Review -

Laplace transform and z transform

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Overview

- Laplace transform
- Z transform
- Transfer function
- Analysis of systems by transform method
- Example analysis

Laplace transform

as a tool for solving continuous-time linear time-invariant systems and electric circuits

Laplace Transform

- ***Laplace transform*** is useful as an analytical tool in the characterization, analysis, and study of **LTI systems** and electrical circuits
- It plays an important role in analyzing **causal** systems specified by linear constant-coefficient **differential** equations

Definition

$$X(s) = \mathcal{L}(x(t)) = \int_0^{\infty} x(t)\mathrm{e}^{-st}\,\mathrm{d}t$$

Laplace transform $X(s)$ of a **causal** signal $x(t)$

$$x(t) = \mathcal{L}^{-1}(X(s)) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\mathrm{e}^{st}\,\mathrm{d}s$$

Inverse Laplace transform

The two functions $\boldsymbol{x(t)}$ and $\boldsymbol{X(s)}$ form a ***Laplace transform pair*** which is designated by $\boldsymbol{x(t) \leftrightarrow X(s)}$. The variable $\boldsymbol{s}$ is called a ***complex frequency*** variable.

Laplace transform pairs

$$\delta(t) \leftrightarrow 1$$

$$u(t) \leftrightarrow \frac{1}{s}$$

$$e^{-at}\, u(t) \leftrightarrow \frac{1}{s+a}$$

$$t^n\, u(t) \leftrightarrow \frac{n!}{s^{n+1}}$$

$$\sin(\omega t)\, u(t) \leftrightarrow \frac{\omega}{s^2+\omega^2}$$

$$\cos(\omega t)\, u(t) \leftrightarrow \frac{s}{s^2+\omega^2}$$

$$e^{-\alpha t} \sin(\omega t)\, u(t) \leftrightarrow \frac{\omega}{(s+\alpha)^2+\omega^2}$$

$$e^{-\alpha t} \cos(\omega t)\, u(t) \leftrightarrow \frac{(s+\alpha)}{(s+\alpha)^2+\omega^2}$$

Properties

Uniqueness

$$x_1(t) = x_2(t) \Leftrightarrow X_1(s) = X_2(s)$$

Homogeneity

$$\mathcal{L}(Kx(t)) = K\mathcal{L}(x(t))$$
$$\mathcal{L}^{-1}(KX(s)) = K\mathcal{L}^{-1}(X(s))$$

Additivity

$$\mathcal{L}(x_1(t) + x_2(t)) = \mathcal{L}(x_1(t)) + \mathcal{L}(x_2(t))$$
$$\mathcal{L}^{-1}(X_1(s) + X_2(s)) = \mathcal{L}^{-1}(X_1(s)) + \mathcal{L}^{-1}(X_2(s))$$

Differentiation

$$\mathrm{D}x(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$$

$$\mathcal{L}(\mathrm{D}x(t)) = s\mathcal{L}(x(t)) - x(0^-) = sX(s) - x(0^-)$$

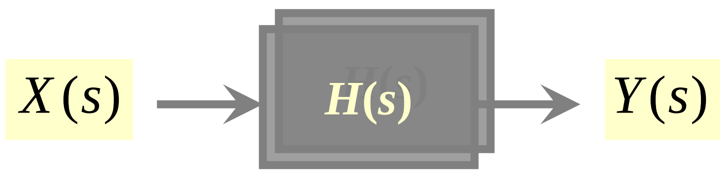
Convolution

$$\mathcal{L}\int_0^\infty x(\tau)h(t-\tau)\,\mathrm{d}\tau = X(s)H(s)$$

Transfer function of continuous-time systems

$$\sum_{m=0}^{M} a_m \mathrm{D}^m y(t) = \sum_{l=0}^{L} b_l \mathrm{D}^l x(t)$$

Relaxed LTI system described by linear constant-coefficients differential equation



$$\left(\sum_{m=0}^{M} a_m s^m\right) Y(s) = \left(\sum_{l=0}^{L} b_l s^l\right) X(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

