first order

$$\frac{\partial}{\partial x^\mu} h^\mu{}_\nu = \frac{1}{2}\frac{\partial}{\partial x^\nu} h^\mu{}_\mu \tag{10.1.9}$$

That this choice is always possible follows from the general argument of Section 7.4; it can also be seen from (10.1.8) that if $h_{\mu\nu}$ does not satisfy (10.1.9), then we can find an $h'_{\mu\nu}$ that does, by performing the coordinate transformation (10.1.7) with

$$\Box^2 \varepsilon_\nu \equiv \frac{\partial}{\partial x^\mu} h^\mu{}_\nu - \frac{1}{2}\frac{\partial}{\partial x^\nu} h^\mu{}_\mu$$

It will be assumed from now on that $h_{\mu\nu}$ does satisfy Eq. (10.1.9).

Using (10.1.9) in (10.1.4), the field equation now read

$$\Box^2 h_{\mu\nu} = -16\pi G S_{\mu\nu} \tag{10.1.10}$$

One solution is the *retarded potential*

$$h_{\mu\nu}(\mathbf{x}, t) = 4G\int d^3\mathbf{x}'\,\frac{S_{\mu\nu}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|} \tag{10.1.11}$$

We have already remarked that the conservation law (10.1.6) for $T^{\mu\nu}$ is equivalent to

$$\frac{\partial}{\partial x^\mu} S^\mu{}_\nu = \frac{1}{2}\frac{\partial}{\partial x^\nu} S^\mu{}_\mu \tag{10.1.12}$$

and in consequence the solution (10.1.11) for a source $S_{\mu\nu}$ confined to a finite volume automatically satisfies the harmonic coordinate conditions (10.1.9). (The proof is identical to that used in electrodynamics in the calculation of the vector potential in Lorentz gauge.) To (10.1.11) we can add any solution of the homogeneous equations

$$\Box^2 h_{\mu\nu} = 0 \tag{10.1.13}$$

$$\frac{\partial}{\partial x^\mu} h^\mu{}_\nu = \frac{1}{2} \frac{\partial}{\partial x^\nu} h^\mu{}_\mu \tag{10.1.14}$$

We interpret (10.1.11) as the gravitational radiation produced by the source $S_{\mu\nu}$, whereas any additional term satisfying (10.1.13), (10.1.14) represents the gravitational radiation coming in from infinity. The occurrence in (10.1.11) of the time argument $t - |\mathbf{x} - \mathbf{x}'|$ shows that gravitational effects propagate with unit velocity, that is, with the speed of light.

2 Plane Waves

We now consider the plane-wave solutions of the homogeneous equations (10.1.13) and (10.1.14), both because they are important in their own right and, as we shall see, because the retarded wave (10.1.11) approaches a plane wave as $r \to \infty$. The general solution of (10.1.13) and (10.1.14) is a linear superposition of solutions of the form

$$h_{\mu\nu}(x) = e_{\mu\nu} \exp(ik_\lambda x^\lambda) + e^*_{\mu\nu} \exp(-ik_\lambda x^\lambda) \tag{10.2.1}$$

This satisfies (10.1.13) if

$$k_\mu k^\mu = 0 \tag{10.2.2}$$

and satisfies (10.1.14) if

$$k_\mu e^\mu{}_\nu = \tfrac{1}{2} k_\nu e^\mu{}_\mu \tag{10.2.3}$$

(Of course we are still raising and lowering indices with $\eta_{\mu\nu}$, so that $k^\mu \equiv \eta^{\mu\nu} k_\nu$.) The matrix $e_{\mu\nu}$ is obviously symmetric:

$$e_{\mu\nu} = e_{\nu\mu} \tag{10.2.4}$$

It will be called the *polarization tensor*.

A symmetric 4×4 matrix would in general have ten independent components, and the four relations (10.2.3) would lower this number to six, but of these six only two represent physically significant degrees of freedom. By a change of coordinates $x^\mu \to x^\mu + \varepsilon^\mu(x)$ we can transform the metric $\eta_{\mu\nu} + h_{\mu\nu}$ into a new metric $\eta_{\mu\nu} + h'_{\mu\nu}$ with $h'_{\mu\nu}$ given by (10.1.8). Suppose that we choose

$$\varepsilon^\mu(x) = i\varepsilon^\mu \exp(ik_\lambda x^\lambda) - i\varepsilon^{\mu *} \exp(-ik_\lambda x^\lambda) \tag{10.2.5}$$

Then (10.1.8) gives

$$h'_{\mu\nu}(x) = e'_{\mu\nu} \exp(ik_\lambda x^\lambda) + e'^*_{\mu\nu} \exp(-ik_\lambda x^\lambda) \tag{10.2.6}$$

where

$$e'_{\mu\nu} = e_{\mu\nu} + k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu \tag{10.2.7}$$

[Note that the wave still satisfies the harmonic coordinate condition (10.2.3).] We conclude that $e'_{\mu\nu}$ and $e_{\mu\nu}$ represent the same physical situation for arbitrary values of the four parameters ε_μ, so of the six independent $e_{\mu\nu}$'s satisfying (10.2.3) and (10.2.4), only $6 - 4 = 2$ are physically significant. For instance, consider a wave traveling in the $+z$-direction, with wave vector

$$k^1 = k^2 = 0 \qquad k^3 = k^\circ \equiv k > 0 \tag{10.2.8}$$

In this case (10.2.3) gives

$$e_{31} + e_{01} = e_{32} + e_{02} = 0$$
$$e_{33} + e_{03} = -e_{03} - e_{00} = \tfrac{1}{2}(e_{11} + e_{22} + e_{33} - e_{00})$$

These four equations allow us to express e_{i0} and e_{22} in terms of the other six $e_{\mu\nu}$:

$$e_{01} = -e_{31}; \qquad e_{02} = -e_{32}; \qquad e_{03} = -\tfrac{1}{2}(e_{33} + e_{00}); \qquad e_{22} = -e_{11} \tag{10.2.9}$$

When the coordinate system is subjected to the transformation defined by (10.1.7) and (10.2.5), these six independent components of $e_{\mu\nu}$ change according to Eq. (10.2.7):

$$\begin{aligned} e'_{11} &= e_{11}, & e'_{12} &= e_{12} \\ e'_{13} &= e_{13} + k\varepsilon_1, & e'_{23} &= e_{23} + k\varepsilon_2 \\ e'_{33} &= e_{33} + 2k\varepsilon_1, & e'_{00} &= e_{00} - 2k\varepsilon_0 \end{aligned}$$

Thus it is only e_{11} and e_{12} that have an absolute physical significance. Indeed, we can arrange that all components of $e'_{\mu\nu}$ vanish except for e'_{11}, e'_{12}, and $e'_{22} = -e'_{11}$, by performing a coordinate transformation with

$$\varepsilon_1 = -\frac{e_{13}}{k}, \qquad \varepsilon_2 = -\frac{e_{23}}{k}, \qquad \varepsilon_3 = -\frac{e_{33}}{2k}, \qquad \varepsilon_0 = \frac{e_{00}}{2k}$$

