solution may be an even or odd function of z. In the first case,

$$\psi^{(i)} = A \cos k_z z \, e^{ik_x x}, \quad \psi^{(e)} = Be^{-k_x|z|} e^{ik_x x},$$

with $\mu k_x^2 = -k_z^2$ (the wave vector is in the xz-plane); the boundary conditions give the relation.

$$\tan k_z L = k_x/k_z. \tag{3}$$

In the second case,

$$\psi^{(i)} = A \sin k_z z \, e^{ik_x x}, \quad \psi^{(e)} = \pm Be^{-k_x|z|} e^{ik_x x},$$

and the boundary conditions give

$$\tan k_z L = -k_z/k_x. \tag{4}$$

The demagnetizing factor of the slab is $n^{(z)} = 1$, and the demagnetizing field is therefore $-4\pi M$. With $\mu(\omega)$ from (1), we find the oscillation frequency

$$\omega^2 = \gamma^2 (M\beta + \mathfrak{H} - 4\pi M)(M\beta + \mathfrak{H} - 4\pi M \cos^2 \theta), \tag{5}$$

where θ is the angle between $\mathbf{k}$ and the z-axis. For any value of k_x there is a discrete infinity of k_z values given by the conditions (3) and (4). The corresponding frequencies are given by (5), and depend only on the ratio k_x/k_z. All possible frequency values lie in the range

$$\gamma(M\beta + \mathfrak{H} - 4\pi M) \leqslant \omega \leqslant \gamma[(M\beta + \mathfrak{H} - 4\pi M)(M\beta + \mathfrak{H})]^{\frac{1}{2}}.$$

As $k_z \to 0$, only symmetrical oscillations are possible, and from (3) $k_x L \sim (k_z L)^2$, i.e. is a second-order small quantity. Accordingly, we put $\theta = 0$ in (5) and obtain a frequency equal to the homogeneous resonance frequency, as it should be.

§80. The field energy in dispersive media

The formula

$$\mathbf{S} = c\mathbf{E} \times \mathbf{H}/4\pi \tag{80.1}$$

for the energy flux density remains valid in variable electromagnetic fields, even if dispersion is present. This is evident from the arguments given at the end of §30: on account of the continuity of the tangential components of $\mathbf{E}$ and $\mathbf{H}$, formula (80.1) follows from the condition that the normal component of $\mathbf{S}$ be continuous at the boundary of the body and the validity of the same formula in the vacuum outside the body.

The rate of change of the energy in unit volume of the body is div $\mathbf{S}$. Using Maxwell's equations, we can write this expression as

$$-\operatorname{div} \mathbf{S} = \frac{1}{4\pi}\left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}\right); \tag{80.2}$$

see (75.15). In a dielectric medium without dispersion, when ε and μ are real constants, this quantity can be regarded as the rate of change of the electromagnetic energy

$$U = (\varepsilon \mathbf{E}^2 + \mu \mathbf{H}^2)/8\pi, \tag{80.3}$$

which has an exact thermodynamic significance: it is the difference between the internal energy per unit volume with and without the field, the density and entropy remaining unchanged.

In the presence of dispersion, no such simple interpretation is possible. Moreover, in the general case of arbitrary dispersion, the electromagnetic energy cannot be rationally defined as a thermodynamic quantity. This is because the presence of dispersion in general signifies a dissipation of energy, i.e. a dispersive medium is also an absorbing medium.

To determine this dissipation, let us consider a monochromatic electromagnetic field. By averaging with respect to time the expression (80.2), we find the steady inflow of energy per unit time and volume from the external sources which maintain the field. Since the amplitude of the field is assumed constant, all of this energy goes to compensate the dissipation. Under the conditions stated, therefore, the time average of (80.2) is the mean quantity Q of heat evolved per unit time and volume.

Since the expression (80.2) is quadratic in the fields, all quantities must be written in real form. If, as is convenient for a monochromatic field, we take **E** and **H** to be complex, then in (80.2) we must substitute for **E** and $\partial\mathbf{D}/\partial t$ respectively $\frac{1}{2}(\mathbf{E}+\mathbf{E}^*)$ and $\frac{1}{2}(-i\omega\varepsilon\mathbf{E}+i\omega\varepsilon^*\mathbf{E}^*)$, and similarly for **H** and $\partial\mathbf{B}/\partial t$. On averaging with respect to time, the products $\mathbf{E}\cdot\mathbf{E}$ and $\mathbf{E}^*\cdot\mathbf{E}^*$, which contain factors $e^{\mp 2i\omega t}$, give zero, leaving

$$Q = \frac{i\omega}{16\pi}\left[(\varepsilon^*-\varepsilon)\mathbf{E}\cdot\mathbf{E}^* + (\mu^*-\mu)\mathbf{H}\cdot\mathbf{H}^*\right] = \frac{\omega}{8\pi}(\varepsilon''|\mathbf{E}|^2+\mu''|\mathbf{H}|^2). \tag{80.4}$$

This expression can also be written

$$Q = \omega(\varepsilon''\overline{\mathbf{E}^2}+\mu''\overline{\mathbf{H}^2})/4\pi, \tag{80.5}$$

where **E** and **H** are the real fields, and the bar denotes an average with respect to time; cf. the last footnote to § 59.

