Sampled Signals

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An experimental set-up in 1909
Geiger & Marsden's experiment

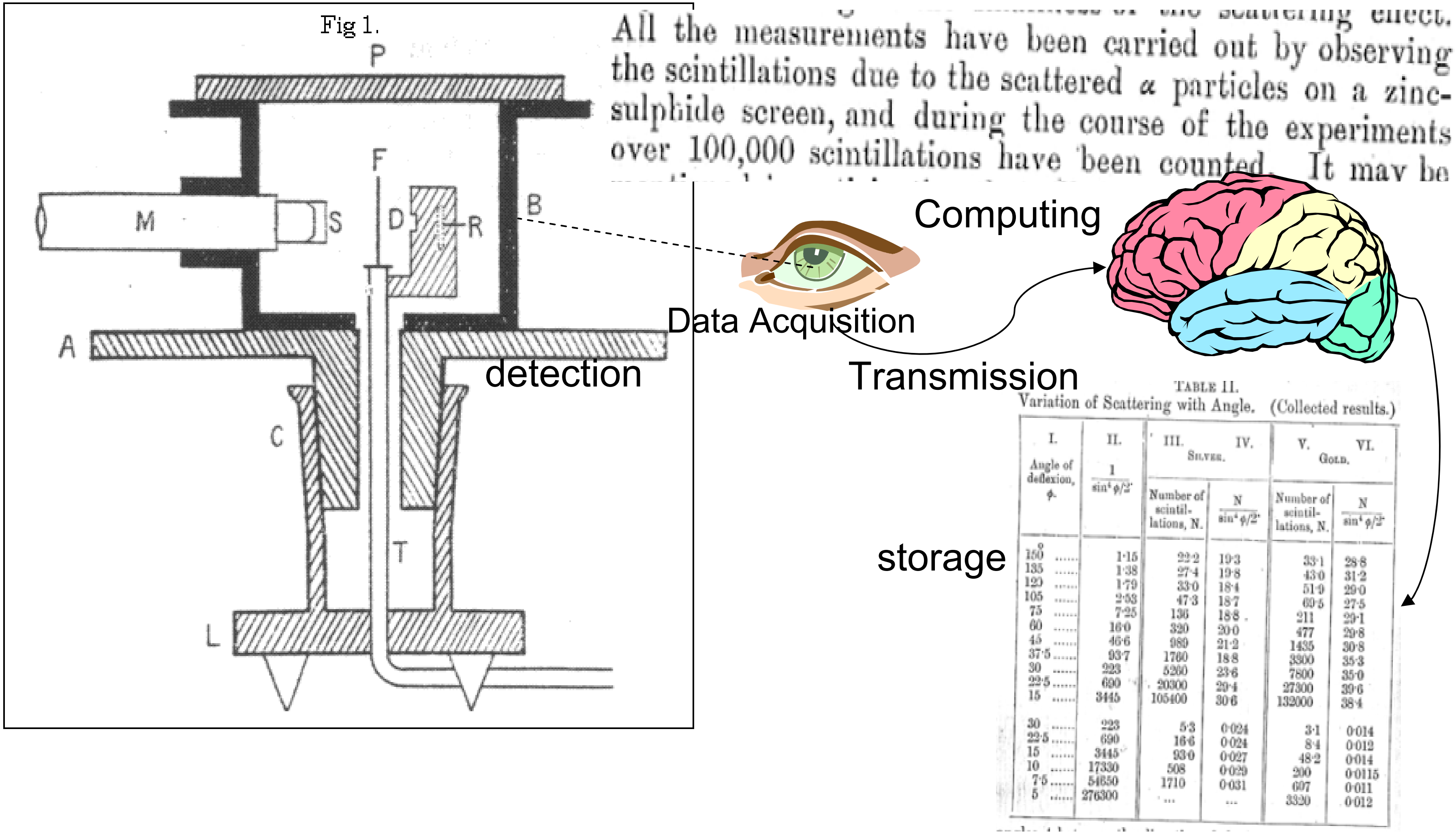


Fig 1.

the scattering effect. All the measurements have been carried out by observing the scintillations due to the scattered α particles on a zinc-sulphide screen, and during the course of the experiments over 100,000 scintillations have been counted. It may be

detection

Data Acquisition

Computing

Transmission

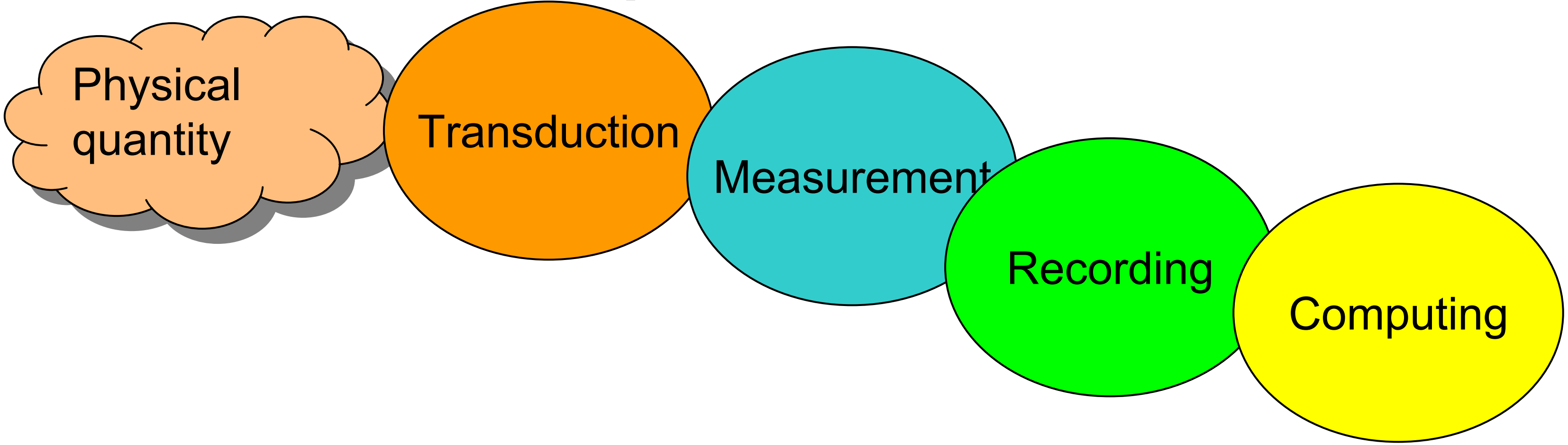
storage

TABLE II.