The distinction between the different components of the polarization tensor is clarified if we ask how $e_{\mu\nu}$ changes when we subject the coordinate system to a rotation about the z-axis. This is just a Lorentz transformation of the form

$$\begin{aligned} R_1{}^1 &= \cos\theta & R_1{}^2 &= \sin\theta \\ R_2{}^1 &= -\sin\theta & R_2{}^2 &= \cos\theta \\ R_3{}^3 &= R_0{}^0 = 1 & \text{other } R_\mu{}^\nu &= 0 \end{aligned} \tag{10.2.10}$$

and since it leaves k_μ invariant (i.e., $R_\mu{}^\nu k_\nu = k_\mu$), the only effect is to transform $e_{\mu\nu}$ into

$$e'_{\mu\nu} = R_\mu{}^\rho R_\nu{}^\sigma e_{\rho\sigma} \tag{10.2.11}$$

Using the relations (10.2.9), we find that

$$e'_\pm = \exp(\pm 2i\theta)e_\pm \tag{10.2.12}$$

$$f'_\pm = \exp(\pm i\theta)f_\pm \tag{10.2.13}$$

$$e'_{33} = e_{33}, \qquad e'_{00} = e_{00} \tag{10.2.14}$$

where

$$e_\pm \equiv e_{11} \mp ie_{12} = -e_{22} \mp ie_{12} \tag{10.2.15}$$

$$f_\pm \equiv e_{31} \mp ie_{32} = -e_{01} \pm ie_{02} \tag{10.2.16}$$

In general, any plane wave ψ, which is transformed by a rotation of any angle θ about the direction of propagation into

$$\psi' = e^{ih\theta}\psi \tag{10.2.17}$$

is said to have *helicity h*. We thus have shown that a gravitational plane wave can be decomposed into parts $e_\pm$ with helicity ± 2, parts $f_\pm$ with helicity ± 1, and parts e_{00} and e_{33} with helicity zero. However, we have also seen that the parts with helicity 0 and ± 1 can be made to vanish by a suitable choice of coordinates, so *the physically significant components are just those with helicity* ± 2.

Once again we find a fruitful analogy with electromagnetism. The Maxwell equations in Lorentz gauge are (2.7.12) and (2.7.13); in empty space they become

$$\Box^2 A_\alpha = 0 \qquad \frac{\partial A^\alpha}{\partial x^\alpha} = 0$$

in analogy with Eqs. (10.1.13) and (10.1.14) for the metric in harmonic coordinates. (We are now in an inertial coordinate system, and so $\Box^2 \equiv \eta^{\alpha\beta}\,\partial^2/\partial x^\alpha\,\partial x^\beta$.) We can find a plane-wave solution of the form

$$A_\alpha = e_\alpha \exp(ik_\beta x^\beta) + e^*_\alpha \exp(-ik_\beta x^\beta)$$

where

$$k_\alpha k^\alpha = 0$$

$$k_\alpha e^\alpha = 0$$

in analogy with Eqs. (10.2.1)–(10.2.3).

In general e^α would have four independent components, but the condition that $k_\alpha e^\alpha$ vanish reduces the number of independent components to three. just as (10.2.3) reduces the number of independent components of $e_{\mu\nu}$ from ten to six.

Furthermore, without changing the physical fields **E** and **B** and without leaving the Lorentz gauge, we can change A_α by a gauge transformation

$$A_\alpha \to A'_\alpha = A_\alpha + \frac{\partial \Phi}{\partial x_\alpha}$$

$$\Phi(x) = i\varepsilon \exp(ik_\beta x^\beta) - i\varepsilon^* \exp(-ik_\beta x^\beta)$$

in analogy with (10.1.8) and (10.2.5). The new potential can be written

$$A'_\alpha = e'_\alpha \exp(ik_\beta x^\beta) + e'^*_\alpha \exp(-ik_\beta x^\beta)$$

$$e'_\alpha = e_\alpha - \varepsilon k_\alpha$$

in analogy with (10.2.6) and (10.2.7). The parameter ε is arbitrary, so of the three algebraically independent components of e_α only $3 - 1 = 2$ are physically significant, just as general covariance renders only two of the six independent components of $e_{\mu\nu}$ physically significant. To identify the two significant components of e_α, we may consider a wave traveling in the z-direction, with k^α given by Eq. (10.2.8). Then the condition that $k_\alpha e^\alpha$ vanish allows us to determine e^0,

$$e_0 = -e_3$$

just as (10.2.3) allows us to determine e_{22} and e_{0i} in terms of the other six $e_{\mu\nu}$. Also, the preceding gauge transformation leaves e_1 and e_2 invariant but changes e_3 into

$$e'_3 = e_3 - \varepsilon k$$

Hence e'_3 can be made equal to zero by choosing $\varepsilon = e_3/k$, so it is only e_1 and e_2 that carry physical significance, just as it was only e_{11} and e_{12} that could not be made equal to zero by a suitable coordinate transformation. Finally, we can work out the meaning of these two components by subjecting the plane electromagnetic wave to the rotation (10.2.10). The polarization vector is then changed into

$$e'_\alpha = R_\alpha{}^\beta e_\beta$$

and therefore

$$e'_\pm = \exp(\pm i\theta) e_\pm$$

$$e'_3 = e_3$$

where

$$e_\pm \equiv e_1 \mp i e_2$$

Thus the electromagnetic wave can be decomposed into parts with helicity ± 1 and 0. However, the physically significant helicities are ± 1, not 0, just as for gravitational waves they are ± 2, not ± 1 or 0. This is what we mean when we say, speaking classically, that electromagnetism and gravitation are carried by waves of spin 1 and spin 2, respectively.

3 Energy and Momentum of Plane Waves

The physical significance of the plane-wave solution (10.2.1) is brought forward by calculating the energy and momentum it carries. According to Eq. (7.6.4), the energy-momentum tensor of gravitation is given to order h^2 by

$$t_{\mu\nu} \simeq \frac{1}{8\pi G}\left[-\tfrac{1}{2}h_{\mu\nu}\eta^{\lambda\rho}R^{(1)}_{\lambda\rho} + \tfrac{1}{2}\eta_{\mu\nu}h^{\lambda\rho}R^{(1)}_{\lambda\rho} + R^{(2)}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\lambda\rho}R^{(2)}_{\lambda\rho}\right]$$

where $R^{(N)}_{\mu\nu}$ is the term in the Ricci tensor of order N in $h_{\mu\nu}$. The metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ satisfies the first-order Einstein equations $R^{(1)}_{\mu\nu} = 0$, so we can drop these terms in $t_{\mu\nu}$ and use

$$t_{\mu\nu} \simeq \frac{1}{8\pi G}\left[R^{(2)}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\lambda\rho}R^{(2)}_{\lambda\rho}\right] \tag{10.3.1}$$

(For the actual metric it is $R_{\mu\nu}$ rather than $R^{(1)}_{\mu\nu}$ that vanishes, and $t_{\mu\nu}$ arises only from the first-order terms in Eq. (7.6.4). Here it is $R^{(1)}_{\mu\nu}$ rather than $R_{\mu\nu}$ that vanishes, because $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ satisfies the first-order Einstein equations rather than the exact equations. The difference is only of order h^3.) To calculate $R^{(2)}_{\mu\nu}$ we must use Eq. (10.2.1) in Eq. (7.6.15); the result is extremely complicated, but can be simplified if we average $t_{\mu\kappa}$ over a region of space and time much larger than $|\mathbf{k}|^{-1}$. (This is the way the energy and momentum of any wave are usually evaluated.) The averaging kills all terms proportional to $\exp(\pm 2ik_\lambda x^\lambda)$, and we are left with only the x^μ-independent cross-terms:

$$\begin{aligned}\langle R^{(2)}_{\mu\nu}\rangle = \mathrm{Re}\,\{&e^{\lambda\rho *}[k_\mu k_\nu e_{\lambda\rho} - k_\mu k_\lambda e_{\nu\rho} - k_\nu k_\rho e_{\mu\lambda} + k_\lambda k_\rho e_{\mu\nu}]\\ &+ [e^\lambda{}_\rho k_\lambda - \tfrac{1}{2}e_\lambda{}^\lambda k_\rho]^*[k_\mu e^\rho{}_\nu + k_\nu e^\rho{}_\mu - k^\rho e_{\mu\nu}]\\ &- \tfrac{1}{2}[k_\lambda e_{\rho\nu} + k_\nu e_{\rho\lambda} - k_\rho e_{\lambda\nu}]^*[k^\lambda e^\rho{}_\mu + k_\mu e^{\rho\lambda} - k^\rho e^\lambda{}_\mu]\}\end{aligned} \tag{10.3.2}$$

