Data Analysis: Noise

1. Random processes and signals
2. Statistics for random processes
3. Examples: Poisson process, shot noise
4. Linear systems
5. Correlation, Autocorrelation, Power spectrum
6. Fluctuation-Dissipation Theorem, thermal noise
7. Discrete sampling and sampling theorem
8. Optimal detection
9. Environmental noise

Random processes

In a measurement we have an instrument which is sensitive to the physical process we want to study.

A transducer is used to convert the physical quantity of interest into one that is easier to measure (voltage, position of a dial, ...)

The yielded value is determined by many factors, not all under control, so that the instrument reading varies with time, giving a signal $s(t)$

$s(t)$ is random: the physical quantity is unknown and there is noise (fluctuations).

$$s(t) = x(t) + n(t)$$

$x(t)$ could be deterministic (data transmission) but for us it is random.

Out of $s(t)$ one has to extract the most likely value of the physical quantity of interest.

Random variables and random processes

Random process: a family of functions $X(t, S)$, where S is a random variable and t is time

Once S is extracted: $S = s_i$, one has a realization of the random process : for all t $x(t)$ has a value defined by $S = s_i$ and t

$X(t_0)$ with a fixed t_0 is a random variable

Quantities used in probability theory can be applied:

$$\mathrm{E}[X(t_0)], \mathrm{var}X(t_0), \sigma[X(t_0)]$$

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If the signal is sampled at $(t_1, t_2, \ldots, t_n)$ the result is a random vector $X = (X_1, X_2, \ldots, X_n)$

On the other hand if the random variable S is drawn, $S = s_i$, $X(t, s_i)$ is a deterministic signal

$X(t)$ at fixed t behaves as a random variable

One will have a cumulative probability distribution function

$$F_X(x;t) = P\{X(t) < x\}$$

and its probability density

$$f_X(x;t) = \frac{\partial F_X(x;t)}{\partial x}$$

Random processes (II)

A random process is a function of at least two variables : t and S

A first order statistics is not enough for a statistical description

Usually there are correlations, described by second order statistics

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) < x_1 \cap X(t_2) < x_2\}$$

or equivalently

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}$$

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Statistics for random processes

One can compute the average over all realizations of $X(t, S)$, weighted according to the probability distribution of S: this is an ensemble average

Example: mean value at time t

$$\eta_X(t) = E[X(t)] = \int_{-\infty}^{+\infty} x f_X(x;t) \mathrm{d}x$$

is a deterministic function of time

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Autocorrelation between the two times (t_1, t_2)

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2)\mathrm{d}x_1\mathrm{d}x_2$$

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Autocovariance at (t_1, t_2)

$$\phi_X(t_1,t_2) = E[(X(t_1)-\eta_X(t_1))(X(t_2)-\eta_X(t_2))] = R_X(t_1,t_2) - \eta_x(t_1)\eta_X(t_2)$$

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Autocovariance at (t_1, t_2)

$$\phi_X(t_1, t_2) = E[(X(t_1) - \eta_X(t_1))(X(t_2) - \eta_X(t_2))] = R_X(t_1, t_2) - \eta_x(t_1)\eta_X(t_2)$$

Variance at a given time

$$\mathrm{var} X(t) = E[|X(t)|^2] = R_X(t, t)$$

These averages can be computed using first and second order statistics

Harmonic process (I)

Consider the random process

$$X(t) = a\cos(2\pi f_0 t + \Theta)$$

with Θ being a random variable uniformly distributed between 0 e 2π
Statistics are

$$\begin{aligned}
\eta_X(t) = E[X(t)] &= \int_0^{2\pi} a\cos(2\pi f_0 t + \Theta)\frac{\mathrm{d}\Theta}{2\pi} = 0 \\
R_X(t_1, t_2) = E[X(t_1)X(t_2)] &= \int_0^{2\pi} a\cos(2\pi f_0 t_1 + \Theta) a\cos(2\pi f_0 t_2 + \Theta)\frac{\mathrm{d}\Theta}{2\pi} \\
&= \frac{a^2}{2}\cos(2\pi f_0(t_1 - t_2)) \\
R_X(t, t) = \mathrm{var}(X(t)) &= \frac{a^2}{2}
\end{aligned}$$

Note that these quantities do not depend on time

Harmonic process (II)

First order statistics is given by

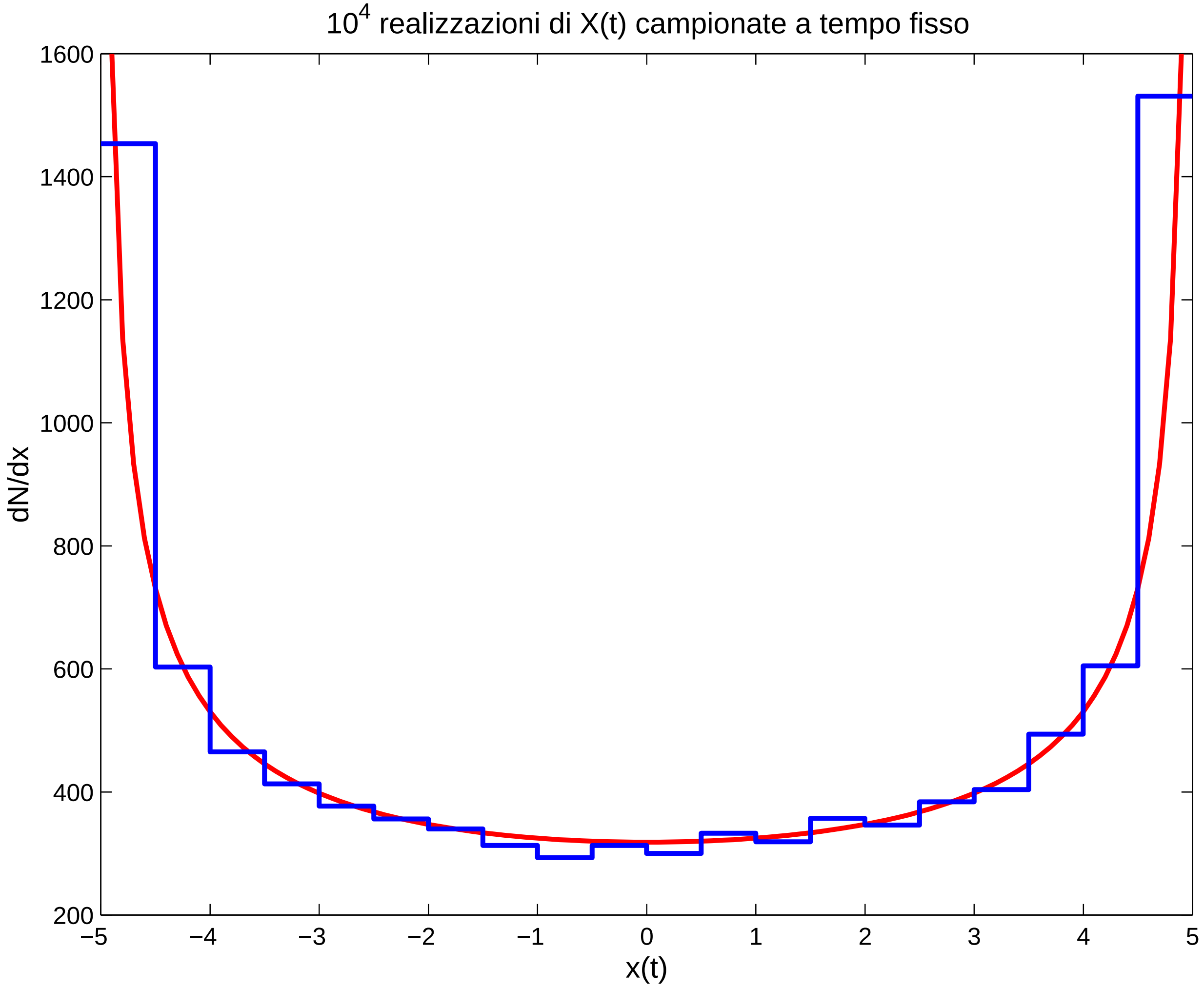
$$
\begin{aligned}
F_X(x;t) &= P\{X(t;\Theta) < x\} \\
&= P\left\{\left[\left(\arccos\left(\frac{x}{a}\right) - 2\pi f_0 t\right) \bmod 2\pi \ < \Theta\right]\right. \\
&\quad \left.\cap \left[\Theta < \left(2\pi - \arccos\left(\frac{x}{a}\right) - 2\pi f_0 t\right) \bmod 2\pi\right]\right\} \\
&= \int_{\arccos(x/a) - 2\pi f_0 t \bmod 2\pi}^{2\pi - \arccos(x/a) - 2\pi f_0 t \bmod 2\pi} \frac{\mathrm{d}\Theta}{2\pi} \\
&= \frac{\pi - \arccos(x/a)}{\pi}
\end{aligned}
$$