Transfer function

$$\boxed{H(s) = \frac{\sum_{l=0}^{L} b_l s^l}{\sum_{m=0}^{M} a_m s^m}}$$

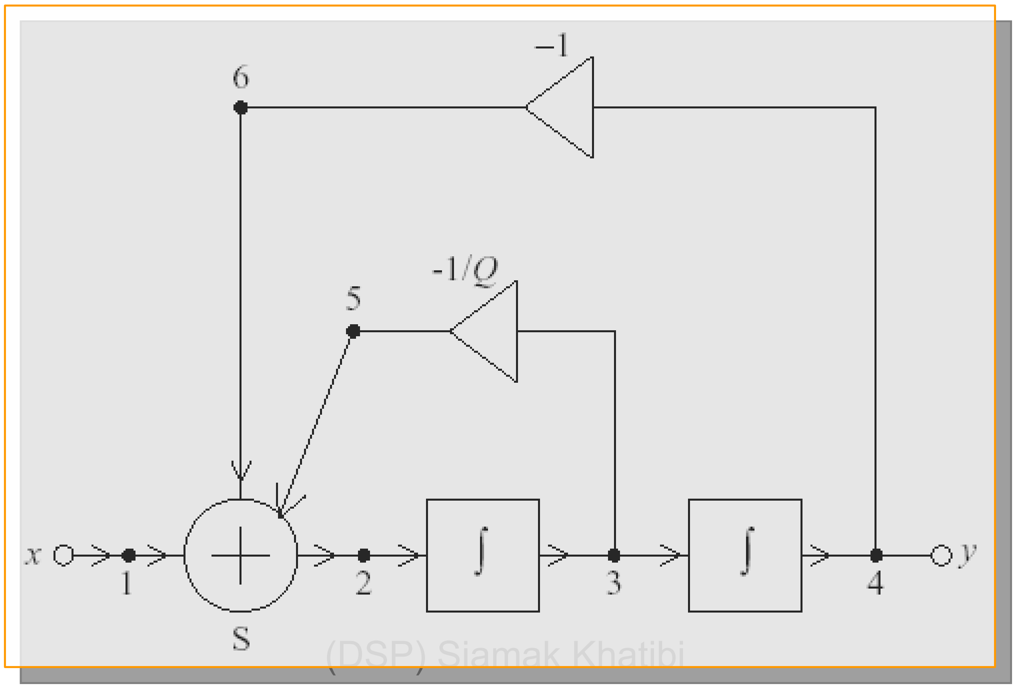
Relaxed system

- A system is said to be ***relaxed*** at time t_0 if the output for $t \geq t_0$ is solely and uniquely determined by the **input** for $t \geq t_0$

- If the concept of **energy** is applicable, the system is said to be relaxed at t_0 if no energy is stored in the system at t_0

Analysis of CTLTI systems by transform method

Consider a continuous-time linear time-invariant (CTLTI) system specified by its block diagram and assume zero initial conditions (the system is at rest) and find the response of the system (signals at all nodes) for an excitation applied at node 1



Analysis procedure

1. Label nodes by integer numbers
2. Assume that signal at some node is $y(t)$
3. Assume that input is a known function of time $x(t)$
4. Write equations characterizing each block
5. Apply the Laplace transform to both sides of each equation
6. Solve the set of algebraic equations
7. Find transfer function by dividing $Y(s)$ by $X(s)$ assuming zero initial conditions
8. Find the inverse Laplace transform of $Y(s)$ to obtain the response $y(t)$

Analysis steps

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$$y_1 = x(t)$$
$$y_2 = y_1 + y_5 + y_6$$
$$y_3 = \omega \int_0^t y_2(\tau)\,d\tau$$
$$y_4 = \omega \int_0^t y_3(\tau)\,d\tau$$
$$y_5 = -\frac{1}{Q} y_3$$
$$y_6 = -y_4$$

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$$Y_1(s) = X(s)$$
$$Y_2(s) = Y_1(s) + Y_5(s) + Y_6(s)$$
$$Y_3(s) = \omega \frac{Y_2(s)}{s}$$
$$Y_4(s) = \omega \frac{Y_3(s)}{s}$$
$$Y_5(s) = -\frac{1}{Q} Y_3(s)$$
$$Y_6(s) = -Y_4(s)$$

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$$Y_1(s) = X(s)$$
$$Y_2(s) = \frac{s^2}{s^2 + \frac{\omega}{Q}s + \omega^2} X(s)$$
$$Y_3(s) = \frac{\omega s}{s^2 + \frac{\omega}{Q}s + \omega^2} X(s)$$
$$Y_4(s) = \frac{\omega^2}{s^2 + \frac{\omega}{Q}s + \omega^2} X(s)$$