(We have not yet made use of the conditions (10.2.2) and (10.2.3) appropriate to harmonic coordinates, so suppose for a moment that we leave the harmonic coordinate system by adding to $h_{\mu\nu}(x)$ a term

$$i(q_\mu\varepsilon_\nu + q_\nu\varepsilon_\mu)\exp(iq_\lambda x^\lambda) - i(q_\mu\varepsilon_\nu^* + q_\nu\varepsilon_\nu^*)\exp(-iq_\lambda x^\lambda) \tag{10.3.3}$$

where $q_\mu q^\mu \neq 0$. After averaging over space-time distances large compared with $|q - k|^{-1}$, the interference between (10.2.1) and (10.3.3) drops out, and we find for $\langle R^{(2)}_{\mu\kappa}\rangle$ the term (10.3.2), plus another term obtained by replacing k with q and $e_{\mu\nu}$ with $q_\mu\varepsilon_\nu + q_\nu\varepsilon_\mu$. Inspection of (10.3.2) shows immediately that this second term *vanishes*, so $\langle R^{(2)}_{\mu\kappa}\rangle$ and hence $\langle t_{\mu\kappa}\rangle$ may be calculated in harmonic coordinates with no loss of generality.)

If we now use in (10.3.2) the harmonic coordinate conditions (10.2.2) and (10.2.3), we find

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{2}\left(e^{\lambda\rho *}e_{\lambda\rho} - \tfrac{1}{2}|e^\lambda{}_\lambda|^2\right) \tag{10.3.4}$$

The quantity $\eta^{\lambda\rho}\langle R^{(2)}_{\lambda\rho}\rangle$ vanishes because $k^\rho k_\rho = 0$, so (10.3.1) now gives the average energy-momentum tensor of a plane wave as

$$\langle t_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{16\pi G}\left(e^{\lambda\rho *}e_{\lambda\rho} - \tfrac{1}{2}|e^\lambda{}_\lambda|^2\right) \tag{10.3.5}$$

Note that a "gauge transformation" (10.2.7) will change the terms in $\langle t_{\mu\kappa}\rangle$ into

$$e'^{\lambda\rho *}e'_{\lambda\rho} = e^{\lambda\rho *}e_{\lambda\rho} + 2\,\mathrm{Re}\,\varepsilon^*_\rho k^\rho e^\lambda{}_\lambda + 2|\varepsilon_\rho k^\rho|^2$$
$$e'^\lambda{}_\lambda = e^\lambda{}_\lambda + 2k^\lambda\varepsilon_\lambda$$

but $\langle t_{\mu\kappa}\rangle$ is gauge-invariant! Thus, as far as energy and momentum are concerned, the polarizations $e_{\mu\nu}$ and $e_{\mu\nu} + k_\mu\varepsilon_\nu + k_\nu\varepsilon_\mu$ represent the same physical wave, and we see again that there are not six but only two physically significant polarization parameters. In particular, a wave traveling in the z-direction, with wave vector and polarization tensor given by (10.2.8) and (10.2.9), has the energy-momentum tensor

$$\langle t_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{8\pi G}\left(|e_{11}|^2 + |e_{12}|^2\right) \tag{10.3.6}$$

or, in terms of the helicity amplitudes (10.2.15),

$$\langle t_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{16\pi G}\left(|e_+|^2 + |e_-|^2\right) \tag{10.3.7}$$

4 Generation of Gravitational Waves

We wish to calculate the energy emitted in the form of gravitational radiation by a system whose energy-momentum tensor can be expressed as a Fourier integral,

$$T_{\mu\nu}(\mathbf{x}, t) = \int_0^\infty d\omega T_{\mu\nu}(\mathbf{x}, \omega)e^{-i\omega t} + \text{c.c.} \tag{10.4.1}$$

or as a sum of Fourier components,

$$T_{\mu\nu}(\mathbf{x}, t) = \sum_\omega e^{-i\omega t}T_{\mu\nu}(\mathbf{x}, \omega) + \text{c.c.} \tag{10.4.2}$$

(Here "+ c.c." means "plus the complex conjugate of the preceding term.") We first do the calculation for a single Fourier component,

$$T_{\mu\nu}(\mathbf{x}, t) = T_{\mu\nu}(\mathbf{x}, \omega)e^{-i\omega t} + \text{c.c.} \tag{10.4.3}$$

and will then return to the more general systems described by (10.4.1) and (10.4.2).

From (10.1.11) we find that the field emitted by the source (10.4.3) is

$$h_{\mu\nu}(\mathbf{x}, t) = 4G \int \frac{d^3\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} S_{\mu\nu}(\mathbf{x}', \omega) \exp\{-i\omega t + i\omega|\mathbf{x} - \mathbf{x}'|\} + \text{c.c.} \quad (10.4.4)$$

where

$$S_{\mu\nu}(\mathbf{x}, \omega) \equiv T_{\mu\nu}(\mathbf{x}, \omega) - \tfrac{1}{2}\eta_{\mu\nu}T^{\lambda}{}_{\lambda}(\mathbf{x}, \omega) \quad (10.4.5)$$

Suppose that we observe this radiation in the *wave zone*, that is, at distances $r \equiv |\mathbf{x}|$ much larger than the dimension $R = |\mathbf{x}'|_{\max}$ of the source, and also much larger than ωR^2 and $1/\omega$. Then the denominator $|\mathbf{x} - \mathbf{x}'|$ can be replaced with r, while in the exponent we may approximate

$$|\mathbf{x} - \mathbf{x}'| \simeq r - \mathbf{x}' \cdot \hat{\mathbf{x}} \qquad \hat{\mathbf{x}} \equiv \frac{\mathbf{x}}{r}$$

and the field becomes

$$h_{\mu\nu}(\mathbf{x}, t) = \frac{4G}{r} \exp(i\omega r - i\omega t) \int d^3\mathbf{x}' S_{\mu\nu}(\mathbf{x}', \omega) e^{-i\omega\hat{\mathbf{x}}\cdot\mathbf{x}'} + \text{c.c.} \quad (10.4.6)$$

Since $r\omega$ is assumed large, this looks just like a plane wave,

$$h_{\mu\nu}(\mathbf{x}, t) = e_{\mu\nu}(\mathbf{x}, \omega) \exp(ik_\mu x^\mu) + \text{c.c.} \quad (10.4.7)$$

with "wave vector" and "polarization tensor" given by

$$\mathbf{k} \equiv \omega\hat{\mathbf{x}} \qquad k^0 \equiv \omega \quad (10.4.8)$$

$$e_{\mu\nu}(\mathbf{x}, \omega) \equiv \frac{4G}{r} \int d^3\mathbf{x}' S_{\mu\nu}(\mathbf{x}', \omega) e^{-i\mathbf{k}\cdot\mathbf{x}'} \quad (10.4.9)$$