The probability density is obtained differentiating the cumulative probability function

$$
f(x;t) = \frac{1}{\pi a\sqrt{1-(x/a)^2}}
$$

The first order statistics does not depend on time

Distribution at fixed time



Stationarity

If all statistical properties of a random process are invariant upon time translation, the process is said to be **Strict Sense Stationary** (SSS)
In other words the realizations of the process $X(t)$ or of $X(t-t_0)$ have the same statistical properties and in particular they have equal probability of being observed.
If the ensemble average of a random process is independent of time and if autocorrelation depends on on $\tau = t_1 - t_2$, the process is said to be **Wide Sense Stationary** (WSS)
The harmonic process is WSS

$$\eta_X(t) = \eta_X = 0,\; R_X(t_1, t_2) = R_X(\tau) = \frac{a^2}{2}\cos(2\pi f_0 \tau))$$

Stationarity

For the first order statistics, using

$$f_X(x;t) = f_X(x;t-t_0)$$

one has that $E[X]$ is independent of time
For the second order statistics

$$f_X(x_1,x_2;t_1,t_2) = f_X(x_1,x_2;t_1-t_0,t_2-t_0) = f_X(x_1,x_2;t_1-t_2,t_2-t_2)$$

therefore the autocorrelation $R_X(t_1,t_2) = R_X(t_1-t_2) = R_X(\tau)$
An SSS process is also WSS but not inversely

Autocorrelation

Given a WSS process one has

$$R_X(-\tau) = E[X(t_1)X(t_1+\tau)] = E[X(t_2-\tau)X(t_2)] = R_X(\tau)$$

Autocorrelation is an even function of τ

$R_X(0) = E[X^2(t)] \geq 0$

It can be shown that $|R_X(\tau)| \leq R_X(0)$

From the definition: autocorrelation measures how much the value of $X(t+\tau)$ is correlated with $X(t)$ averaged over all times

If autocorrelation vanishes for $|\tau| > M$ then the time interval M can be considered as the memory duration of the process.

Pulse response

Let $h(t)$ function of time with $h(t) = 0$ per $t < 0$, for example

$$h(t) = C\exp(-\lambda t),\ t \geq 0$$

Let's consider the random process $X(t) = h(t-\tau)$ with τ random variable
This is the output signal of a system with pulse response $h(t)$ and input signal $\delta(t-\tau)$
If t is fixed, $h(t-\tau)$ is random

Poisson Process (I)

Random choice of points in time t_i, on average λ points per unit time. The probability to have one point in some given time interval Δt is independent of what has happened before.

The probability to have k points per unit time is

$$\frac{\exp(-\lambda t)(\lambda t)^k}{k!}$$

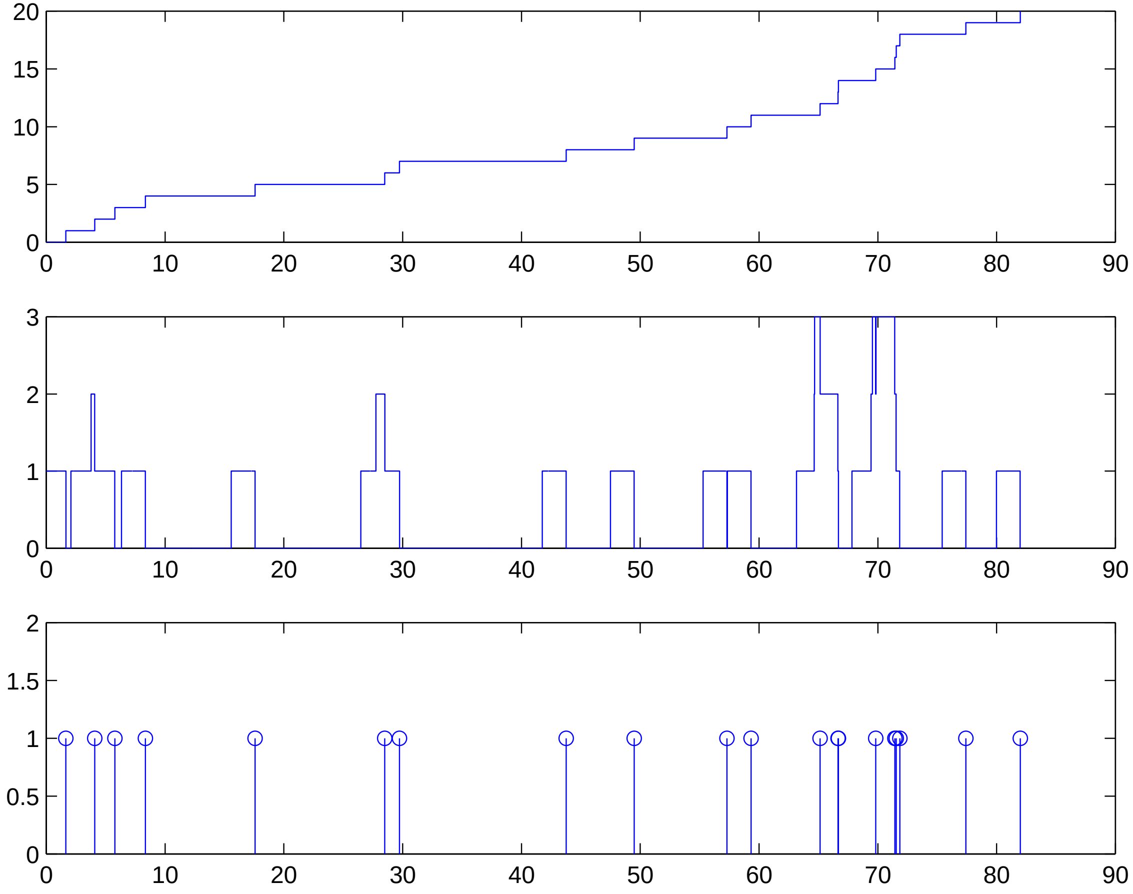
Let $X(t)$ such that $X(0) = 0$ e $X(t_2) - X(t_1) =$ number of points in (t_1, t_2).

$X(t)$ is given by the realization of t_i and is given by the number of points since 0 to t

$X(t)$, at a given t is a random variable with Poisson distribution with parameter λt.