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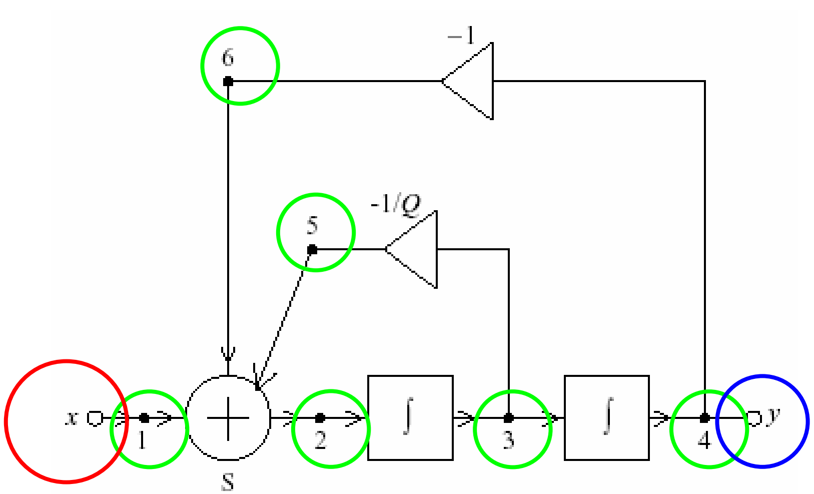
$$H_2(s) = \frac{Y_2(s)}{X(s)} = \frac{s^2}{s^2 + \frac{\omega}{Q}s + \omega^2}$$
$$H_3(s) = \frac{Y_3(s)}{X(s)} = \frac{\omega s}{s^2 + \frac{\omega}{Q}s + \omega^2}$$
$$H_4(s) = \frac{Y_4(s)}{X(s)} = \frac{\omega^2}{s^2 + \frac{\omega}{Q}s + \omega^2}$$



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$$y_3(t) = \mathcal{L}^{-1}(H_3(s)X(s)) = \mathcal{L}^{-1}\frac{H_3(s)}{s} = \frac{3}{2\sqrt{2}} e^{\frac{-1}{3}t} \sin\left(\frac{2\sqrt{2}}{3}t\right) u(t)$$

Z transform

as a tool for solving discrete-time linear time-invariant systems

Z Transform

- ***Z transform*** is useful as an analytical tool in the characterization, analysis, and study of discrete-time **LTI systems**

- It plays an important role in analyzing **causal** systems specified by linear constant-coefficient **difference** equations

Definition

$$X(z) = Z\{x_{(n)}\} = \sum_{n=0}^{\infty} x_{(n)} z^{-n}$$

Z transform $X(z)$
of a **causal**
sequence $\{x_{(n)}\}$

$$x_{(n)} = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1}\, \mathrm{d}z$$

Inverse *z* transform

Γ is a contour in the counterclockwise sense enclosing all the singularities of $\boldsymbol{X(z)}$

The sequence $\boldsymbol{\{x_{(n)}\}}$ and the function $\boldsymbol{X(z)}$ form a ***z transform pair*** which is designated by $\boldsymbol{\{x_{(n)}\}} \leftrightarrow \boldsymbol{X(z)}$. The variable $\boldsymbol{z}$ is called a ***complex*** variable.

Z transform pairs

$$\delta_{(n)} \leftrightarrow 1$$

$$u_{(n)} \leftrightarrow \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

$$a^n u_{(n)} \leftrightarrow \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

$$n\, u_{(n)} \leftrightarrow \frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2}$$

$$\sin(\theta n)\, u_{(n)} \leftrightarrow \frac{z \sin(\theta)}{z^2 - 2z\cos(\theta) + 1}$$

$$\cos(\theta n)\, u_{(n)} \leftrightarrow \frac{z(z - \cos(\theta))}{z^2 - 2z\cos(\theta) + 1}$$

$$\mathrm{e}^{-\alpha n} \sin(\theta n)\, u_{(n)} \leftrightarrow \frac{z\, \mathrm{e}^{-\alpha} \sin(\theta)}{z^2 - 2z\, \mathrm{e}^{-\alpha} \cos(\theta) + \mathrm{e}^{-2\alpha}}$$

$$\mathrm{e}^{-\alpha n} \cos(\theta n)\, u_{(n)} \leftrightarrow \frac{z(z - \mathrm{e}^{-\alpha} \cos(\theta))}{z^2 - 2z\, \mathrm{e}^{-\alpha} \cos(\theta) + \mathrm{e}^{-2\alpha}}$$