It will be convenient to write $e_{\mu\nu}$ explicitly in terms of the Fourier transform of $T_{\mu\nu}$:

$$e_{\mu\nu}(\mathbf{x}, \omega) = \frac{4G}{r} [T_{\mu\nu}(\mathbf{k}, \omega) - \tfrac{1}{2}\eta_{\mu\nu}T^{\lambda}{}_{\lambda}(\mathbf{k}, \omega)] \quad (10.4.10)$$

$$T_{\mu\nu}(\mathbf{k}, \omega) \equiv \int d^3\mathbf{x}' T_{\mu\nu}(\mathbf{x}', \omega) e^{-i\mathbf{k}\cdot\mathbf{x}'} \quad (10.4.11)$$

The conservation equation for $T_{\mu\nu}(\mathbf{x}, t)$ is

$$\frac{\partial}{\partial x^\mu} T^\mu{}_\nu(\mathbf{x}, t) = 0$$

Applying this to (10.4.3) gives

$$\frac{\partial}{\partial x^i} T^i{}_\nu(\mathbf{x}, \omega) - i\omega T^0{}_\nu(\mathbf{x}, \omega) = 0$$

Multiplying with $e^{-i\mathbf{k}\cdot\mathbf{x}}$ and integrating over $\mathbf{x}$, we find that $T_{\mu\nu}(\mathbf{k}, \omega)$ is subject to the algebraic relations

$$k_\mu T^\mu{}_\nu(\mathbf{k}, \omega) = 0 \tag{10.4.12}$$

where k^μ is the vector given by (10.4.8). This, incidentally, verifies that (10.4.10) obeys the harmonic coordinate condition (10.2.3).

Now let us calculate the power per unit solid angle emitted in a direction $\hat{x}$. Since $r \gg 1/\omega$, we can use for the energy flux vector the value $\langle t^{i0} \rangle$ obtained by averaging over space-time dimensions large compared with $1/\omega$, so that the power per solid angle is

$$\frac{dP}{d\Omega} = r^2 \hat{x}^i \langle t^{i0} \rangle$$

We use for $\langle t^{\mu\nu} \rangle$ the value (10.3.5), so this gives

$$\frac{dP}{d\Omega} = \frac{r^2(\mathbf{k}\cdot\hat{\mathbf{x}})k^0}{16\pi G}\,[e^{\lambda\nu *}(\mathbf{x}, \omega)e_{\lambda\nu}(\mathbf{x}, \omega) - \tfrac{1}{2}|e^\lambda{}_\lambda(\mathbf{x}, \omega)|^2]$$

and inserting the values (10.4.8) and (10.4.10) for k^μ and $e_{\lambda\nu}$, we find that the r^2 factors cancel, and

$$\frac{dP}{d\Omega} = \frac{G\omega^2}{\pi}\,[T^{\lambda\nu *}(\mathbf{k}, \omega)T_{\lambda\nu}(\mathbf{k}, \omega) - \tfrac{1}{2}|T^\lambda{}_\lambda(\mathbf{k}, \omega)|^2] \tag{10.4.13}$$

The problem is thus solved once we have calculated the Fourier transform (10.4.11).

It will be convenient to express this result in terms of the purely spacelike components of $T^{\lambda\nu}(\mathbf{k}, \omega)$. From (10.4.12) we have

$$T_{0i}(\mathbf{k}, \omega) = -\hat{k}^j T_{ji}(\mathbf{k}, \omega)$$
$$T_{00}(\mathbf{k}, \omega) = \hat{k}^i \hat{k}^j T_{ji}(\mathbf{k}, \omega)$$

where $\hat{\mathbf{k}} \equiv \mathbf{k}/\omega \equiv \hat{\mathbf{x}}$. Using this in (10.4.13) gives

$$\frac{dP}{d\Omega} = \frac{G\omega^2}{\pi}\,\Lambda_{ij,lm}(\hat{k})T^{ij*}(\mathbf{k}, \omega)T^{lm}(\mathbf{k}, \omega) \tag{10.4.14}$$

where

$$\Lambda_{ij,lm}(\hat{k}) \equiv \delta_{il}\delta_{jm} - 2\hat{k}_j\hat{k}_m\delta_{il} + \tfrac{1}{2}\hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m - \tfrac{1}{2}\delta_{ij}\delta_{lm} + \tfrac{1}{2}\delta_{ij}\hat{k}_l\hat{k}_m + \tfrac{1}{2}\delta_{lm}\hat{k}_i\hat{k}_j \tag{10.4.15}$$

If the energy-momentum tensor is a *sum* of individual Fourier components as in (10.4.2), then the field $h_{\mu\nu}$ in the wave zone will look like a sum of the plane waves (10.4.7). The gravitational energy-momentum tensor will then be given by a double sum over these Fourier components, but all cross-terms drop out when we average over a time interval long compared with the longest "beat period," that is, the reciprocal of the shortest frequency difference. The power is thus given by a sum of terms like (10.4.14), one for each frequency in the source.

Suppose, on the other hand, that the energy-momentum tensor is a Fourier *integral*, as in (10.4.1). Then $h_{\mu\nu}$ in the wave zone will look like an integral over ω of the individual plane waves (10.4.7), and the gravitational energy-momentum tensor will be given by a double integral $\iint d\omega\, d\omega'$ of products of these terms. The integrand again has time dependence $\exp(-i(\omega - \omega')t)$, but now there is no "longest beat period," so instead of computing the average power we calculate the total energy emitted. This is given by integrating the power over all time, and the effect is to replace the factors $e^{-i\omega t}e^{i\omega' t}$ in the double integral for the power with

$$\int_{-\infty}^{\infty} \exp(-i(\omega - \omega')t)\, dt = 2\pi\, \delta(\omega - \omega')$$

The energy per solid angle emitted in a direction $\hat{\mathbf{k}}$ is thus a single integral:

$$\frac{dE}{d\Omega} = 2G \int_0^{\infty} \omega^2 [T^{\lambda\nu *}(\mathbf{k}, \omega) T_{\lambda\nu}(\mathbf{k}, \omega) - \tfrac{1}{2} |T^{\lambda}{}_{\lambda}(\mathbf{k}, \omega)|^2]\, d\omega \qquad (10.4.16)$$

or, in terms of the space-space components,

$$\frac{dE}{d\Omega} = 2G\Lambda_{ij,lm}(\hat{k}) \int_0^{\infty} \omega^2 T^{ij*}(\mathbf{k}, \omega) T^{lm}(\mathbf{k}, \omega)\, d\omega$$

As an example, consider a system of free particles n that initially move at constant velocity $\mathbf{v}_n$, collide at the origin at $t = 0$, and then move off again at velocities $\tilde{\mathbf{v}}_n$. The energy-momentum tensor is then

$$T^{\mu\nu}(\mathbf{x}, t) = \sum_n \frac{P_n{}^{\mu} P_n{}^{\nu}}{E_n} \delta^3(\mathbf{x} - \mathbf{v}_n t)\theta(-t)$$
$$+ \sum_n \frac{\tilde{P}_n{}^{\mu} \tilde{P}_n{}^{\nu}}{\tilde{E}_n} \delta^3(\mathbf{x} - \tilde{\mathbf{v}}_n t)\theta(t) \qquad (10.4.17)$$

where $P_n{}^0 = E_n = m_n(1 - \mathbf{v}_n{}^2)^{-1/2}$ and $\mathbf{P}_n = E_n \mathbf{v}_n$ are the energy and momentum of the nth incoming particle, $\tilde{P}_n{}^0 = \tilde{E}_n$ and $\tilde{\mathbf{P}}_n$ are the corresponding quantities for the outgoing particles, and θ is the step function

$$\theta(s) = \begin{cases} +1 & s > 0 \\ 0 & s < 0 \end{cases} \qquad (10.4.18)$$