Poisson Process (II)

Poisson Process, finite difference ratio $\frac{X(t+\epsilon)-X(t)}{\epsilon}$ and Poisson pulses



Poisson Process (III)

Statistics of $X(t)$.
Given t_a and t_b the random variable

$$X(t_b) - X(t_a), \quad t_b > t_a$$

has a Poisson distribuzione with parameter $\lambda(t_b - t_a)$:

$$P\{X(t_b) - X(t_a) = k\} = \exp\left(-\lambda(t_b - t_a)\right) \frac{[\lambda(t_b - t_a)]^k}{k!}$$

One deduces also that

$$\begin{aligned} E[X(t_b) - X(t_a)] &= \lambda(t_b - t_a) \\ E[(X(t_b) - X(t_a))^2] &= \lambda^2(t_b - t_a)^2 + \lambda(t_b - t_a) \end{aligned}$$

Poisson Process (IV)

Consider $t_d > t_c > t_b > t_a$, then since the intervals are non overlapping $X(t_d) - X(t_c)$ e $X(t_b) - X(t_a)$ are independent. The expectation value of the product is equal to the product of the expectation values.

$$E[(X(t_d) - X(t_c))(X(t_b) - X(t_a))] = \lambda^2 (t_d - t_c)(t_b - t_a)$$

If on the other hand the intervals are overlapping: $t_d > t_b > t_c > t_a$, one has

$$E[(X(t_d) - X(t_c))(X(t_b) - X(t_a))] = \lambda^2 (t_d - t_c)(t_b - t_a) + \lambda(t_b - t_c)$$

$t_b - t_c$ is the ovelapping time

From this one has

$$\begin{aligned} E[X(t)] &= \lambda t \\ R_X(t_1, t_2) = E[X(t_1)X(t_2)] &= \begin{cases} \lambda^2 t_2 t_1 + \lambda t_2 & t_1 \geq t_2 \\ \lambda^2 t_1 t_2 + \lambda t_1 & t_2 \geq t_1 \end{cases} \end{aligned}$$

Poisson Increments and Pulses (I)

Consider the process

$$Y(t) = \frac{X(t+\epsilon) - X(t)}{\epsilon}$$

One obtains

$$\begin{aligned} E[Y(t)] &= \lambda \\ R_Y(t_1, t_2) &= \begin{cases} \lambda^2 & |t_1 - t_2| > \epsilon \\ \lambda^2 + \frac{\lambda}{\epsilon} - \frac{\lambda |t_1 - t_2|}{\epsilon^2} & |t_1 - t_2| \leq \epsilon \end{cases} \end{aligned}$$

Poisson Increments and Pulses (II)

If one considers the Poisson pulses

$$Z(t) = \sum_i \delta(t - T_i)$$

one has

$$Z(t) = \frac{\mathrm{d}X(t)}{\mathrm{d}t} = \lim_{\epsilon \to 0} Y(t)$$

From the results for $Y(t)$ one deduces that

$$\begin{aligned} E[Z(t)] &= \lambda \\ R_Z(t_1, t_2) &= \lambda^2 + \lambda\delta(t_1 - t_2) \end{aligned}$$

Shot Noise (I)

Given the function $h(t)$ we consider the process

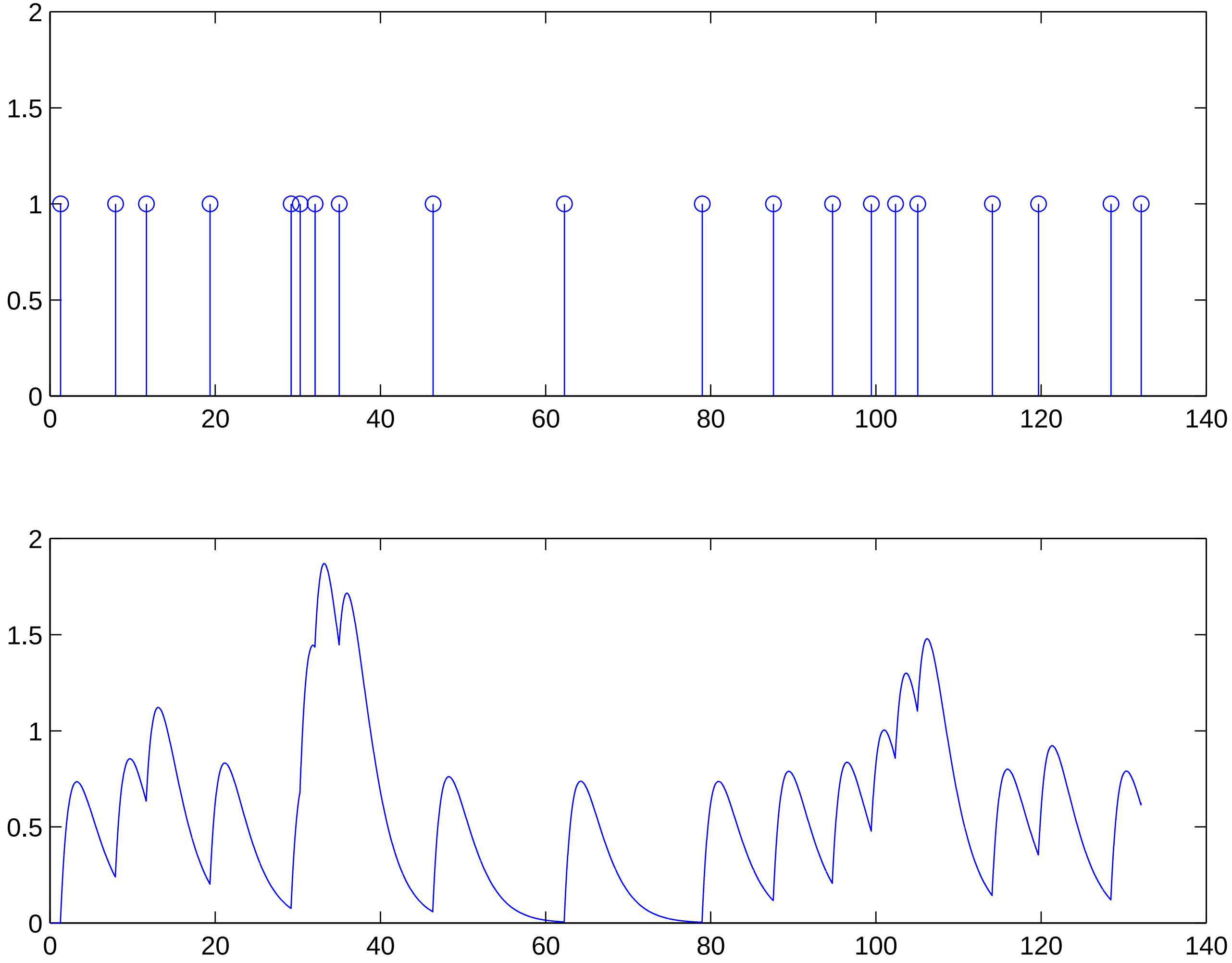
$$S(t) = \sum_i h(t - T_i)$$

The output signal of a linear system with input signal $Z(t)$ and pulse response $h(t)$

$$S(t) = Z(t) * h(t)$$

Example: monochromatic photons counting

Shot Noise (II)



Correlation, Orthogonality, independence

Two processes $X(t)$ e $Y(t)$ are said to be non correlated if

$$R_{XY}(t_1, t_2) = \eta_X(t_1)\eta_Y^*(t_2)$$

that is

$$C_{XY}(t_1, t_2) = 0$$

They are said to be orthogonals

$$R_{XY}(t_1, t_2) = 0$$

Finally we have independent processes if the random variables

$$X(t_1), \ldots, X(t_n)$$

are independent from

$$Y(t'_1), \ldots, Y(t'_m)$$

for any $t_1, \ldots, t_n, t'_1, \ldots, t'_m$

Normal Processes

One process is normal if the random variables

$$X(t_1), \ldots, X(t_n)$$

are mixed normal for any $n, t_1, \ldots, t_n$. Probability densities of any order must be normal. It can be shown that the statistics of a normal process is completely determined if one knows

$$E[X(t))] = \eta_X(t)$$

e

$$R_X(t_1, t_2)$$

The first order probability density is given by

$$f(x;t) = \frac{1}{\sqrt{2\pi C(t,t)}} \exp\left[-[x - \eta(t)]^2 / 2C(t,t)\right]$$

Fundamental observation: a WSS normal process is also SSS because its statistics in completely determined by expectation value and covariance matrix, which are first and second order statistics.