Variation of Scattering with Angle. (Collected results.)

I.	II.	III.	IV.	V.	VI.
		Silver.		Gold.	
Angle of deflexion, ϕ.	$\frac{1}{\sin^4 \phi/2}$.	Number of scintillations, N.	$\frac{N}{\sin^4 \phi/2}$.	Number of scintillations, N.	$\frac{N}{\sin^4 \phi/2}$.
150°	1·15	22·2	19·3	33·1	28·8
135	1·38	27·4	19·8	43·0	31·2
120	1·79	33·0	18·4	51·9	29·0
105	2·53	47·3	18·7	69·5	27·5
75	7·25	136	18·8	211	29·1
60	16·0	320	20·0	477	29·8
45	46·6	989	21·2	1435	30·8
37·5	93·7	1760	18·8	3300	35·3
30	223	5260	23·6	7800	35·0
22·5	690	20300	29·4	27300	39·6
15	3445	105400	30·6	132000	38·4
30	223	5·3	0·024	3·1	0·014
22·5	690	16·6	0·024	8·4	0·012
15	3445	93·0	0·027	48·2	0·014
10	17330	508	0·029	200	0·0115
7·5	54650	1710	0·031	607	0·011
5	276300	...	...	3320	0·012

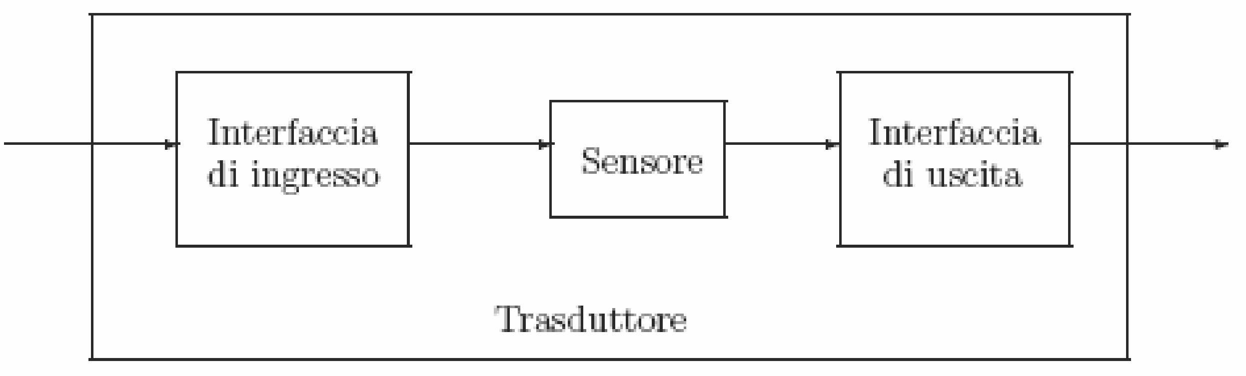
From Physical quantity to computed data



- Sensors: systems that excited by a certain form of energy, representing the physical phenomenon to observe, give in exit a different form of energy 2 (e.g. light into electricity).
- Transducers: systems that convert (translate) a form of energy into another (e.g. electrical into mechanical);
- Recorders: systems to translate the acquired signal in a series of numbers or graphs that could be studied in a second moment. They could record continuously or at discrete times and could be also humans with paper and pencils.
- Computers: (formerly) Humans that take the recorded data and elaborate them to obtain an aggregate result.

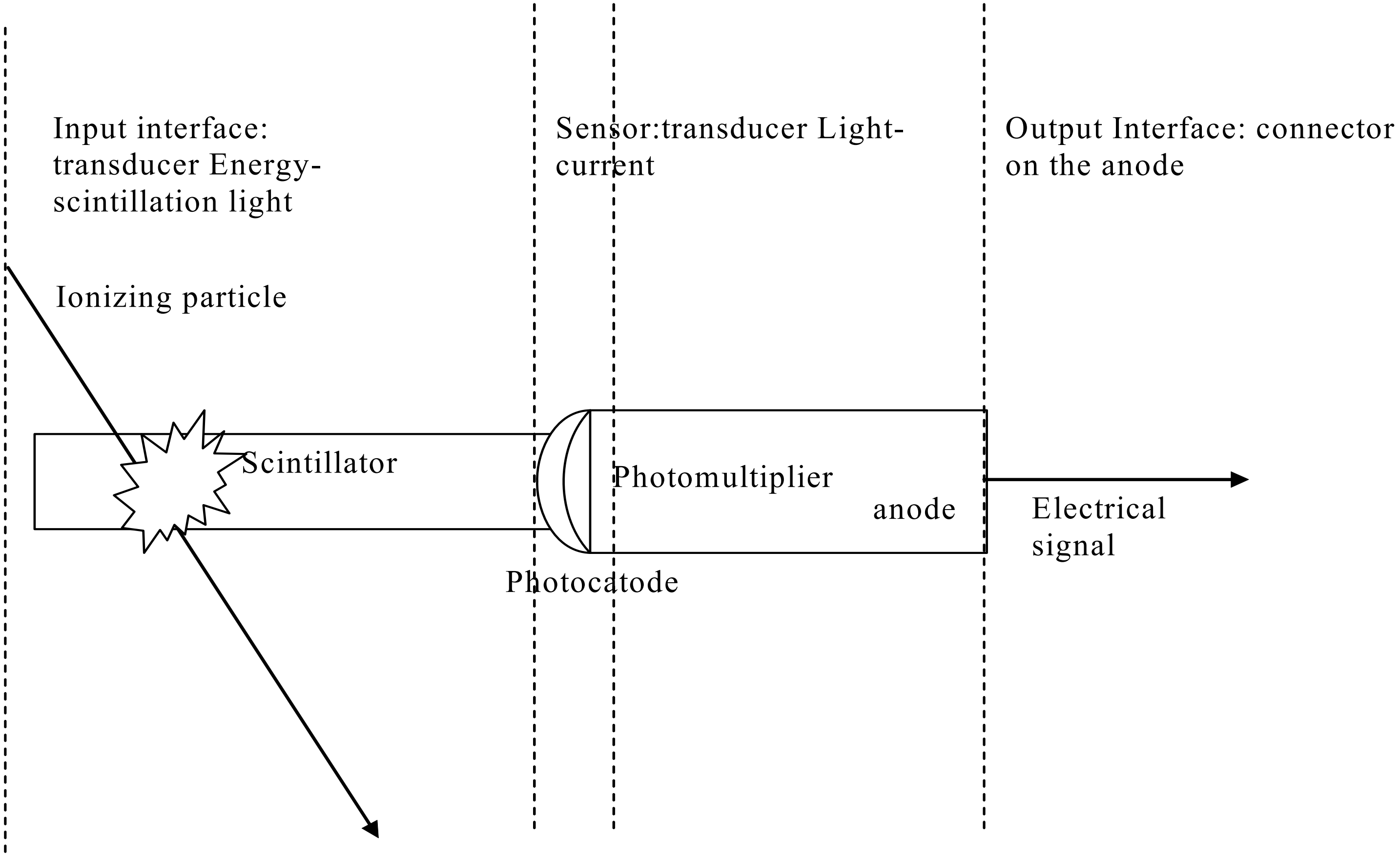
Transducer/Sensor

- A transducer consists, in general, of 3 parts one of which is the sensor. Often the distinction among sensor and transducer is not net and in the practice the two terms are used as synonyms