The functions θ and δ^3 have the well-known integral representations

$$\theta(s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{+i\omega s}}{\omega - i\varepsilon}\, d\omega \qquad \varepsilon \to 0+ \qquad (10.4.19)$$

$$\delta^3(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \qquad (10.4.20)$$

(To prove (10.4.19), note that the contour can be closed with a large semicircle in the lower or upper half-plane, according to whether $s < 0$ or $s > 0$. To prove (10.4.20). take the Fourier transform of both sides.) We see then that $T^{\mu\nu}(\mathbf{x}, t)$ is of the form (10.4.1), with

$$T^{\mu\nu}(\mathbf{x}, \omega) = \frac{1}{(2\pi)^4 i}\left[\sum_n \frac{P_n{}^\mu P_n{}^\nu}{E_n}\int d^3\mathbf{k}\,\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\omega - \mathbf{v}_n\cdot\mathbf{k} - i\varepsilon} - \sum_n \frac{\tilde{P}_n{}^\mu \tilde{P}_n{}^\nu}{\tilde{E}_n}\int d^3\mathbf{k}\,\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\omega - \tilde{\mathbf{v}}_n\cdot\mathbf{k} + i\varepsilon}\right]$$

and the Fourier transform (10.4.11) is

$$T^{\mu\nu}(\mathbf{k}, \omega) = \frac{1}{2\pi i}\left[\sum_n \frac{P_n{}^\mu P_n{}^\nu}{E_n(\omega - \mathbf{v}_n\cdot\mathbf{k} - i\varepsilon)} - \sum_n \frac{\tilde{P}_n{}^\mu \tilde{P}_n{}^\nu}{\tilde{E}_n(\omega - \tilde{\mathbf{v}}_n\cdot\mathbf{k} + i\varepsilon)}\right]$$

We can now drop the $\pm i\varepsilon$ in the denominators, for $\omega - \mathbf{v}_n\cdot\mathbf{k}$ cannot vanish if $\omega = |\mathbf{k}|$ and $|\mathbf{v}_n| < 1$. (For the case of particles traveling at the speed of light, see below.) Also, $E_n(\mathbf{v}_n\cdot\mathbf{k} - \omega) = P_n{}^\lambda k_\lambda \equiv (P_n\cdot k)$, so we can write

$$T^{\mu\nu}(\mathbf{k}, \omega) = \frac{1}{2\pi i}\sum_N \frac{P_N{}^\mu P_N{}^\nu \eta_N}{(P_N\cdot k)} \tag{10.4.21}$$

where N runs over particles in both the initial and the final states, the sign factor η_N being

$$\eta_N = \begin{cases} +1 & N \text{ in final state} \\ -1 & N \text{ in initial state}\end{cases}$$

We note that (10.4.12) is satisfied, because

$$k_\mu T^{\mu\nu}(\mathbf{k}, \omega) = \frac{1}{2\pi i}\sum_N P_N{}^\mu \eta_N$$

and this must vanish because $\sum_N P_N{}^\mu \eta_N$ is simply the change in the total P^μ, which is conserved.

The gravitational energy per solid angle and per unit frequency interval emitted at frequency ω and direction $\hat{\mathbf{k}}$ is now given by (10.4.16) as

$$\left(\frac{dE}{d\Omega\, d\omega}\right) = \frac{G\omega^2}{2\pi^2}\sum_{N,M}\frac{\eta_N\eta_M}{(P_N\cdot k)(P_M\cdot k)}\left[(P_N\cdot P_M)^2 - \tfrac{1}{2}m_N{}^2 m_M{}^2\right] \tag{10.4.22}$$

If we tried to compute the total emitted energy by integrating ω from 0 to ∞, we would get a result that diverges like $\int^\infty d\omega$. This is just due to our approximation that the collision occurs instantaneously: actually it must take up some finite time Δt, and the ω-integral will be cut off at ω of order $1/\Delta t$.

Note that if none of the momenta $P_N{}^\mu$ are changed by the collision, then the contributions of the incoming and outgoing particles in (10.4.21) will cancel, and so $T_{\mu\nu}(\mathbf{k}, \omega)$ will vanish. Gravitational radiation is only emitted when the particles actually undergo accelerations.

Note also that (10.4.22) seems to become infinite if one of the particles participating in the reaction (say, $N = 1$) has zero mass, and has momentum approaching a direction parallel to $\mathbf{k}$, since then $P_1 \cdot k = E_1\omega(\hat{\mathbf{P}}_1 \cdot \hat{\mathbf{k}} - 1) \to 0$. However, this singularity is spurious, for when $\mathbf{P}_1$ becomes parallel to $\mathbf{k}$ we can treat $(P_1 \cdot P_M)$ in (10.4.22) as proportional for all $M \neq 1$ to $(k \cdot P_M)$, so the singular part of (10.4.22) is

$$\frac{G\omega^2}{\pi^2} \frac{\eta_1}{(P_1 \cdot k)} \sum_{M \neq 1} \frac{\eta_M}{(P_M \cdot k)} (P_1 \cdot P_M)^2 \propto \sum_{M \neq 1} \eta_M (P_1 \cdot P_M)$$

We have already remarked that $\sum_M \eta_M P_M{}^\mu$ must vanish when the sum is extended over all particles, so the right-hand side is simply $-\eta_1 P_1{}^2$, and this vanishes because particle 1 is assumed to have zero mass. Thus no difficulty is encountered in applying (10.4.22) to collisions involving photons, neutrinos, or even (to run ahead of ourselves a bit) gravitons.

The total energy per unit frequency interval emitted as gravitational radiation in a collision is obtained by integrating Eq. (10.4.22) over the directions of $\hat{\mathbf{k}}$. We then find

$$\frac{dE}{d\omega} = \frac{G}{2\pi} \sum_{NM} \eta_N \eta_M m_N m_M \frac{1 + \beta_{NM}^2}{\beta_{NM}(1 - \beta_{NM}^2)^{1/2}} \ln\left(\frac{1 + \beta_{NM}}{1 - \beta_{NM}}\right) \tag{10.4.23}$$

with β_{NM} the relative speed of particles N and M:

$$\beta_{NM} \equiv \left[1 - \frac{m_N^2 m_M^2}{(P_N \cdot P_M)^2}\right]^{1/2}$$

For nonrelativistic two-body elastic scattering this reduces to

$$\frac{dE}{d\omega} = \frac{8G}{5\pi} \mu^2 v^4 \sin^2\theta \tag{10.4.24}$$

where μ is the reduced mass, v is the relative velocity, and θ is the scattering angle in the center-of-mass reference frame.