Power of a random process

Instantaneous power dissipated in an electric circuit or in general in a linear dynamic system is proportional to the square modulus of some coordinate

For a random process one uses $|X(t)|^2$

The average power for a single *single realization* is given by

$$P_X(s_i) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |X(t; s_i)|^2 \mathrm{d}t$$

which is a random variable

Averaging on the ensemble one obtains

$$P_X = E[P_X(s_i)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} E[|X(t; s_i)|^2] \mathrm{d}t$$

For a WSS process the integrand is constant

$$P_X = R_X(0)$$

The process power is equal to the autocorrelation with null delay

White noise

White noise is the result of a WSS process where $X(t)$ in uncorrelated with $X(t')$, $t \neq t'$ ($X(t)$ is assumed to be with zero mean)

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = 0$$

for $t_1 \neq t_2$

On the other side $R_X(0) = \mathrm{var}\, X$

This is an ideal case, there is always some memory in a physical system

Often it is required to simulate white noise with a computer using discrete sampling.

There is a normalization issue, because in discrete sampling there is some memory time, at least from one sample to the other.

Transforms of random processes (I)

When a signal is fed into some system we have a transform of that signal

If we assign to each realization of process $X(t)$ a function $Y(t)$ we have defined a transform

$$Y(t) = T[X(t)]$$

where $T[\cdot]$ is an operator that describes how a system works, using in principle all values of $X(t)$

$T[\cdot]$ shall be deterministic

Example: time invariant systems without memory

$$Y(t) = g[X(t)]$$

$g[\cdot]$ is a function that depends only on the value of $X(t)$. The system described by $g[\cdot]$ is time invariant and has no memory

Statistical properties of transformed signals are deduced as in the case of functions of random variables

It can be shown that if $X(t)$ is SSS (or stationary up to order k)then $Y(t)$ is also stationary to the same order

If $X(t)$ is only WSS, $Y(t)$ is not necessarily stationary.

Trasforms of random processes (II)

One uses frequently *detection* transforms

$$
\begin{aligned}
Y(t) &= X^2(t) \text{ Quadratic detector} \\
Z(t) &= |X(t)| \text{ Linear detection, full wave} \\
W(t) &= \frac{1}{2}(X(t) + |X(t)|) \text{ Linear detector, half wave}
\end{aligned}
$$

Linear transforms (I)

Let's consider

$$Y(t) = L[X(t)]$$

with $L[\cdot]$ linear: a linear combination of input signals gives the corresponding linear combination of output signals
If

$$Y(t - t_0) = L[X(t - t_0)]$$

whatever t_0, the system corresponding to $L[\cdot]$ is time invariant.
One has this fundamental theorem

$$E[Y(t)] = L[E[X(t)]]$$

The expected value for output is obtained transforming the expected value of the input random process

Linear Trasforms (II)

Autocorrelation of transformed signals is obtained as follows

$$R_{XY}(t_1, t_2) = L_{t_2}[R_{XX}(t_1, t_2)]$$
$$R_{YY}(t_1, t_2) = L_{t_1}[R_{XY}(t_1, t_2)]$$

The lower index t_i for the linear transform indicates that the argument is to be taken as function of t_i only.

Power spectrum (I)

The power spectrum $S_{XX}(\omega)$ of a random process $X(t)$ is defined to be the Fourier transform of the process autocorrelation

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} \exp\left[-j\omega\tau\right] R_{XX}(\tau)\mathrm{d}\tau$$

The units of $S_{XX}(\omega)$ are $[X^2]t = \frac{[X^2]}{Hz}$
Often one uses the linear power spectrum (LPS)

$$\tilde{x}(\omega) = \sqrt{S_{XX}(\omega)}$$

Units of $\tilde{x}(\omega)$ are $\frac{[X]}{\sqrt{Hz}}$
$\tilde{x}(\omega)$ is NOT the Fourier transform of the signal, as one can see from the measurement units
The inverse Fourier transform gives the autocorrelation out of the power spectrum

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left[j\omega\tau\right] S_{XX}(\omega)\mathrm{d}\omega$$

If $X(t)$ is real, $R_{XX}(\tau)$ is even and so is $S_{XX}(\omega)$:

$$S_{XX}(-\omega) = S_{XX}(\omega)$$

Power Spectrum (II)

One can define the crossed power spectrum $S_{XY}(\omega)$ using cross correlation

$$S_{XY}(\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau) \exp\left[-j\omega\tau\right] \mathrm{d}\omega = S^{*}_{YX}(\omega)$$

This expression can be inverted:

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left[j\omega\tau\right] S_{XY}(\omega) \mathrm{d}\omega$$

For $\tau = 0$ one has

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XY}(\omega) \mathrm{d}\omega = R_{XY}(0) = E[X(t)Y^{*}(t)]$$

If $X(t)$ is the voltage at the terminals of a device and $Y(t)$ is the current circulating through the terminals, $R_{XY}(0)$ is the expected value of the dissipated power

Power spectrum (III)

If processes are orthogonal

$$R_{XY}(\tau) = 0 \quad S_{XY}(\omega) = 0$$

and power spectra can be summed to give the total power spectrum

$$R_{X+Y}(\tau) = R_X(\tau) + R_Y(\tau) \quad S_{X+Y}(\omega) = S_X(\omega) + S_Y(\omega)$$

Power spectrum (IV)

Process transforms and power spectra

$X(t)$	$R_X(\tau)$	$S_X(\omega)$
$aX(t)$	$\|a\|^2 R_X(\tau)$	$\|a\|^2 S_X(\omega)$
$\frac{\mathrm{d}X(t)}{\mathrm{d}t}$	$-\frac{\mathrm{d}^2 R_X(\tau)}{\mathrm{d}\tau^2}$	$\omega^2 S_X(\omega)$
$\frac{\mathrm{d}^n X(t)}{\mathrm{d}t^n}$	$(-1)^n \frac{\mathrm{d}^{2n} R_X(\tau)}{\mathrm{d}\tau^{2n}}$	$\omega^{2n} S_X(\omega)$
$X(t)\exp[\pm j\omega_0 t]$	$R_X(\tau)\exp[\pm j\omega_0 \tau]$	$S_X(\omega \mp \omega_0)$

Some power spectra

$R_X(\tau)$	$S_X(\omega)$
$\exp\left(-\alpha\lvert\tau\rvert\right)$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$1-\lvert\tau/T\rvert \quad -T<\tau<T$	$\frac{4\sin^2 2\omega T/2}{T\omega^2}$
$\exp\left[-\alpha\lvert\tau\rvert\right]\cos\omega_0\tau$	$\frac{\alpha}{\alpha^2+(\omega-\omega_0)^2}+\frac{\alpha}{\alpha^2+(\omega+\omega_0)^2}$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$\cos\omega_0 t$	$\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$

Spectrum as time average

In many engineering books the power spectrum is defined as the limit

$$S(\omega) = \lim_{T\to\infty} \frac{1}{2T} \left| \int_{-\infty}^{+\infty} X(t) \exp\left[-j\omega t\right] \mathrm{d}t \right|^2$$

of the random power averaged over the interval $(-T, T)$

$$S_T(\omega) = \frac{1}{2T} \left| \int_{-T}^{+T} X(t) \exp\left[-j\omega t\right] \mathrm{d}t \right|^2$$

This last expression is a random variable.

The condition of validity for expressing a power spectrum as a time average is given by the following theorem.