An example

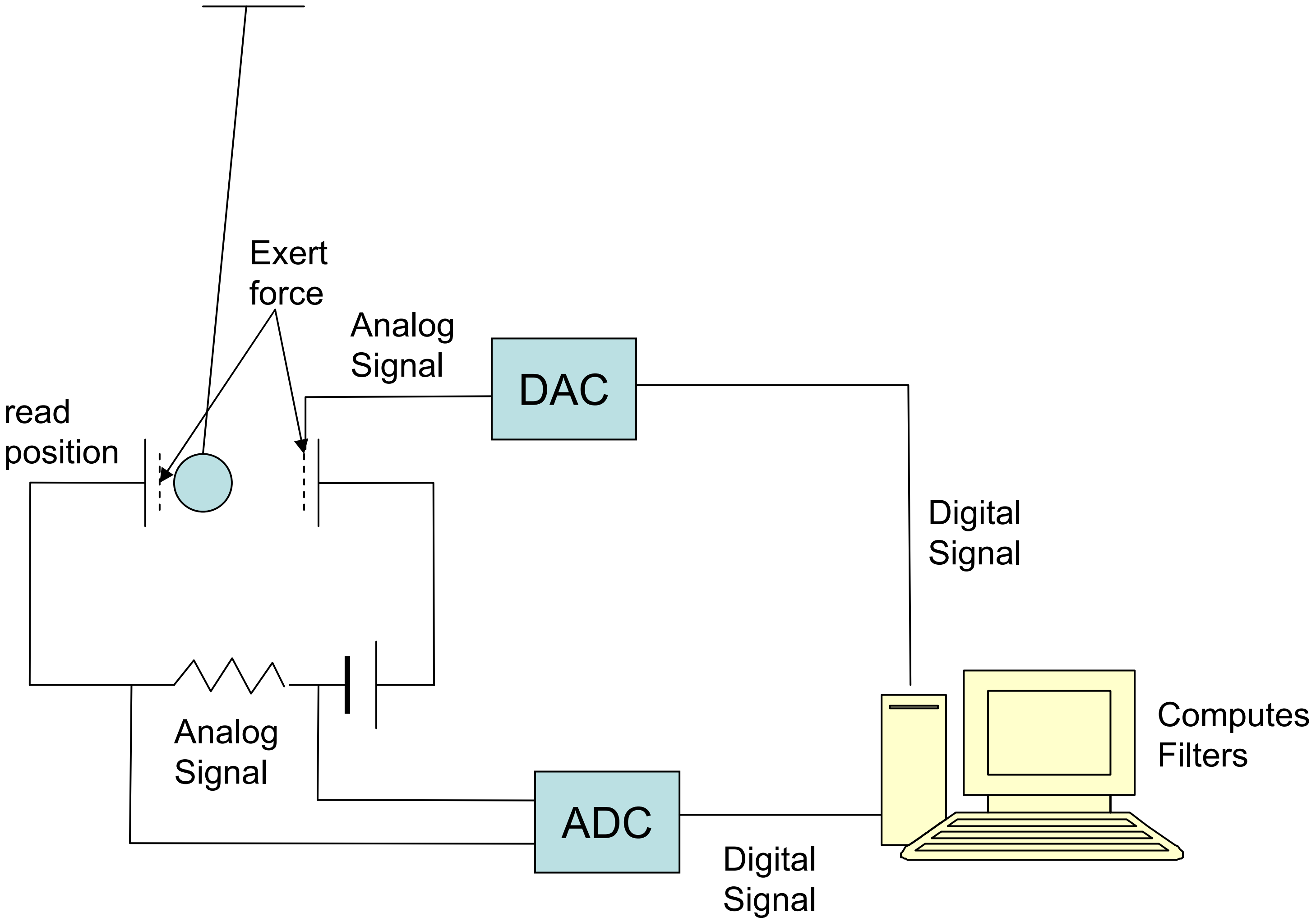
- Detection of cosmic rays with scintillators



Acquisition of signals for digital handling

- When computers started to be electronic devices, the data acquisition chain changed, but the elements remained essentially the same. Recorders and computers became electronic devices, linked to one another.
- data entering into the computing system had to be formatted in a way understandable to computers, i.e. they had to be in digital form.
- This new requirement to the data acquisition chain introduced a new element: the Converters, i.e. electronic devices that convert a continuous-time, continuous-value (Analogical) signal to discrete time, numerical (digital) data: the Analog to Digital Converter (ADC)
- The inverse converter – the Digital to Analog Converter (DAC) – is needed to get an electrical signal from a numerical data stream, in order to realise controlled systems that, from the acquired data, derive a signal that is used to take actions (actuate) on the system itself.

Example: damping pendulum oscillations



Sampled Signals

- The computer takes some time to elaborate input signals and thus cannot follow the signal continuous evolution; it will, instead, take the signal value at discrete times. This operation is the signal sampling.
- A computer has a limited resolution given by the number of bits of its word, thus it will not give the actual value of the signal at the time of sampling, but only the nearest value that can be expressed with this word: **Quantization**
- The information linked with the sampled and quantized analogic signal is called *digital signal*.