The gravitational radiation produced by the collisions occurring in a gas can be determined by summing up the radiated energies per collision given by Eq. (10.4.23) or (10.4.24), provided that there is enough time between collisions so that they do not interfere. This conditions can be expressed as

$$\omega \gg \omega_c \tag{10.4.25}$$

where ω_c is the collision frequency of a typical gas particle. (If $\omega \ll \omega_c$, then the gas behaves as a fluid rather than as a collection of independent particles.) When (10.4.25) is satisfied, the power per unit volume and per unit frequency interval is

$$\frac{dP}{d\omega} = \frac{8G}{5\pi} \sum_{(a,b)} \mu_{ab}^2 n_a n_b \left\langle v_{ab}^5 \int \frac{d\sigma_{ab}}{d\Omega} \sin^2\theta \, d\Omega \right\rangle \tag{10.4.26}$$

where n_a is the number density of gas particles of type a, $d\sigma_{ab}/d\Omega$ is the center-of-mass-system differential scattering cross-section, the sum runs over all different pairs of particle types, and the average $\langle \cdots \rangle$ is taken over all collisions.

As an example, let us calculate the gravitational radiation emitted by Coulomb collisions in a plasma. The Rutherford scattering cross-section is

$$\frac{d\sigma_{ab}}{d\Omega} = \frac{e_a{}^2 e_b{}^2}{4v_{ab}^4 \mu_{ab}^2 \sin^4(\theta/2)} \tag{10.4.27}$$

The integral over θ must be cut off at a minimum angle $1/\Lambda$, with $\Lambda \gg 1$ determined by the Debye screening of the Coulomb force at large impact parameters; we have then

$$\int \frac{d\sigma_{ab}}{d\Omega} \sin^2\theta \, d\Omega \simeq \frac{4\pi e_a{}^2 e_b{}^2 \ln \Lambda}{\mu_{ab}{}^2 v_{ab}{}^4} \tag{10.4.28}$$

We are left with an average of v_{ab}, which for a Maxwell-Boltzmann distribution is

$$\langle v_{ab} \rangle = 2\left(\frac{2kT}{\pi \mu_{ab}}\right)^{1/2} \tag{10.4.29}$$

Putting (10.4.28) and (10.4.29) into (10.4.26) gives the power per unit volume and per unit frequency interval (in c.g.s. units) as

$$\frac{dP}{d\omega} = \frac{64G}{5c^5}\left(\frac{2kT}{\pi}\right)^{1/2} \ln \Lambda \sum_{(a,b)} \frac{n_a n_b e_a{}^2 e_b{}^2}{\sqrt{\mu_{ab}}} \tag{10.4.30}$$

Typically $\ln \Lambda$ is of order 10. For a plasma of completely ionized hydrogen we must take into account electron-electron and electron-proton collisions, and (10.4.30) gives

$$\frac{dP}{d\omega} = \frac{64Gn_e{}^2 e^4}{5c^5}\left(\frac{2kT}{\pi m_e}\right)^{1/2} (1 + \sqrt{2}) \ln \Lambda \tag{10.4.31}$$

The electron collision frequency may in this case be estimated as

$$\omega_c \approx \frac{e^4 n_e \langle v \rangle}{(kT)^2} \approx \frac{e^4 n_e}{(kT)^{3/2}\sqrt{m_e}} \tag{10.4.32}$$

Equation (10.4.30) or (10.4.31) holds for $\omega \gg \omega_c$ and $\hbar\omega \ll kT$.

These results can be applied to the hydrogen plasma in the solar core. Within a volume V of order 2×10^{31} cm^3 this plasma has $T \simeq 10^7$°K, $n_e \simeq 3 \times 10^{25}$ cm^{-3}, and $\ln \Lambda \simeq 4$. The collision frequency (10.4.32) is 10^{15} sec^{-1}, three orders of magnitude less than the thermal frequency $kT/\hbar \approx 10^{18}$ sec^{-1}, so the total power produced in gravitational radiation can be estimated by multiplying (10.4.31) by $VkT/\hbar$. In this way we find that the thermal collisions in the solar core produce about 10^8 watts of gravitational radiation.

5 Quadrupole Radiation

Up to this point we have made no approximations beyond the basic assumption that the fields are weak. (Our use of the wave zone limits $r \gg R, r \gg 1/\omega, r \gg \omega R^2$ was not really an approximation, since we can always choose r large enough to make these assumptions true; and by the conservation of energy, the power passing through a sphere at large r must equal that passing through any surface enclosing the radiating system.) We now make a further approximation. and assume that the source radius R is much smaller than the wavelength $1/\omega$:

$$\omega R \ll 1 \tag{10.5.1}$$

Most of the radiation is emitted at frequencies of order $\bar{v}/R$, where $\bar{v}$ is some typical velocity within the system, so we are really making the same sort of approximation as that made in the previous chapter, that is, $\bar{v} \ll 1$.

When (10.5.1) holds we may approximate the Fourier transforms needed in (10.4.14) and (10.4.16) by the $\mathbf{k}$-independent integral

$$T_{ij}(\mathbf{k}, \omega) \simeq \int d^3x T_{ij}(\mathbf{x}, \omega) \tag{10.5.2}$$

This can be rewritten in a useful way by using the conservation laws in the form

$$\frac{\partial^2}{\partial x^i \, \partial x^j} T^{ij}(\mathbf{x}, \omega) = -\omega^2 T^{00}(\mathbf{x}, \omega)$$

Multiplying with $x^i x^j$ and integrating over $\mathbf{x}$, we find

$$T_{ij}(\mathbf{k}, \omega) \simeq -\frac{\omega^2}{2} D_{ij}(\omega) \tag{10.5.3}$$

$$D_{ij}(\omega) \equiv \int d^3x x^i x^j T^{00}(\mathbf{x}, \omega) \tag{10.5.4}$$

The power per solid angle is therefore

$$\frac{dP}{d\Omega} = \frac{G\omega^6}{4\pi} \Lambda_{ij,lm}(\hat{k}) D_{ij}^*(\omega) D_{lm}(\omega) \tag{10.5.5}$$

If the source is a sum of Fourier components, then the power radiated is a sum of terms such as (10.5.5). If the source is a Fourier integral like (10.4.1), then the energy emitted per solid angle is

$$\frac{dE}{d\Omega} = \tfrac{1}{2} G \Lambda_{ij,lm}(\hat{k}) \int_0^\infty \omega^6 D_{ij}^*(\omega) D_{lm}(\omega) \, d\omega \tag{10.5.6}$$

The coefficients $D_{ij}(\omega)$ in (10.5.5) and (10.5.6) do not depend on the direction $\hat{k}$ of the emitted radiation, so we can do the integral over solid angle once and for all. We use the formulas

$$\int d\Omega \hat{k}_i \hat{k}_j = \frac{4\pi}{3}\delta_{ij}$$

$$\int d\Omega \hat{k}_i \hat{k}_j \hat{k}_l \hat{k}_m = \frac{4\pi}{15}(\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})$$

(The form of the right-hand sides is dictated by symmetry and rotational invariance; the numerical coefficients can be calculated by contracting i with j and l with m.) We then find

$$\int d\Omega \Lambda_{ij,lm}(\hat{k}) = \frac{2\pi}{15}[11\delta_{il}\delta_{jm} - 4\delta_{ij}\delta_{lm} + \delta_{im}\delta_{jl}]$$

so the power emitted at a single discrete frequency ω is

$$P = \frac{2G\omega^6}{5}[D_{ij}^*(\omega)D_{ij}(\omega) - \tfrac{1}{3}|D_{ii}(\omega)|^2] \tag{10.5.7}$$

whereas for a smooth distribution of frequencies the total emitted energy is

$$E = \frac{4\pi G}{5}\int_0^\infty \omega^6[D_{ij}^*(\omega)D_{ij}(\omega) - \tfrac{1}{3}|D_{ii}(\omega)|^2]\,d\omega \tag{10.5.8}$$

Before going on to calculate the quadrupole radiation emitted in a few special cases, it will be necessary to pause for a few comments on the method of calculation:

(A) The quadrupole approximation is usually applicable to nonrelativistic systems, and for these systems the energy density $T^{00}(\mathbf{x}, \omega)$ is dominated by the rest-mass density of the system. It may be surprising that we do not need to take explicit account of the potential and kinetic energy terms in the full tensor $T^{\mu\nu}$, because such terms must be included if $T^{\mu\nu}$ is to be conserved! Indeed, for a system of particles bound by gravitational forces we should in principle take $T^{\mu\nu}$ as the total "tensor" $\tau^{\mu\nu}$ constructed in Section 7.6, including terms nonlinear in the gravitational fields. However, we have already exploited energy and momentum conservation in our derivation of Eqs. (10.5.3)–(10.5.6), and since this has given a result involving only T^{00}, we are now free to approximate T^{00} with the rest-mass density.