If

$$\int_{-\infty}^{+\infty} |\tau R(\tau)| \mathrm{d}\tau < \infty$$

then

$$\lim_{T\to\infty} E[S_T(\omega)] = S(\omega) = \int_{-\infty}^{+\infty} R(\tau) \exp\left[-j\omega\tau\right] \mathrm{d}\tau$$

The harmonic process doesn't fulfill this condition

Power spectrum and linear systems (I)

Consider a linear system with pulse response $h(t)$ and let $H(\omega)$ be its Fourier transform

$$H(\omega) = \int_{-\infty}^{+\infty} h(t) \exp\left[-j\omega t\right] \mathrm{d}t$$

is also called *transfer function*.

If we apply to the input a signal described by process $X(t)$, the output signal $Y(t)$ is given by

$$Y(t) = \int_{-\infty}^{+\infty} X(t-\alpha) h(\alpha) \mathrm{d}\alpha$$

The expected value is

$$E[Y(t)] = \int_{-\infty}^{+\infty} E[x(t-\alpha)] h(\alpha) \mathrm{d}\alpha$$

If $X(t)$ is stationary

$$\eta_Y = \eta_X \int_{-\infty}^{+\infty} h(\alpha) \mathrm{d}\alpha = H(0) \eta_X$$

Power spectrum and linear systems (II)

Autocorrelation of $Y(t)$ is obtained using the cross correlation $R_{XY}(\tau)$

$$Y(t)X^*(t-\tau) = \int_{-\infty}^{+\infty} X(t-\alpha)X^*(t-\tau)h(\alpha)\mathrm{d}\alpha$$

taking into account that

$$E[X(t-\alpha)X^*(t-\tau) = R_{XX}((t-\alpha)(t-\tau)) = R_{XX}(\tau-\alpha)$$

one obtains

$$E[Y(t)X^*(t-\tau)] = \int_{-\infty}^{+\infty} R_{XX}(\tau-\alpha)h(\alpha)\mathrm{d}\alpha$$

that is

$$R_{YX}(\tau) = R_{XX}(\tau) \otimes h(\tau)$$

Power spectrum and linear systems (III)

On the other hand one can write

$$Y(t+\tau)Y^*(t) = \int_{-\infty}^{+\infty} Y(t+\tau)X^*(t-\alpha)h^*(\alpha)\mathrm{d}\alpha$$

therefore

$$R_{YY}(\tau) = \int_{-\infty}^{+\infty} R_{YX}(\tau+\alpha)h^*(\alpha)\mathrm{d}\alpha = R_{YX}(\tau) \otimes h^*(-\tau)$$

From this

$$R_{XY}(\tau) = R_{XX}(\tau) \otimes h^*(-\tau) \quad R_{YY}(\tau) = R_{XY}(\tau) \otimes h(\tau)$$

and finally

$$R_{YY}(\tau) = R_{XX}(\tau) \otimes h^*(-\tau) \otimes h(\tau)$$

Power spectrum and linear systems (IV)

Application: if $X(t)$ is a white noise ($R_{XX}(\tau) = \delta(\tau)$) and if the system is causal and real ($h(t) = 0$ per $t < 0$) then

$$R_{XY}(\tau) = h(-\tau) = 0 \quad \tau > 0$$

that is $Y(t)$ is orthogonal to $X(t)$.
One can measure a transfer function using a white noise input $R_{XX}(\tau) = \delta(\tau)$ and for $\tau < 0$ $h(-\tau) = R_{XY}(\tau)$:

$$h(-\tau) = R_{XY}(\tau) \simeq \frac{1}{T} \int_0^T x(t+\tau) y(t) \mathrm{d}t$$

Power spectrum

Switching to frequency domain

$$S_{XY}(\omega) = S_{XX}(\omega)H^*(\omega) \quad S_{YY}(\omega) = S_{XY}(\omega)H(\omega)$$

The fundamental equality is obtained

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

The power spectrum of a real or complex process $X(t)$ is

$$S_{XX}(\omega) \geq 0$$

Power spectrum and linear systems

If $S_{XX}(\omega)$ were negative in an interval, $\omega_1 < \omega_0 < \omega_2$, one can choose a transfer function

$$H(\omega) = \begin{cases} 1 & \omega_1 < \omega < \omega_2 \\ 0 & \text{elsewhere} \end{cases}$$

then

$$S_{YY}(\omega) = \begin{cases} S_{XX}(\omega) & \omega_1 < \omega < \omega_2 \\ 0 & \text{elsewhere} \end{cases}$$

One would have $S_{YY}(\omega) < 0$ for eacn ω. But the average process power is always positive

$$E[|Y(t)|^2] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{YY}(\omega) \mathrm{d}\omega$$

which is in contradiction with the hypothesis.

Power spectrum properties

From the previous demonstration one obtains

1. The area under $S_{XX}(\omega)/2\pi$ is equal to the power of $X(t)$.
   Power is localized on the x-axis, as can be seen using a bandpass filter with unity gain

$$E[|Y(t)|^2] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{YY}(\omega)\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\omega_1}^{+\omega_2} S_{XX}(\omega)\mathrm{d}\omega$$

2. If $X(t)$ is real

$$|R_{YY}(\tau)| \leq R(0)$$

   indeed

$$|R_{YY}(\tau)| \leq \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{YY}(\omega)\mathrm{d}\omega = R_{YY}(0)$$

Example: a derivation box. Let $Y(t) = X'(t)$. The transfer function is $H(\omega) = j\omega$. So one has

$$S_{XX'} = S_{XX}(\omega)(-j\omega), \quad S_{X'X'} = \omega^2 S_{XX}(\omega)$$

therefore

$$R_{XX'}(\tau) = -\frac{\mathrm{d}R_{XX}(\tau)}{\mathrm{d}\tau}, \quad R_{X'X'}(\tau) = -\frac{\mathrm{d}^2 R_{XX}(\tau)}{\mathrm{d}\tau^2}$$

Shot nose spectrum (I)

Consider the random process $S(t) = \sum_i h(t - T_i)$, with T_i randomly distributed in time with density λ and $h(t)$ a real valued function. If we consider the succession of Poisson pulses

$$Z(t) = \sum_i \delta(t - t_i)$$

$S(t)$ is the output of a linear system with response $h(t)$.
We have for $Z(t)$

$$E[Z(t)] = \lambda, \quad R_{ZZ}(\tau) = \lambda^2 + \lambda\delta(\tau)$$

with power spectrum of the Poisson process

$$S_{ZZ}(\omega) = 2\pi\lambda^2\delta(\omega) + \lambda$$

Shot noise spectrum (II)

Using the theorem on the expected value of the output of a linear system

$$E[S(t)] = \int_{-\infty}^{+\infty} h(t)E[Z(t)]\mathrm{d}t = E[Z(t)]\int_{-\infty}^{+\infty} h(t)\mathrm{d}t = \lambda H(0)$$

if $H(\omega)$ is the system transfer function.