Properties

Uniqueness

$$\{x_{(n)}\} = \{y_{(n)}\} \Leftrightarrow X(z) = Y(z)$$

Homogeneity

$$Z(K\{x_{(n)}\}) = KX(z)$$
$$Z^{-1}(KX(z)) = K\{x_{(n)}\}$$

Additivity

$$Z(\{x_{(n)}\} + \{y_{(n)}\}) = X(z) + Y(z)$$
$$Z^{-1}(X(z) + Y(z)) = \{x_{(n)}\} + \{y_{(n)}\}$$

Shifting

$$Z\{x_{(n-\mu)}\} = z^{-\mu}X(z)$$
$$Z^{-1}z^{-\mu}X(z) = \{x_{(n-\mu)}\}$$

Convolution

$$Z\{x_{(n)}\} * \{h_{(n)}\} = Z\{h_{(n)}\} * \{x_{(n)}\} = X(z)H(z)$$
$$Z^{-1}(X(z)H(z)) = \{x_{(n)}\} * \{h_{(n)}\} = \{h_{(n)}\} * \{x_{(n)}\}$$

Convolution

Convolution of two causal sequences is defined as

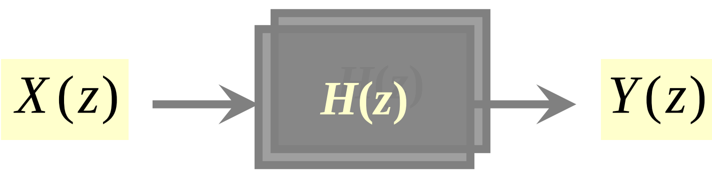
$$\{y_{(n)}\} = \{x_{(n)}\} * \{h_{(n)}\}$$

$$y_{(n)} = \sum_{\mu=0}^{\infty} x_{(\mu)} h_{(n-\mu)}$$

Transfer function of discrete-time systems

$$\sum_{m=0}^{M} a_m y_{(n-m)} = \sum_{l=0}^{L} b_l x_{(n-l)}$$

Relaxed LTI system described by linear constant-coefficients difference equation



$$\left(\sum_{m=0}^{M} a_m z^{-m}\right) Y(z) = \left(\sum_{l=0}^{L} b_l z^{-l}\right) X(z)$$

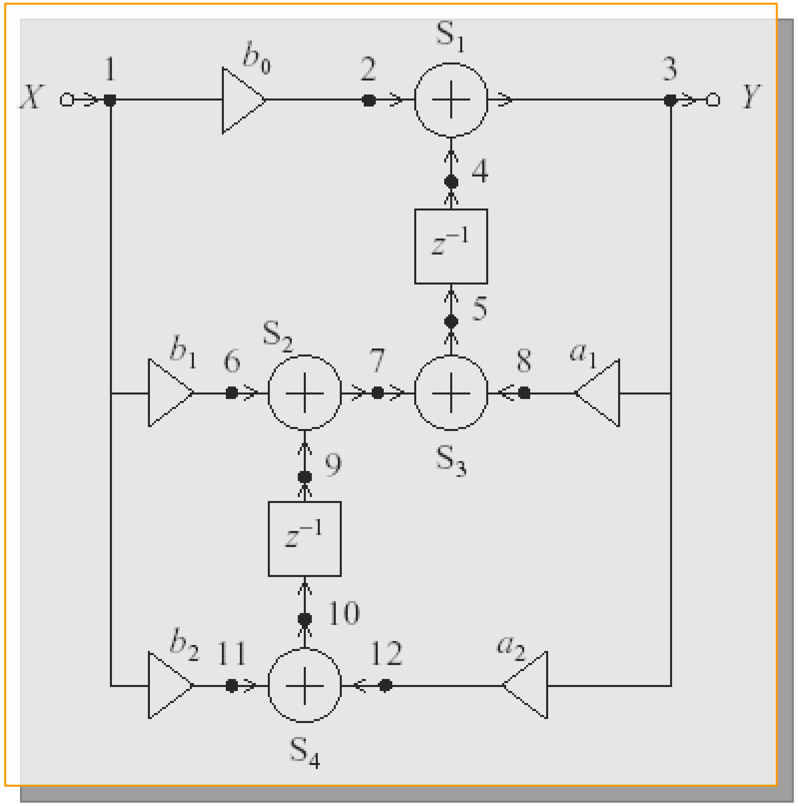
$$H(z) = \frac{Y(z)}{X(z)}$$

Transfer function

$$H(z) = \frac{\sum_{l=0}^{L} b_l z^{-l}}{\sum_{m=0}^{M} a_m z^{-m}}$$

Analysis of DTLTI systems by transform method

Consider a discrete-time linear time-invariant (DTLTI) system specified by its block diagram and assume zero initial conditions (the system is at rest) and find the response of the system (signals at all nodes) for an excitation applied at node 1