(B) For general systems of vibrating and/or rotating solids, it is often quite difficult to evaluate the Fourier transform $T^{00}(\mathbf{x}, \omega)$, defined by Eq. (10.4.1) or (10.4.2). It is much easier first to evaluate the moments

$$D_{ij}(t) \equiv \int d^3x\, x^i x^j T^{00}(\mathbf{x}, t) \tag{10.5.9}$$

and then evaluate $D_{ij}(\omega)$ by expressing $D_{ij}(t)$ as a Fourier integral,

$$D_{ij}(t) = \int_0^\infty d\omega D_{ij}(\omega) e^{-i\omega t} + \text{c.c.} \tag{10.5.10}$$

or as a sum of Fourier components,

$$D_{ij}(t) = \sum_\omega e^{-i\omega t} D_{ij}(\omega) + \text{c.c.} \tag{10.5.11}$$

(C) The question may arise: What origin should be taken for the coordinates x^i in the integral (10.5.4) for D_{ij}? In principle, it doesn't matter. When we shift the origin of coordinates by an amount a_i, we change D_{ij} into

$$\begin{aligned}\int (x^i - a^i)(x_j - a_j) T^{00}(\mathbf{x}, t)\, d^3x \\ = \int x^i x^j T^{00}(\mathbf{x}, t)\, d^3x - a^i \int x^j T^{00}(\mathbf{x}, t)\, d^3x - a^j \int x^i T^{00}(\mathbf{x}, t)\, d^3x \\ + a^i a^j \int T^{00}(\mathbf{x}, t)\, d^3x\end{aligned}$$

But conservation of energy and momentum tells us that the last three terms are at most linear functions of time, because

$$\begin{aligned}\frac{\partial}{\partial t}\int T^{00}(x, t)\, d^3x &= -\int \frac{\partial}{\partial x^i} T^{i0}(x, t)\, d^3x = 0 \\ \frac{\partial^2}{\partial t^2}\int x^i T^{00}(x, t)\, d^3x &= \int x^i \frac{\partial^2}{\partial x^j\, \partial x^k} T^{jk}(x, t)\, d^3x \\ &= -\int \frac{\partial}{\partial x^j} T^{ij}(x, t)\, d^3x = 0\end{aligned}$$

Thus the shift in origin does not affect the Fourier components with $\omega \neq 0$, that is,

$$D_{ij}(\omega) \equiv \int x^i x^j T^{00}(\mathbf{x}, \omega)\, d^3x = \int (x^i - a^i)(x^j - a^j) T^{00}(\mathbf{x}, \omega)\, d^3x \tag{10.5.12}$$

However, it is only when we take T^{00} as the energy density of the entire system that we can shift origins freely in computing $D_{ij}(\omega)$.

As a first example, let us calculate the gravitational radiation produced by a sound wave in a tube lying in the z-direction. The density of the vibrating material can be written

$$\rho = \rho_0 + \rho_1$$

where ρ_0 is the constant unperturbed value and ρ_1 is a small perturbation. We also treat the material velocity v (in the z-direction) as a small perturbation, and neglect dissipative effects. The equations of motion are then

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial v}{\partial z} = 0$$

$$\rho_0 \frac{\partial v}{\partial t} + v_s^{\,2} \frac{\partial \rho_1}{\partial z} = 0$$

where v_s is the speed of sound. The tube is not supported at its ends (otherwise we would have to take into account the gravitational radiation emitted by the support!), so the pressure $v_s^{\,2}\rho_1$ must vanish at the tube ends. With this boundary condition, the general solution for a tube extending from $z = 0$ to $z = L$ is a superposition of the normal modes

$$v = -\varepsilon v_s \cos kz \sin(\omega t + \phi) \tag{10.5.13}$$

$$\rho_1 = \varepsilon \rho_0 \sin kz \cos(\omega t + \phi) \tag{10.5.14}$$

where ε is a small dimensionless number, ϕ is an arbitrary phase, and

$$k = N\frac{\pi}{L} \qquad \omega = N\pi \frac{v_s}{L} \tag{10.5.15}$$

with N any positive integer. Since v is not constrained to vanish at the tube ends, these ends will in general be displaced by amounts $\delta(0, t)$ and $\delta(L, t)$, respectively, where

$$\delta(z, t) \equiv \int v(z, t)\, dt = \varepsilon v_s \omega^{-1} \cos kz \cos(\omega t + \phi)$$

The time-dependent part of the second-moment of the mass density is given here by

$$D_{ij}(t) = n_i n_j A \left(\int_0^L \rho_1(z, t) z^2\, dz + L^2 \rho_0\, \delta(L, t) \right)$$

where A is the cross-sectional area of the tube, and $\mathbf{n} = (0, 0, 1)$ is a unit vector in the z-direction. This vanishes for N even, whereas for N odd we find

$$D_{ij}(t) = -\left(\frac{4 n_i n_j M L^2 \varepsilon}{N^3 \pi^3} \right) \cos(\omega t + \phi)$$

where $M \equiv \rho_0 A L$ is the mass of the tube. (The reader can easily check that $D_{ij}(t)$ would be the same if the second moment of the mass distribution were evaluated using as origin some point other than $z = 0$.) Comparing with Eq. (10.5.11), we see that $D_{ij}(t)$ has a Fourier component

$$D_{ij}\left(N\pi \frac{v_s}{L} \right) = -\frac{2 n_i n_j M L^2 \varepsilon}{N^3 \pi^3} \tag{10.5.16}$$

The radiated power is thus given by Eq. (10.5.7) (in c.g.s. units) as

$$P = \frac{16GM^2{v_s}^6\varepsilon^2}{15L^2c^5} \tag{10.5.17}$$

for each odd N. This may be compared with the total energy of the oscillation, which is simply the kinetic energy at the times when ρ_1 vanishes, that is, when v is greatest:

$$E = \tfrac{1}{2}\rho_0 A \int_0^L v^2{}_{\text{max}}(z)\, dz = \tfrac{1}{4}M{v_s}^2\varepsilon^2$$

Evidently the emission of gravitational radiation will cause the oscillator to lose energy at a rate

$$\Gamma_{\text{grav}} \equiv \frac{P}{E} = \frac{64GM{v_s}^4}{15L^2c^5} \tag{10.5.18}$$

For instance, let us calculate the rate of gravitational radiation by acoustic oscillations in the large aluminum cylinders used as antennas in Weber's experiments on gravitational radiation.[1] (As we shall see, the effective cross-section of such antennas is determined by Γ_{grav}.) Weber's cylinders have the parameters

$$L = 153 \text{ cm} \qquad v_s = 5.1 \times 10^5 \text{ cm/sec} \qquad M = 1.4 \times 10^6 \text{ gm}$$

Hence, if gravitational radiation were the only loss mechanism, the oscillations (10.5.13), (10.5.14) with N odd would lose energy at a rate

$$\Gamma_{\text{grav}} = 4.7 \times 10^{-35} \text{ sec}^{-1} \tag{10.5.19}$$

In contrast, the actual decay rate Γ of the $N = 1$ mode in this cylinder is about 0.15 sec^{-1}, owing primarily to viscous dissipation within the aluminum. Hence the "branching ratio" of gravitational radiation here is of order

$$\eta \equiv \frac{\Gamma_{\text{grav}}}{\Gamma} \simeq 3 \times 10^{-34} \qquad (N = 1) \tag{10.5.20}$$

Any ordinary mechanical oscillation will always give up vastly more of its energy to heat than to gravitational radiation.