The power spectrum is obtained using the fundamental relation and applying the properties of $\delta(\omega)$

$$S_{SS}(\omega) = 2\pi\lambda^2 H^2(0)\delta(\omega) + \lambda|H(\omega)|^2$$

Conversely the autocorrelation $S(t)$ is given by

$$R_{SS}(\tau) = \lambda^2 H^2(0) + \lambda\int_{-\infty}^{+\infty} h(\tau+\beta)h(\beta)\mathrm{d}\beta$$

while the autocovariance is

$$C_{SS}(\tau) = \lambda\int_{-\infty}^{+\infty} h(\tau+\beta)h(\beta)\mathrm{d}\beta$$

Finally for this process one has

$$E[S(t)] = \lambda\int_{-\infty}^{+\infty} h(t)\mathrm{d}t, \;\; \sigma^2 = \lambda\int_{-\infty}^{+\infty} h^2(t)\mathrm{d}t$$

Poisson process and shot noise

Shot noise: consider the random process

$$S(t) = \sum_i h(t - T_i)$$

where $h(t)$ is the pulse response of the system.
If λ is constant one has

$$E[S(t)] = \lambda \int_{-\infty}^{\infty} h(t)\mathrm{d}t \quad E[S^2(t)] = \lambda^2 \left[\int_{-\infty}^{\infty} h(t)\mathrm{d}t\right]^2 + \lambda \int_{-\infty}^{\infty} h^2(t)\mathrm{d}t$$

The autocorrelation is given by

$$R(\tau) = \lambda^2 \left[\int_{-\infty}^{\infty} h(t)\mathrm{d}t\right]^2 + \lambda \int_{-\infty}^{\infty} h(\alpha)h(\tau + \alpha)\mathrm{d}\alpha$$

Poisson process and shot noise (II)

The power spectrum is instead

$$S(\omega) = 2\pi\lambda^2 H(0)\delta(\omega) + \lambda |H(\omega)|^2$$

If the area of each puls is q, shot noise becomes:

$$S(t) = q \sum_i h(t - T_i)$$

while mean and variance are respectively

$$E[S(t)] = \lambda q \int_{-\infty}^{\infty} h(t)\mathrm{d}t \quad E[S^2(t)] = \lambda q^2 \int_{-\infty}^{\infty} h^2(t)\mathrm{d}t$$

Poisson process and shot noise (III)

Consider a current generator that sends pulses

$$i(t) = \sum_i q\delta(t - T_i)$$

to a capacitor in parallel with a resistor.

$$V(t) = \sum_i qh(t - T_i)$$

with

$$h(t) = \frac{1}{C}\exp\left[-t/RC\right]U(t)$$

We have

$$\int_{-\infty}^{\infty} h(t)\mathrm{d}t = R \quad \int_{-\infty}^{\infty} h^2(t)\mathrm{d}t = \frac{R}{2C}$$

For comparison

$$\lambda = \frac{E^2[V(t)]}{2RC\sigma_V^2} \quad q = \frac{2C\sigma_V^2}{E[V(t)]}$$

It is possible to deduce microscopic propertis of the shot noise from integrated measurements.

Poisson process and shot noise (IV)

A similar problem comes from microcreep in stressed metals, like the Virgo mirror suspension wires. Suppose to have a cantilever spring that suspends a mass and that inside the spring there is microcreep. The mechanical response to a microcreep event is

$$x(t) = x_0 \exp\left[-t/\tau\right] \cos \omega_0 t$$

with

$$\omega_0^2 = \frac{k}{m} \quad \tau = \frac{1}{\omega_0 \phi}$$

Poisson process and shot noise (V)

Suppose that microcreep events are Poisson distributed (not at all obvious). The interesting quantities are

$$\begin{aligned} \int_{-\infty}^{\infty} h(t)\mathrm{d}t &= \frac{x_0\tau}{1+{\omega_0}^2\tau^2} \\ \int_{-\infty}^{\infty} h^2(t)\mathrm{d}t &= 1/4\,\frac{\tau\, x_0^2\left(2+{\omega_0}^2\tau^2\right)}{1+{\omega_0}^2\tau^2} \end{aligned}$$

and one obtains

$$\lambda = \frac{E^2[S(t)]\int_{-\infty}^{\infty} h^2(t)\mathrm{d}t}{\sigma^2_{S(t)}\left[\int_{-\infty}^{\infty} h(t)\mathrm{d}t\right]^2} \quad q = \frac{\sigma^2_{X(t)}\int_{-\infty}^{\infty} h(t)\mathrm{d}t}{E^2[S(t)]\int_{-\infty}^{\infty} h^2(t)\mathrm{d}t}$$

Thermal noise

The mirror is part of a pendulum in thermal equilibrium. The average energy is $k_B T = 3.9 \times 10^{-21}\,\mathrm{J}$. For a 40 kg mirror of a 1 m pendulum, the elastic constant is $mg/l = 400\,\mathrm{N/\,m}$ and the corresponding oscillation amplitude x_T is

$$x_T = \sqrt{\frac{k_B T}{k}} = \sqrt{\frac{kTl}{mg}} = 3.1 \times 10^{-12}\,\mathrm{m}$$

This is the rms value, integrated over all frequencies.

The Fluctuation-Dissipation Theorem

Links the dynamical response of the system to the thermal noise frequency spectrum.

$$S_{\dot{x}_T}(\omega) = 2k_B T \Re Y(\omega), \quad Y = \frac{1}{Z(\omega)} = \frac{\dot{x}}{F}$$

The equation of motion for a pendulum in vacuum is

$$\ddot{x} + \omega_0^2[1 + j\phi(\omega)]x = F/m$$

with $\phi(\omega) \sim 10^{-6}$. Solving

$$x(\omega) = \frac{F}{m} \frac{\omega_0^2 - \omega^2 - j\phi(\omega)\omega_0^2}{(\omega_0^2 - \omega^2)^2 + \phi^2(\omega)\omega_0^4}$$

The real part of the mechanical admittance Y, $F(\omega)Y(\omega) = x(\omega)$, is

$$\Re Y = \frac{\omega}{m} \frac{\phi(\omega)\omega_0^2}{(\omega_0^2 - \omega^2)^2 + \phi^2(\omega)\omega_0^4}$$

The linear power density for the pendulum thermal noise is

$$\tilde{x}_T(\omega) = \frac{\sqrt{S_{\dot{x}_T}(\omega)}}{\omega} = \sqrt{\frac{4k_B T\phi(\omega)\omega_0^2}{m\omega[(\omega_0^2 - \omega^2)^2 + \phi^2(\omega)\omega_0^4]}}$$

computing spectrum over positive frequencies only.

Thermal noise in circuits (I)

The Fluctuation Dissipation theorem states that the voltage power spectrum is proportional to the real part (dissipative part) of the circuit impedance.
A resistor can then be represented as an ideal resistor in series with a voltage source $N_e(t)$ such that

$$E[N_e(t)] = 0, \quad S_{N_e}(\omega) = 2k_B T R$$

where T is the absolute temperature and k_B is the Boltzmann constant.
$N_e(t)$ is a normal process.
In terms of current generator $N_e(t)$ corresponds to $N_i(t) = N_e(t)/R$. Noise can be represented with

$$S_{N_i}(\omega) = \frac{S_{N_e}(\omega)}{R^2} = 2k_B T G$$

where G is the circuit admittance.
Noise contributions coming from different dissipative circuit elements are independent one from the other.

Thermal noise in circuits (II)

Power spectrum at the ends of a capacitor C in parallel with a resistor R.
The circuit is made of a voltage generator in series with R and C. The system transfer function is

$$H(\omega) = \frac{1}{j\omega C}\left(\frac{1}{j\omega C} + R\right)^{-1} = \frac{1}{1 + j\omega RC}$$

The spectrum is

$$S_V(\omega) = S_{N_e}(\omega)|H(\omega)|^2 = \frac{2k_B T R}{1 + \omega^2 R^2 C^2}$$

The autocorrelation is

$$R_V(\tau) = \frac{k_B T}{C}\exp\left[-|\tau|/RC\right]$$

Thermal noise in circuits (III)

Observation: if we consider the impedance seen from the output port

$$Z(\omega) = \frac{R}{1 + j\omega RC}$$

we obtain

$$S_V(\omega) = 2k_B T \mathrm{Re} Z(\omega)$$

On the other hadn

$$E[V^2(t)] = R_V(0) = \frac{k_B T}{C}$$

and C is defined by

$$\frac{1}{C} = \lim_{\omega \to \infty} j\omega Z(\omega)$$

Mean noise power

The mean noise power can computed using the Wiener-Khinchine theorem

$$E[V^2(t)] = R_V(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_V(\omega)\mathrm{d}\omega$$

Consider a aystem with a capacitance seen from the output terminals defined by

$$\frac{1}{C} = \lim_{p\to\infty} pZ(p)$$

Using a theorem on Laplace transforms applied to the pulse response of the system

$$z(0^+) = \lim_{p\to\infty} pZ(p)$$

The limiting value is $1/C$ so

$$R_V(0) = k_B T z(0^+) = \frac{k_B T}{C}$$

This does not depend on R, as expected from equipartition of energy.