Sampled Signals

- The transducer range Y_s is the full range the values of the output signal can assume from the minimum Y_m to the maximum Y_M

$$Y_s = Y_M - Y_m$$

- A Computer with a N bits word, can represent this interval in 2^N values.
- The least significant bit is:

$$LSB = Y_s / 2^N$$

- All the values differing by less than one LSB will be represented by the same value.
- The error introduced by the quantization process is ½ LSB:

$$\varepsilon_q = LSB/2 = Y_s / 2^{N+1}$$

Spectral behaviour of sampled signals

- For the continuous f(t) = $e^{i\omega t}$ the larger is ω the larger is rate of oscillations and is periodic for any value of ω.
- For the discrete counterpart F(n): consider the function shifted of 2π:

$$F(n) = e^{i(\Omega+2\pi)n} = e^{i\Omega n}e^{2i\pi n} = e^{i\Omega n}$$

Has the SAME period as F(n), and the same is for shifts of ±4π, ±6π…

- We only need to consider an interval of width 2π.
- For F(n) to be periodic: F(n+N)=F(n). Thus:

$$e^{i\Omega(n+N)} = e^{i\Omega n} \Rightarrow e^{i\Omega N} = 1$$

$$\Omega N = 2k\pi \Leftrightarrow \Omega = \frac{2k\pi}{N}$$

i.e. F(n) is periodic only for a discrete number of frequencies: $\Omega=2k\pi/N$

The Sampling theorem

- Any continuous signal x(t), sampled with a period T can be represented in terms of a sum

$$x_p(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

Applying Fourier transform to both members:

$$X_p(\omega) = \int_{-\infty}^{\infty}\left(x(t)\sum_n \delta(t-nT)\right)e^{-i\omega t}dt$$

The sum of delta functions is periodic and can be expanded in Fourier series as

$$\sum_{n=-\infty}^{\infty}\delta(t-nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{2in\pi\frac{t}{T}} = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{in\omega_s t}$$

Thus:

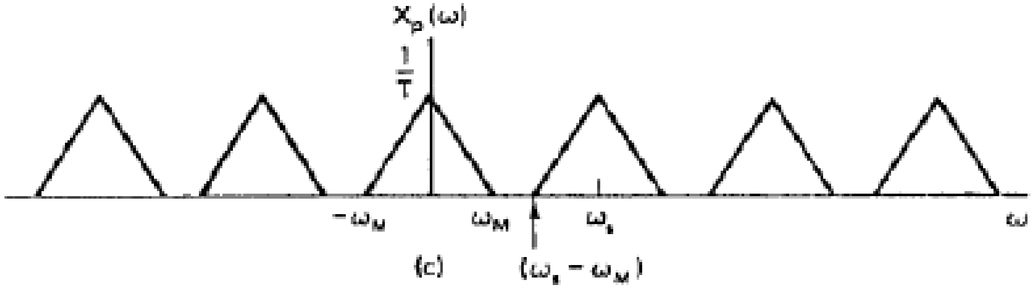
$$\int_{-\infty}^{\infty} x(t)\sum_n\delta(t-nT)e^{-i\omega t}dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\left(\sum_n e^{-i(\omega-n\omega_s)t}\right)dt =$$

$$=\frac{1}{T}\sum_n\int_{-\infty}^{\infty} x(t)e^{-i(\omega-n\omega_s)t}dt = \frac{1}{T}\sum_{n=-\infty}^{\infty}X(\omega-n\omega_s)$$

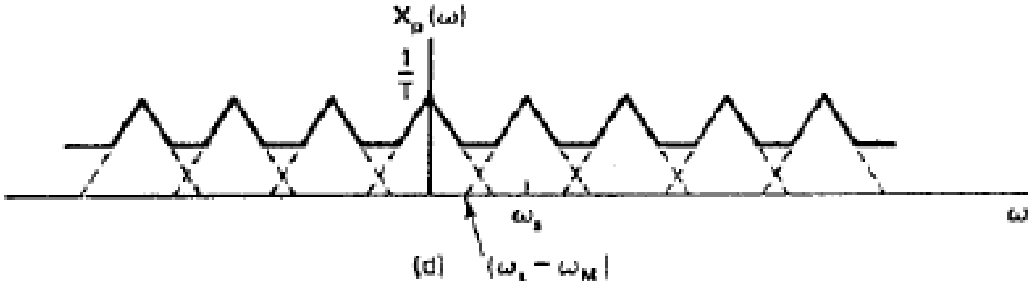
The Sampling theorem

- Thus $X_p(\omega)$ is a sum of replicas of $X(\omega)$ scaled of $1/T$ and shifted of ω_s, being $\omega_s = 2\pi/T_s$ the *sampling frequency*.
- If $X(\omega)$ is *band limited* i.e. has a frequency span comprised between a minimum $-\omega_M$ and a maximum ω_M, then the replicas do not overlap only if:

$\omega_M < (\omega_s - \omega_M)$, i.e. $\omega_M < \omega_s/2$, or conversely: $\omega_s = 2\omega_M$



(c)



(d)

The Sampling theorem

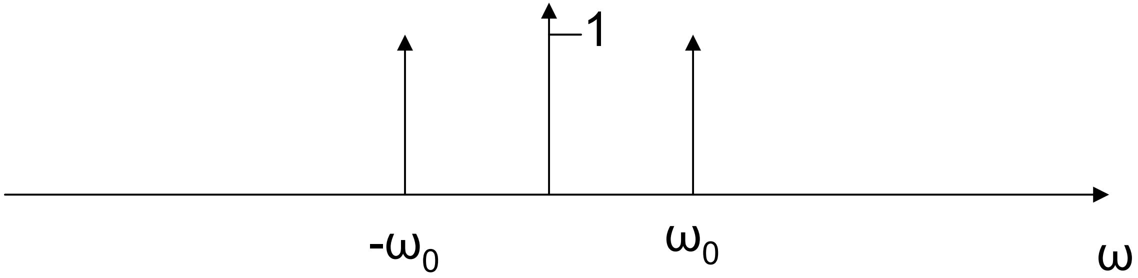
- Theorem: Given a band-limited signal x(t) with $X(\omega) = 0$ for each $|\omega| > \omega_M$, then x(t) is uniquely determined by its samples $x_p(nT)$ if $\omega_s > 2\omega_M$.
- The frequency $\omega_s = 2\omega_M$ is called the *Nyquist frequency*.