Analysis procedure

1. Label nodes by integer numbers
2. Assume that signal at some node is $y_{(n)}$
3. Assume that input is a known sequence $x_{(n)}$
4. Write equations characterizing each block
5. Apply the *z* transform to both sides of each equation
6. Solve the set of algebraic equations
7. Find transfer function by dividing $Y(z)$ by $X(z)$ assuming zero initial conditions
8. Find the inverse *z* transform of $Y(z)$ to obtain the response $y_{(n)}$

Analysis steps

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$$\begin{aligned} y_1(n) &= x(n) \\ y_2(n) &= b_0 y_1(n) + x_{b0}(n) \\ y_3(n) &= y_2(n) + y_4(n) \\ y_4(n) &= y_5(n-1) \end{aligned}$$

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$$\begin{aligned} Y_1(z) &= X(z) \\ Y_2(z) &= b_0 Y_1(z) + X_{b0}(z) \\ Y_3(z) &= Y_2(z) + Y_4(z) \\ Y_4(z) &= z^{-1} Y_5(z) \end{aligned}$$

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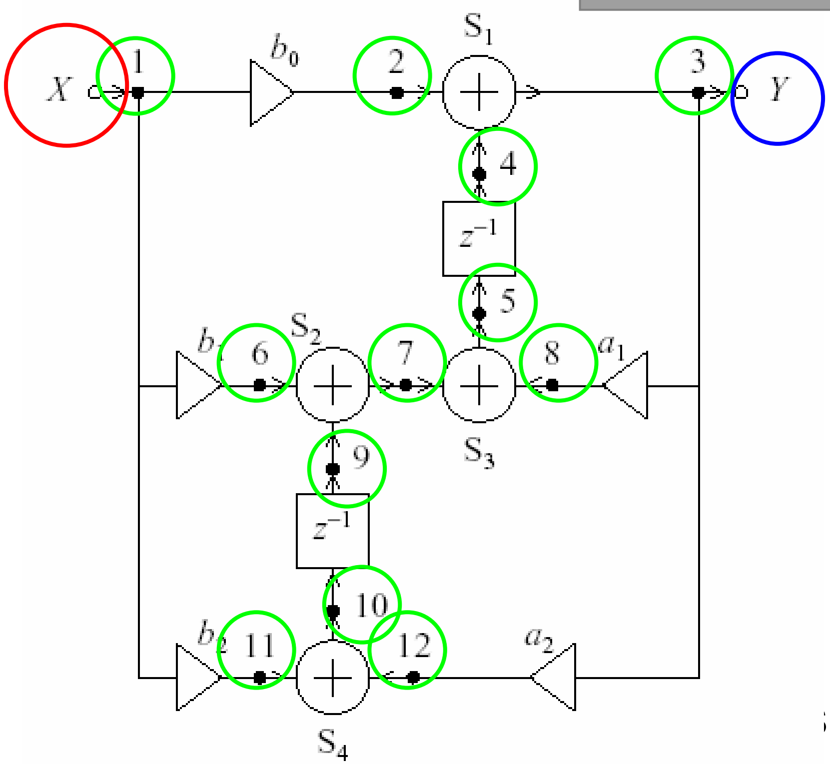
$$Y_3(z) = \frac{(b_0 + b_1 z^{-1} + b_2 z^{-2})X(z) + z^{-1}X_{a1}(z) + z^{-2}X_{a2}(z) + X_{b0}(z) + z^{-1}X_{b1}(z) + z^{-2}X_{b2}(z)}{1 - a_1 z^{-1} - a_2 z^{-2}}$$



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$$H_3(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

8

$$\begin{aligned} y_3(n) &= Z^{-1}(H_3(z)X(z)) = Z^{-1}(H_3(z)) \\ &= 2^{-n/2}\left(-\cos(n\frac{\pi}{2}) + \sqrt{8}\sin(n\frac{\pi}{2})\right)u(n) + 2\delta(n) \end{aligned}$$