Sampling of signals from deterministic systems

Consider a signal $s(t)$ and its value s_i sampled at $t_i = i\Delta T$. Its Fourier transform is given by:

$$
\begin{aligned}
\hat{s}(\omega) &= \int_{-\infty}^{+\infty} s(t) \exp\left[j\omega t\right] \mathrm{d}t \\
&= \sum_{i=-\infty}^{+\infty} \int_{t_i-\Delta T/2}^{t_i+\Delta T/2} s(t) \exp\left[j\omega t\right] \mathrm{d}t \\
&\approx \sum_{i=-\infty}^{+\infty} \left(\int_{t_i-\Delta T/2}^{t_i+\Delta T/2} s(t) \mathrm{d}t \right) \exp\left[j\omega t_i\right] \\
&\approx \Delta T \sum_{i=-\infty}^{+\infty} s_i \exp\left[j\omega t_i\right]
\end{aligned}
$$

This leads to define the Fourier series

$$
\hat{s}(\omega) = \Delta T \sum_{i=-\infty}^{+\infty} s_i \exp\left[j\omega t_i\right] \quad \omega \text{ real}
$$

Sampled signals (II)

Notice in particular the inversion formula

$$s_i = \frac{1}{2\pi} \int_{-\pi/\Delta T}^{\pi/\Delta T} \hat{s}(\omega) \exp\left[-ji\Delta T\omega\right] \mathrm{d}\omega.$$

Now we relate the Fourier transform of a signal and the Fourier series build out of its samples.

$$\begin{aligned}
s_i = s(i\Delta T) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{s}(\omega) \exp\left[-ji\Delta T\omega\right] \mathrm{d}\omega \\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} \int_{(2k-1)\pi/\Delta T}^{(2k+1)\pi/\Delta T} \hat{s}(\omega) \exp\left[-ji\Delta T\omega\right] \mathrm{d}\omega \\
&= \frac{1}{2\pi} \int_{-\pi/\Delta T}^{\pi/\Delta T} \sum_{k=-\infty}^{+\infty} \hat{s}(\omega - \frac{2k\pi}{\Delta T}) \exp\left[-ji\Delta T\omega\right] \mathrm{d}\omega
\end{aligned}$$

Sampled signals (III)

Recalling the inversion formula and calling $\hat{s}^{(d)}(\omega)$ the Fourier series of the sampled signal

$$s_i = \frac{1}{2\pi}\int_{-\pi/\Delta T}^{\pi/\Delta T} \hat{s}^{(d)}(\omega) \exp\left[-ji\Delta T\omega\right]\mathrm{d}\omega$$

one sees that

$$\hat{s}^{(d)}(\omega) = \sum_{k=-\infty}^{+\infty} \hat{s}(\omega - \frac{2k\pi}{\Delta T}).$$

The contribution at the frequency ω of the transform of the sampled signal comes from all frequencies $\omega - 2k\pi/\Delta T$ per $k = \pm 1, \pm 2, \cdots$. These frequencies are an *alias* of ω and this phenomenon is called *aliasing*.

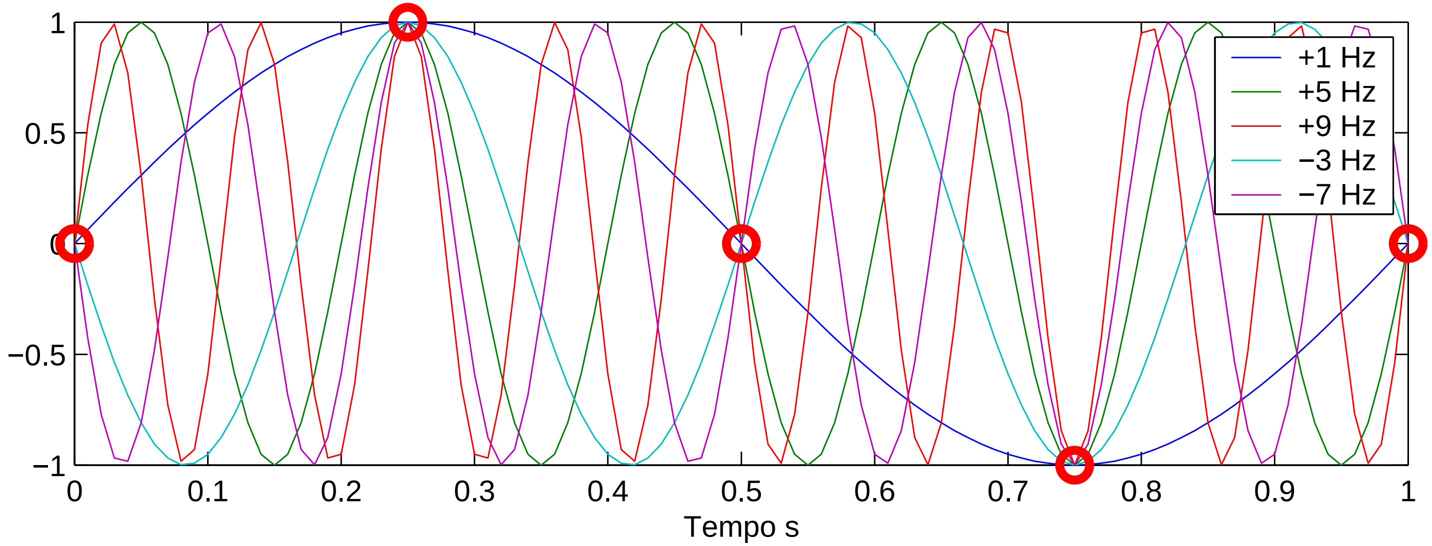
The frequency $\omega_N = \pm\pi/\Delta T$ is called Nyquist frequency and is equal to half the sampling frequency. It defines the frequency band and how this enters the Fourier transform of the sampled signal.

Aliasing (I)

Consider a family of sines

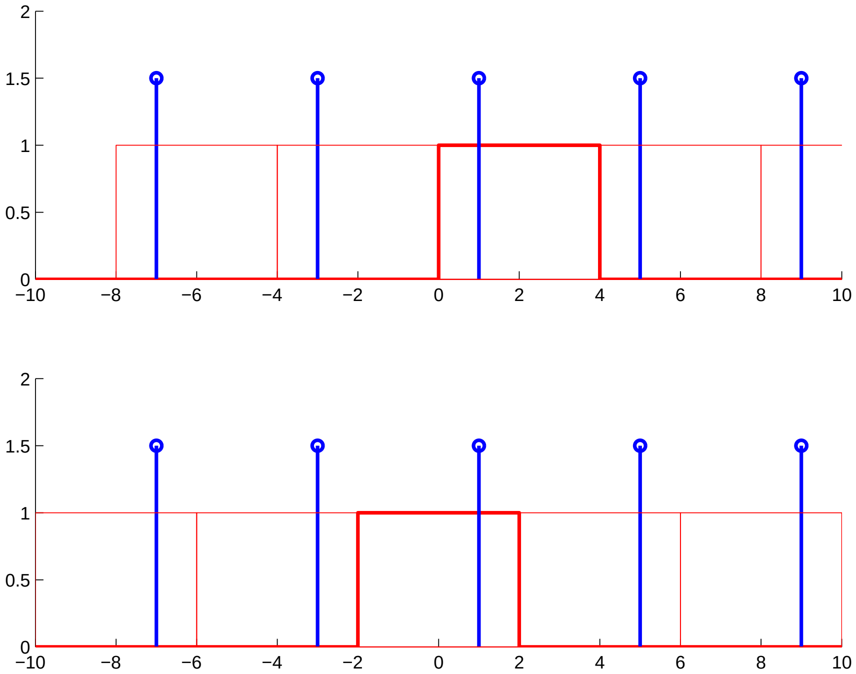
$$s(t) = \sin(k2\pi t)$$

con $k = -7, -3, +1, +5, +9\,\mathrm{Hz}$



Aliasing (II)