Aliasing

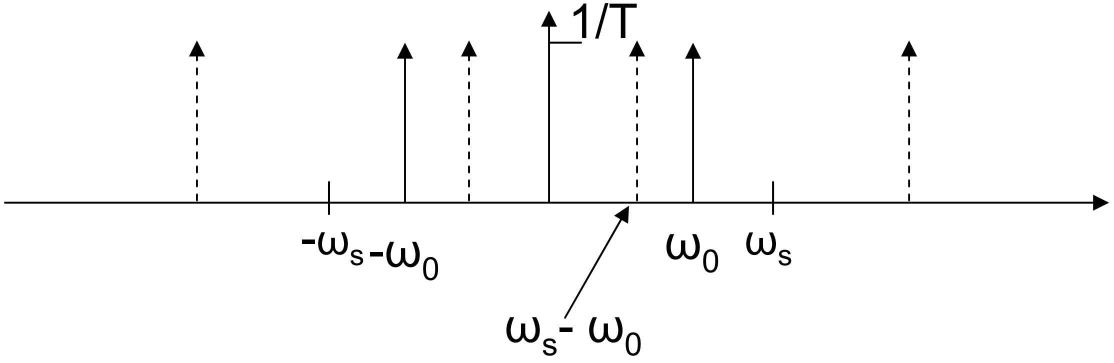
- Let us consider a sinusoidal signal:

$$x(t) = \cos(\omega_0 t)$$

Its spectrum is only two lines at $\pm\omega_0$



- The same function, sampled at a frequency $\omega_s < 2\,\omega_0$ is:

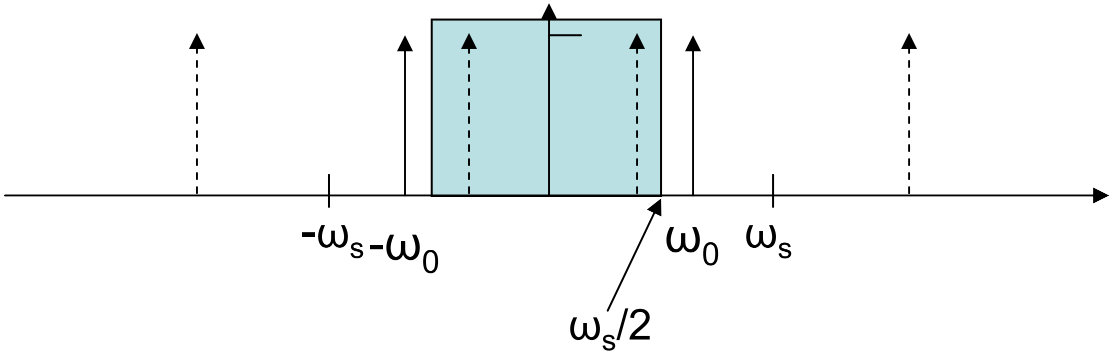


Aliasing

If we apply a low-pass filter with a cut at $\omega_c = \omega_s/2$, the only frequency that survives will be:

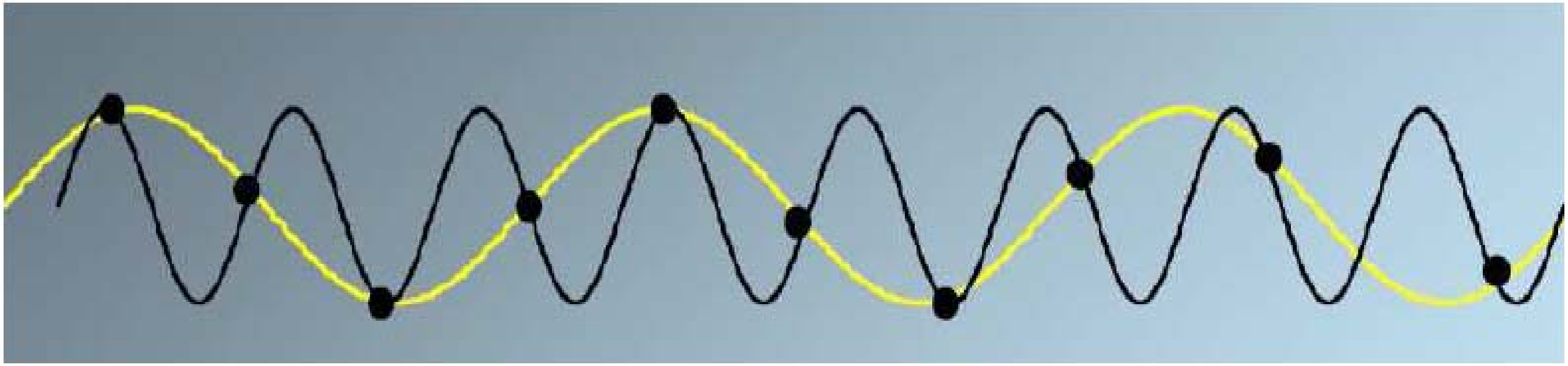
$$x_f(t) = \cos(\omega_0 - \omega_s) \neq x(t)$$

being, by hypothesis $\omega_0 > \omega_s/2$.



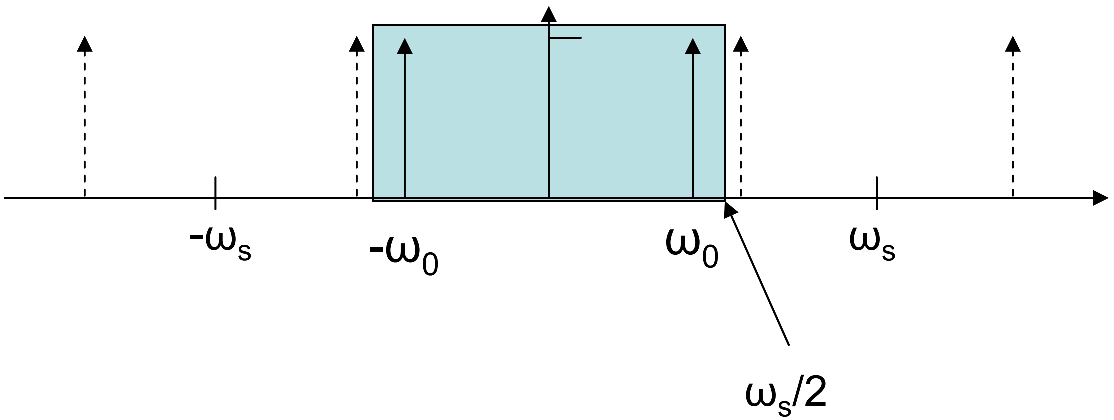
Aliasing

In other terms what comes out from the filter is a sinusoidal function with a lower frequency than the original, as shown in figure.