As a second example, let us calculate the power radiated by a rotating body. If the body rotates rigidly about the 3-axis with angular frequency T, then the mass density T^{00} will take the form

$$T^{00}(\mathbf{x}, t) = \rho(\mathbf{x}')$$

where $\rho(\mathbf{x}')$ is the mass density expressed in coordinates $\mathbf{x}'$ fixed in the body, defined by

$$\begin{aligned} x_1 &\equiv x_1' \cos \Omega t - x_2' \sin \Omega t \\ x_2 &\equiv x_1' \sin \Omega t + x_2' \cos \Omega t \\ x_3 &\equiv x_3' \end{aligned}$$

Hence, by changing coordinates in (10.5.9), we may express $D_{ij}(t)$ in terms of the moment-of-inertia tensor in body-fixed coordinates

$$I_{ij} \equiv \int d^3x' x'_i x'_j \rho(\mathbf{x}') \tag{10.5.21}$$

For simplicity, let us consider rotation around one of the principal axes of the ellipsoid of inertia, so that $I_{13} = I_{23} = 0$. We can also choose the x'_1 and x'_2 axes along the two other principal axes, so that $I_{12} = 0$. With I_{ij} diagonal, we now find

$$\begin{aligned}
D_{11}(t) &= \tfrac{1}{2}(I_{11} + I_{22}) + \tfrac{1}{2}(I_{11} - I_{22}) \cos 2\Omega t \\
D_{12}(t) &= \tfrac{1}{2}(I_{11} - I_{22}) \sin 2\Omega t \\
D_{22}(t) &= \tfrac{1}{2}(I_{11} + I_{22}) - \tfrac{1}{2}(I_{11} - I_{22}) \cos 2\Omega t \\
D_{13}(t) &= D_{23}(t) = 0 \\
D_{33}(t) &= I_{33}
\end{aligned}$$

The nonvanishing Fourier coefficients for $\omega = 2\Omega$ in Eq. (10.5.11) are then

$$D_{11}(2\Omega) = -D_{22}(2\Omega) = iD_{12}(2\Omega) = \tfrac{1}{4}(I_{11} - I_{22})$$

According to Eq. (10.5.7), the total power emitted at twice the rotation frequency is then (in c.g.s. units)

$$P(2\Omega) = \frac{32G\Omega^6 I^2 e^2}{5c^5} \tag{10.5.22}$$

where I and e are the moment of inertia and equatorial ellipticity,

$$I \equiv I_{11} + I_{22} \tag{10.5.23}$$

$$e \equiv \frac{I_{11} - I_{22}}{I} \tag{10.5.24}$$

A body with circular symmetry around the axis of rotation will have $e = 0$, and therefore will not emit gravitational radiation. (Indeed, this conclusion does not even depend on the quadrupole approximation, since such a body, though rotating, has a time-independent energy-momentum tensor.) On the other hand, for a point mass m fixed in the *rotating* coordinate system at a point $x'_1 = r$, $x'_2 = x'_3 = 0$, the only nonvanishing element of I_{ij} will be $I_{11} = mr^2$, so that $I = mr^2$ and $e = 1$, and Eq. (10.5.23) gives a radiated power

$$P(2\Omega) = \frac{32G\Omega^6 m^2 r^4}{5c^5} \tag{10.5.25}$$

For instance, for the orbital motion of the planet Jupiter, we have

$$\Omega = 1.68 \times 10^{-8}\ \text{sec}^{-1}, \qquad m = 1.9 \times 10^{30}\ \text{g}, \qquad r = 7.78 \times 10^{13}\ \text{cm}$$

and Eq. (10.5.25) gives a gravitational radiation power of only 5.3 kW, less even than the solar thermal gravitation power calculated in the last section. At this rate, it would take very much longer than the age of the solar system to observe any effects of this energy loss on Jupiter's orbit.

The negligibility of gravitational radiation in celestial mechanics can be stated in more general terms. For a system consisting of particles with typical mass $\bar{M}$, typical separations $\bar{r}$, and typical velocities $\bar{v}$, the power radiated at a frequency ω of order $\bar{v}/\bar{r}$ will be of order [compare Eq. (10.5.7)]

$$P \sim G\left(\frac{\bar{v}}{\bar{r}}\right)^6 \bar{M}^2\bar{r}^4$$

or, since $G\bar{M}/\bar{r}$ is of order $\bar{v}^2$,

$$P \sim \bar{M}\,\frac{\bar{v}^8}{\bar{r}}$$

The typical deacceleration $\bar{a}_{\text{rad}}$ of particles in the system owing to this energy loss is given by the power P divided by the momentum $\bar{M}\bar{v}$, or

$$\bar{a}_{\text{rad}} \sim \frac{\bar{v}^7}{\bar{r}}$$

This may be compared with the accelerations computed in Newtonian mechanics, which are of order $\bar{v}^2/\bar{r}$, and with the post-Newtonian corrections discussed in the last chapter, which are of order $\bar{v}^4/\bar{r}$. [Radiation effects enter with an *odd* power of $\bar{v}$ because they represent an irreversible process, as shown by our use of an outgoing wave solution in Eq. (10.4.4).] Since radiation reaction is smaller than the post-Newtonian effects by a factor $\bar{v}^3 < 10^{-12}$, the neglect of radiation reaction in the last section was perfectly justified. Indeed, if we had strength we could even compute the post-post-Newtonian accelerations,[2] which are of order $\bar{v}^6/\bar{r}$, without encountering the effects of gravitational radiation!

The discovery of the pulsars has provided us with a more promising source of gravitational radiation. As discussed in Section 11.4, pulsars are probably neutron stars,[3] with masses of the order of one solar mass, radii of the order of 10 km, and hence moments of inertia I of the order of 10^{45} g cm^2. A newborn pulsar, formed in a supernova, may be rotating with Ω of order 10^4 sec^{-1}, so according to Eq. (10.5.22), it would be emitting gravitational radiation at a rate of order $10^{55}\, e^2$ ergs/sec. For comparison, the total rotational energy of the pulsar would be about 10^{53} ergs, so most of the pulsar's kinetic energy would be radiated away as gravitational waves[4] within a few years, provided that the equatorial ellipticity e is greater than about 10^{-4}. This is too large a static ellipticity to be maintained in the huge gravitational field of a neutron star, but it might be possible for dynamical effects to produce a mean ellipticity this large, particularly in the early period before the pulsar has settled down to its equilibrium configuration. Eventually the pulsar will slow down sufficiently so that other loss mechanisms. such as magnetic dipole radiation (for which $P \propto \Omega^4$) become more important than gravitational radiation.