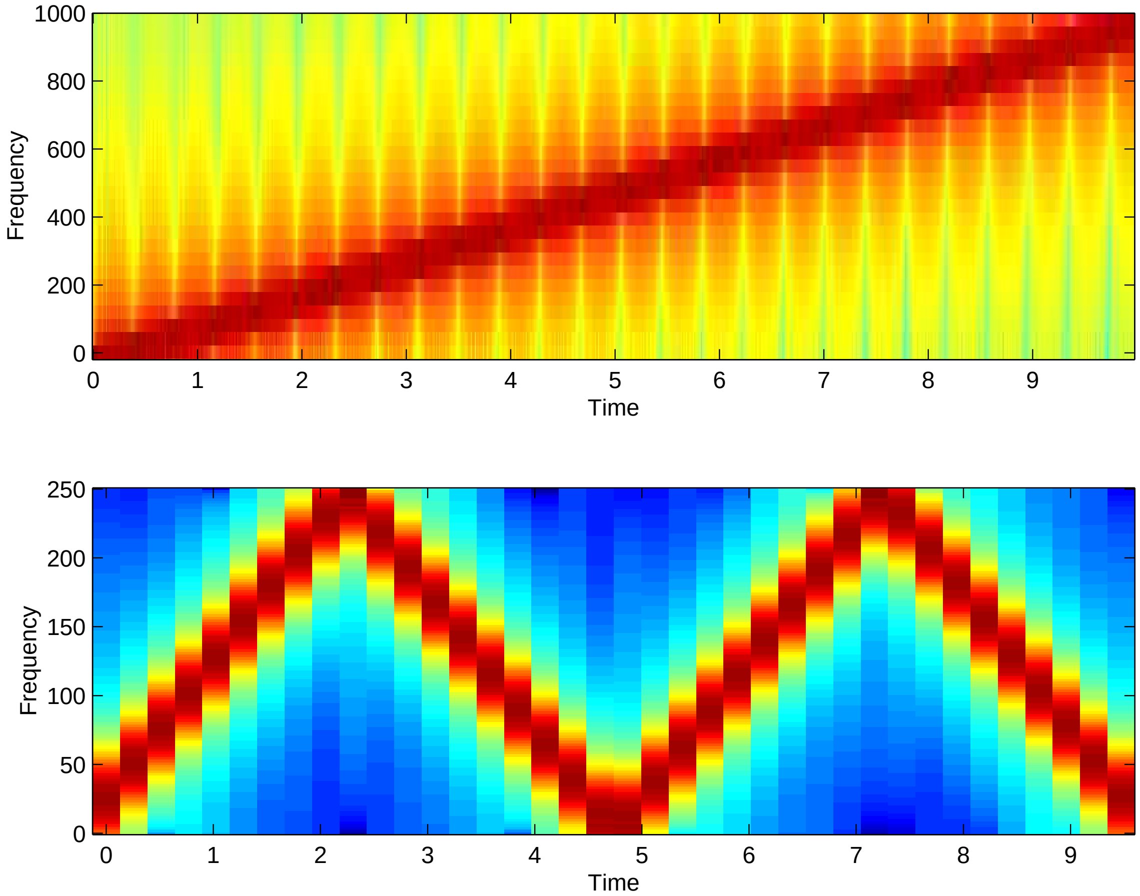
We see that for $t = k/4\,\mathrm{s}$, $k = 0, \pm 1, \pm 2, \cdots\ \mathrm{Hz}$, values are equal. With 4 Hz sampling the 1 Hz sine would have contributions from the 5 e 9 Hz sines as well as, with opposite sign, from 3 e 7 Hz.



At an effective sampling frequency of 4 Hz corrisponds a band of $[-2; 2]\ \mathrm{Hz}$.
The frequency $\omega_N = \pm\pi/\Delta T$ is called Nyquist frequency and is equal to half the sampling frequency. The figures shows which frequencies contribute to the Fourier transform of the sampled signal.

Aliasing (III)

Example: chirp generation $s(t_i) = \cos(t_i \omega(t_i))$
con $t_i = i\Delta t, \ \omega(t) = t\omega_0/2$



Above: time-frequency plot of the original signal
Sotto: same plot for a decimated signal: $s'(t_i) = s(10i\Delta t)$

Sampling theorem (I)

Consideri a signal $s(t)$ that has non null Fourier transform only for $|\omega| < \omega_c$ (band-limited signal). This is an ideal case, as this signal would have an infinite length. Si tratta di una The sampling theorem says that it is possible to fully reconstruct a band limited signal out of its samples if the sampling frequency is higher than $\omega_c/2\pi$.
In practice the sampled signal must go through a low pass filter with cutoff less than ω_c/π.
If the hypothesis are valid there is no aliasing and the Fourier series of the continuous signal coincide with the one of the sampled signal in the range $[-\omega;\omega]$. Therefore:

$$\hat{s}(\omega) = \hat{s}^{(d)}(\omega)\hat{\phi}(\omega)$$

with

$$\hat{\phi}(\omega) = \begin{cases} 1 & \omega \in [-\omega_c;\omega_c] \\ 0 & \text{fuori} \end{cases}$$

These are two analytic functions that coincide over a finite range. So they coincide for all real numbers. The original signal is obtained by applying the inverse Fourier transform to $\hat{s}^{(d)}(\omega)\hat{\phi}(\omega)$.

Sampling theorem (II)

Recalling that the inverse Fourier transform of $\hat{\phi}(\omega)$ is :

$$\phi(t) = \frac{\sin \omega_c t}{\omega_c t}$$

the convolution theorem allows to state that

$$s(t) = \sum_{i=-\infty}^{+\infty} s_i \frac{\sin \omega_c (t - i\Delta T)}{\omega_c (t - i\Delta T)} = \sum_{i=-\infty}^{+\infty} s_i \frac{\sin (\omega_c t - i\pi)}{\omega_c t - i\pi}$$

This gives an interpolation formula which is valid under the hypotheses of the sampling theorem.
For real signals it is essential to use a low pass filter BEFORE the analog-to-digital conversion to avoid aliasing.
This true as well if one resamples at a lower frequency some signal generated at higher rate by a computer simulation.
The interpolation formula is necessary in the digital to analog conversion.

Optimal detection

Detector output : $s(t) = n(t) + h(t)$

$h(t)$ is a deterministic function of parameters (coalescence time, masses, . . .)

Optimal filtering according to the Wiener theory requires to compute

$$\begin{aligned} s_W &= 2\int_0^{+\infty} \frac{\tilde{s}(f)\tilde{h}^*(f)}{S_n(f)}\,\mathrm{d}f \\ &= \int h(t) \int s(\tau)w(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t \end{aligned}$$

$w(t)$ weighs $s(t)$ at those frequencies where the detector is more sensitive

s_W is gaussain with rms 1 in absence of signal

SNR is given by

$$\mathrm{SNR}^2 = 4\int_0^{+\infty} \frac{|\tilde{h}(f)|^2}{S_n(f)}\,\mathrm{d}f$$

SNR^2 is integrated over the frequency spectrum:

Advantage of a wideband detector