Aliasing

If, instead the sampling frequency is $\omega_s > 2\omega_0$,



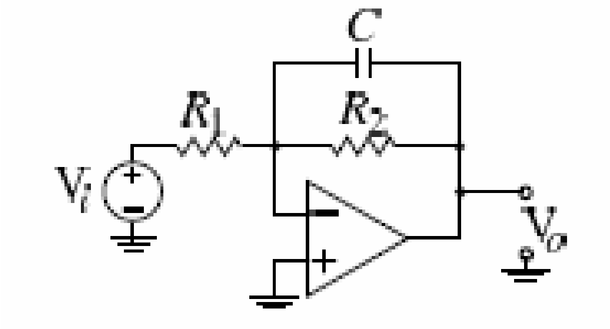
The original function survives the filter, while the replicas are cut away.

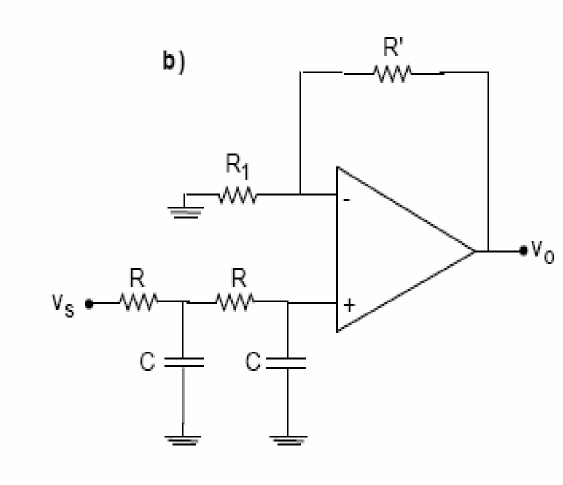
Low pass filters

- Filtro analogico realizzabile con un OpAmp e una capacità
- La tensione V_0 è: $(-R_2/R_1)v_s/(1+j\omega RC)=A_v(0)v_s/[1+j\omega/\omega_0]$
- La frequenza di taglio è $\omega_0=1/RC$ e si ha la usuale discesa di -20dB a decade.
- Il filtro b) è un filtro del secondo ordine.
- La sua funzione di trasferimento è del tipo :

$A_v(s)=A_0/[(s/\omega_0)^2+2k(s/\omega_0)+1]$

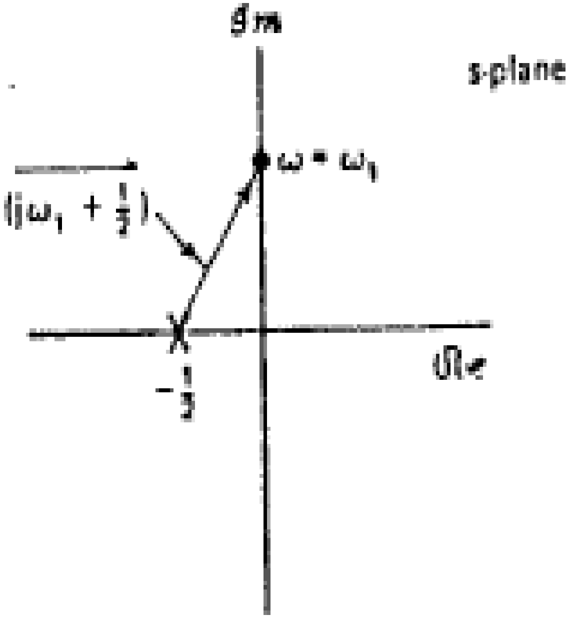
- La discesa è di -40dB per decade.





Rappresentazione dei poli e degli zeri

- Le funzioni di trasferimento possono essere rappresentate come rapporti di polinomi complessi: A=P(n)/Q(m) ove n ed m sono il grado dei polinomi.
- Gli zeri del denominatore si chiamano **Poli**
- Zeri e poli possono essere rappresentati nel piano complesso come vettori: per es. Se X(s)= 1/(s+1/2), la rappresentazione è come in figura ove con la X sull'asse reale si è rappresentato il polo ad s=-1/2.
- Il filtro ideale passa-basso è della forma $A_v(s)$=1/P(s) con P(s) avente zeri nel semipiano sinistro.



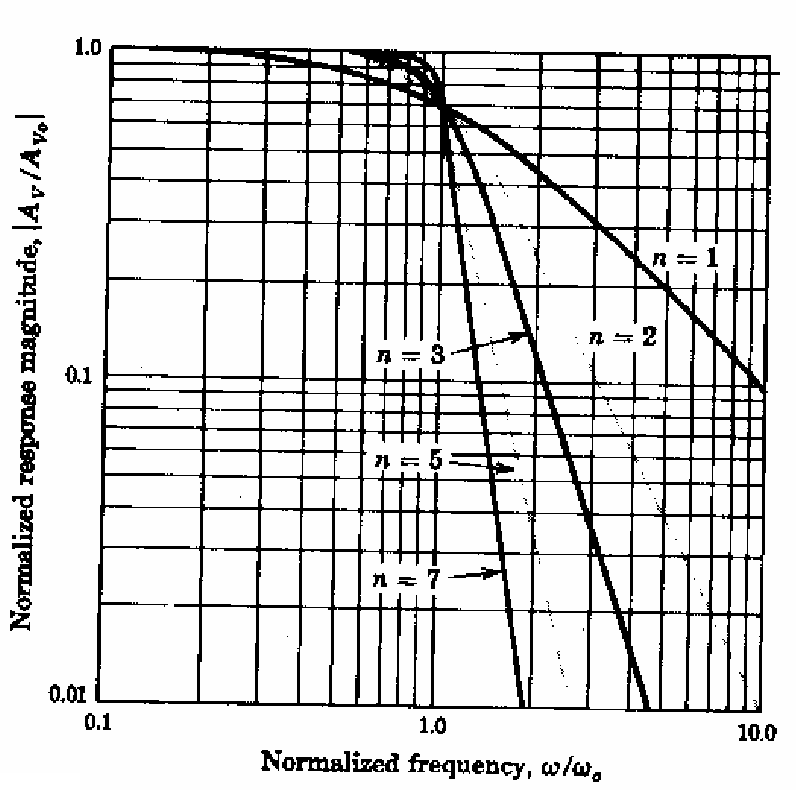
Filtri di Butterworth

- Sono usati come approssimazione di riferimento per la progettazione di filtri analogici e digitali
- La funzione di trasferimento dei filtri di Butterworth è data da:

$$G^2(\omega) = |H(j\omega)|^2 = \frac{G_0^2}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}$$

- I poli saranno equispaziati sul cerchio di raggio ω_c nel semipiano negativo e sono dati dall'espressione:

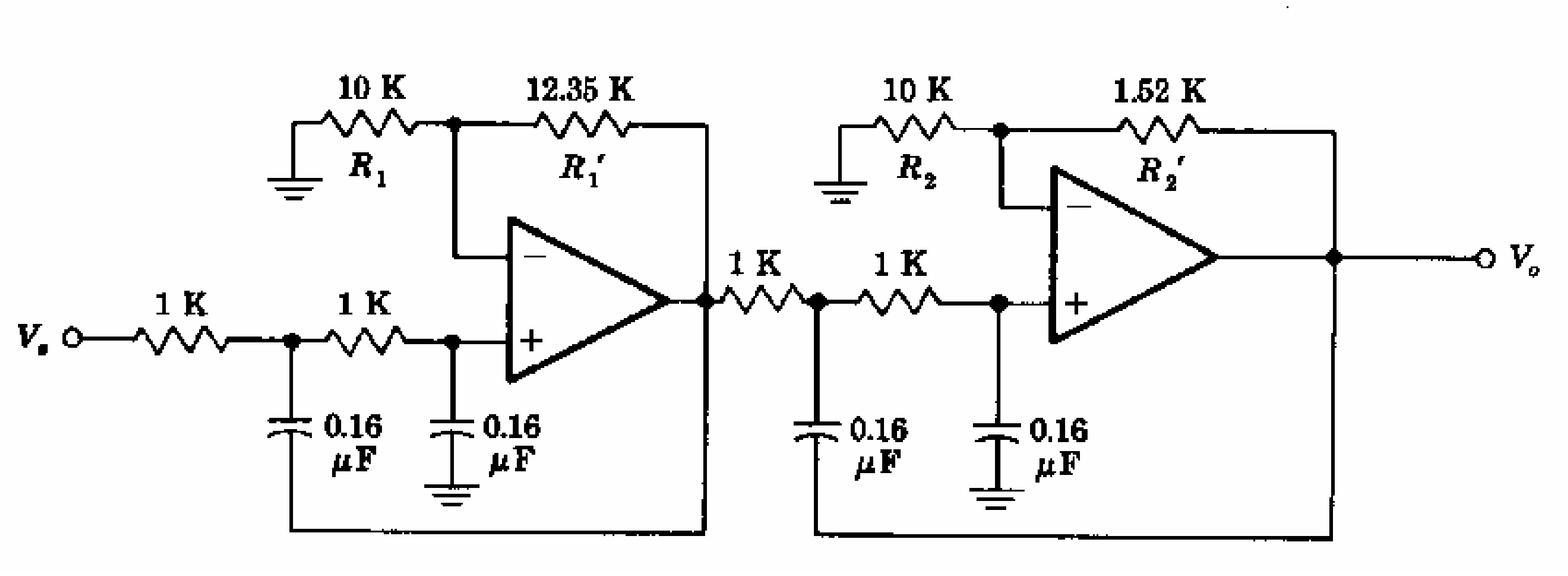
$$s_k = \omega_c e^{\frac{j(2k+n-1)\pi}{2n}} \qquad \mathrm{k} = 1, 2, 3, \ldots, \mathrm{n}$$



…essendo n l'ordine del filtro e ω_c la frequenza di taglio.

Realizzazione analogica del filtro di Butterworth

- Esempio di ordine 4 dal Millman Halkias con f_0= 1kHz



Filtri digitali (numerici)

Sono filtri che agiscono su una sequenza numerica come per es. un segnale campionato.

Se indichiamo la sequenza di ingresso nel filtro con x[n], ed il filtro è linare time invariant (LTI), l’uscita sarà del tipo:

y[n]=h[n]*x[n], essendo h[n] la risposta all’impulso del filtro considerato. Dunque:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Se x[n] è del tipo z^n, sarà:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

$$\Rightarrow y[n] = H(z)z^n = H(z)x[n]$$

Filtri digitali

Più in generale, se x[n] è una fuzione qualsiasi, sviluppata come una combinazione lieare di esponenziali complesse:

$$x[n] = \sum_{k=-\infty}^{\infty} a_k z_k^n$$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{\infty} a_k H(z_k) z_k^n$$

Nel caso della serie di Fourier, le z_k sono tutte uguali e del tipo $e^{i\omega(k)}$, dunque:

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k N / n}$$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{\infty} a_k H\left(\frac{2\pi k}{n}\right) e^{2\pi i k N / n}$$

Filtri digitali

Una equazione differenziale a coefficienti costanti, di ordine N del tipo: $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$

In termini discreti (cioé quando x(t) = x(nT)=x[n]) diventa:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

Equazione alle differenze

$$\Rightarrow y[n] = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right)$$

Equazione ricorsiva: dobbiamo conoscere tutti gli y[n-k]. Se N=0, l’equazione è non-ricorsiva.

Filtri digitali (numerici)

- Le caratteristiche di banda possono essere determinate tramite la trasformata tempo-discreta z: z = $e^{i\omega}$ con ω frequenza di campionamento.
- FIR: $y(k) = \sum_{k=0}^{n} a_k x(n-k)$

$$Y(z) = \sum_{k=0}^{n} a_k z^{-k} X(z)$$

- IIR: ricorsivi

$$y(n) = \sum_{l=0}^{L} a_l x(n-l) + \sum_{m=1}^{M} b_m y(n-m)$$

$$Y(z) = \sum_{l=0}^{L} a_l z^{-l} X(z) + \sum_{m=1}^{M} b_m z^{-m} Y(z)$$

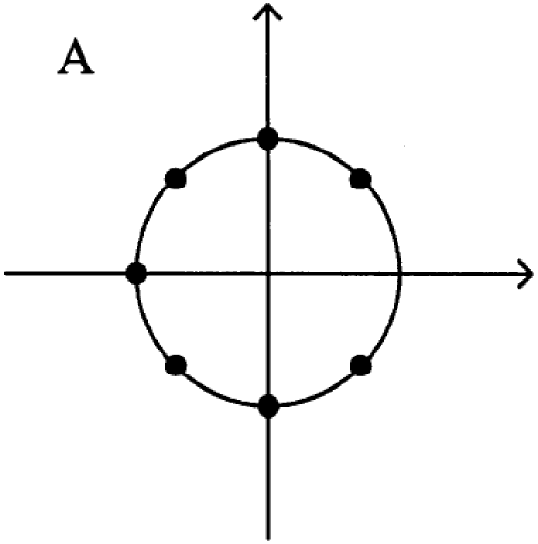
- Funzione di trasferimento H(z)

$$Y(z) = \mathbf{H}(z) X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{n=0}^{N} a_n z^{-n}}$$

Filtri numerici: esempi

- Moving average(FIR):



$$y_n = \frac{1}{L+1}\sum_{l=0}^{L} x_{n-l}$$

$$Y(z) = \frac{1}{L+1}\sum_{l=0}^{L} z^{-l} X(z)$$

$$H(z) \propto \sum_{l=0}^{L} z^{-l} = \frac{1-(z^{-1})^{L+1}}{1-z^{-1}} = \frac{z^{L+1}-1}{z-1} z^{-L}$$

Trascurando gli L poli nell'origine abbiamo L zeri equispaziati sul cerchio unitario. Le corrispondenti sinusoidi sono bloccate dal filtro MA

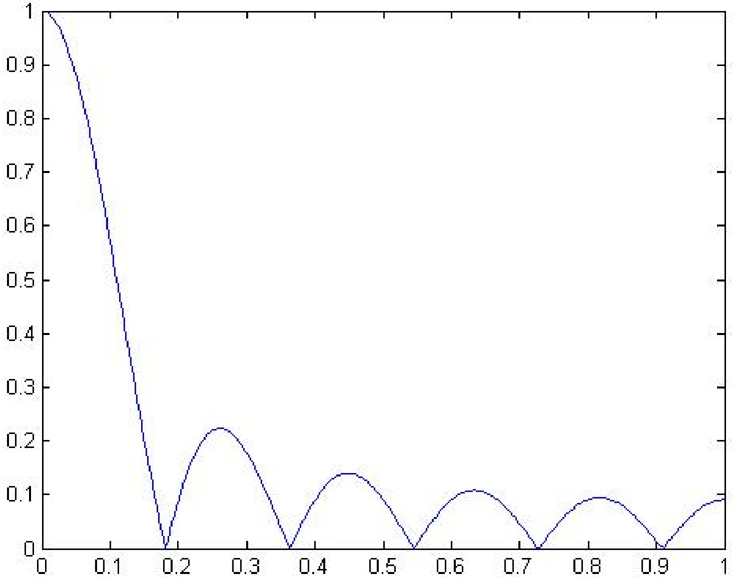
Media mobile (continua)

A meno di fasi, sostituendo a z la sua espressione: $e^{i\omega}$

$$\frac{1}{N+1}\frac{z^{N+1}-1}{(z-1)}=\frac{1}{N+1}\frac{e^{i\omega(N+1)}-1}{(e^{i\omega}-1)}=\frac{1}{N+1}\frac{e^{\frac{i\omega(N+1)}{2}}-e^{-\frac{i\omega(N+1)}{2}}}{(e^{\frac{i\omega}{2}}-e^{\frac{i\omega}{2}})}e^{\frac{i\omega N}{2}}\approx$$

$$\approx\frac{\sin\frac{N+1}{2}\omega}{(N+1)\sin\frac{\omega}{2}}$$

Si azzera quando ω(N+1)/2=kπ ovvero quando ω=2π k/(N+1)



Esempio IIR:DC Blocker

- Si vuole bloccare la componente a frequenza 0 e far passare le altre.
- $H(z)= z-1=z(1-z^{-1})$ ha uno zero banale in $z=0$ e uno in $z=1$ =>
  $|H(\omega)|^2 = 2(1-\cos(\omega))$
  non è un taglio molto netto…
- Aggiungiamo un polo sull'asse reale all'interno del cerchio unitario ma vicino al bordo:

$$H(z) = \frac{z-1}{z-\beta} = \frac{1-z^{-1}}{1-\beta z^{-1}}$$

$$\beta = 1-\varepsilon$$

$$\varepsilon << 1$$

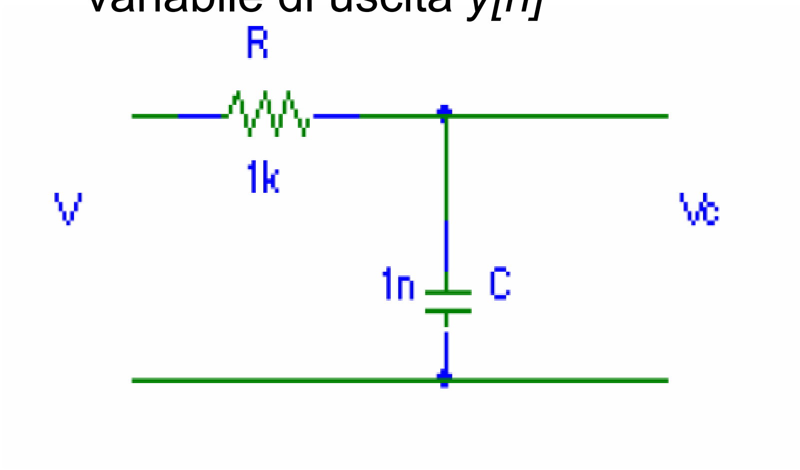
$$y(n) = \beta y(n-1) + x(n) - x(n-1)$$

Il filtro passa basso: dal circuito RC al filtro numerico

- L'equazione differenziale che descrive un circuito RC è:

$$V = RC\frac{dV_c}{dt} + V_c$$

Consideriamo V come la nostra variabile di ingresso campionata *x[n]* e Vc la variabile di uscita *y[n]*



Se T è il periodo di campionamento, allora:

$x[n]=RC/T(y[n]-y[n-1])+y[n]$.

Raccogliendo I termini:

$x[n]=(RC/T+1)y[n]-RC/T\ y[n-1]$

…e, passando alla trasformata Z:

$X(z)=(RC/T+1)Y(z)-z^{-1}(RC/T)Y(Z)$

Dunque:

$$H = \frac{1}{\left(1+\frac{RC}{T}\right)-\frac{RC}{T}z^{-1}} = \frac{\frac{1}{1+\frac{\omega}{\omega_0}}}{1-\left(\frac{\frac{\omega}{\omega_0}}{1+\frac{\omega}{\omega_0}}\right)z^{-1}}$$

Il filtro passa basso: dal circuito RC al filtro numerico

Dette:

$$a = \frac{1}{1 + \frac{RC}{T}}; \; b = \frac{\frac{RC}{T}}{1 + \frac{RC}{T}}$$

$$\Rightarrow \; H = \frac{a}{1 - bz^{-1}} \Rightarrow \; y[n] = ax[n] - by[n-1]$$

Per es.: per ω=1/T=1 kHz e ω_0=100 Hz=> a= 1/11; b= 10/11

y[n]=0.09x[n]-0.9y[n-1]