It is easy to derive also a formula for the energy dissipation in a non-monochromatic field which tends sufficiently rapidly to zero as $t \to \pm\infty$. In this case it is appropriate to consider the dissipation not per unit time but over the whole duration of the existence of the field.

Expanding the field $\mathbf{E}(t)$ as a Fourier integral, we write

$$\mathbf{E}(t) = \int_{-\infty}^{\infty} \mathbf{E}_\omega e^{-i\omega t}\, d\omega/2\pi,$$

$$\frac{\partial\mathbf{D}(t)}{\partial t} = -i\int_{-\infty}^{\infty} \omega\,\varepsilon(\omega)\mathbf{E}_\omega e^{-i\omega t}\, d\omega/2\pi,$$

with $\mathbf{E}_{-\omega} = \mathbf{E}_\omega^*$. Writing the products of these quantities as a double integral and then integrating with respect to time, we have

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} = -\frac{i}{4\pi}\int_{-\infty}^{\infty} \omega\varepsilon(\omega)\mathbf{E}_\omega\cdot\mathbf{E}_{\omega'}\, e^{-i(\omega+\omega')t}\frac{d\omega\, d\omega'}{(2\pi)^2}\, dt.$$

The integration with respect to t is effected by means of the formula

$$\int_{-\infty}^{\infty} e^{-i(\omega+\omega')t}\, dt = 2\pi\delta(\omega+\omega'),$$

and the delta function is then eliminated by the integration over ω'. The result is

$$-\frac{i}{4\pi}\int_{-\infty}^{\infty} \omega\varepsilon(\omega)|\mathbf{E}_\omega|^2\frac{d\omega}{2\pi}.$$

On substituting $\varepsilon = \varepsilon' + i\varepsilon''$, the term in $\varepsilon'(\omega)$ vanishes in the integration, since the integrand is an odd function of ω. With a similar expression for the magnetic field, we have finally

$$\int_{-\infty}^{\infty} Q \, dt = \frac{1}{4\pi} \int_{-\infty}^{\infty} \omega \left[\varepsilon''(\omega)|\mathbf{E}_\omega|^2 + \mu''(\omega)|\mathbf{H}_\omega|^2\right] \frac{d\omega}{2\pi}; \tag{80.6}$$

the integral from $-\infty$ to ∞ can be replaced by twice the integral from 0 to ∞.

These formulae show that the absorption (dissipation) of energy is determined by the imaginary parts of ε and μ. The two terms in (80.5) are called the *electric* and *magnetic losses* respectively. On account of the law of increase of entropy, the sign of these losses is determinate: the dissipation of energy is accompanied by the evolution of heat, i.e. $Q > 0$. It therefore follows that the imaginary parts of ε and μ are always positive:

$$\varepsilon'' > 0, \qquad \mu'' > 0 \tag{80.7}$$

for all substances and at all (positive) frequencies.† The signs of the real parts of ε and μ for $\omega \neq 0$ are subject to no physical restriction.

Any non-steady process in an actual body is to some extent thermodynamically irreversible. The electric and magnetic losses in a variable electromagnetic field therefore always occur to some extent, however slight. That is, the functions $\varepsilon''(\omega)$ and $\mu''(\omega)$ are not exactly zero for any frequency other than zero. We shall see in § 81 that this statement is of fundamental importance, although it does not exclude the possibility of only very small losses in certain frequency ranges. Such ranges, in which ε'' and μ'' are very small in comparison with ε' and μ', are called *transparency ranges*. It is possible to neglect the absorption in these ranges and to introduce the concept of the internal energy of the body in the electromagnetic field, in the same sense as in a static field. To determine this quantity, it is not sufficient to consider a purely monochromatic field, since the strict periodicity results in no steady accumulation of electromagnetic energy. Let us therefore consider a field whose components have frequencies in a narrow range about some mean value ω_0. The field strengths can be written

$$\mathbf{E} = \mathbf{E}_0(t)e^{-i\omega_0 t}, \qquad \mathbf{H} = \mathbf{H}_0(t)e^{-i\omega_0 t}, \tag{80.8}$$

where $\mathbf{E}_0(t)$ and $\mathbf{H}_0(t)$ are functions of time which vary only slowly in comparison with the factor $e^{-i\omega_0 t}$. The real parts of these expressions are to be substituted on the right-hand side of (80.2), and we then average with respect to time over the period $2\pi/\omega_0$, which is small compared with the time of variation of the factors $\mathbf{E}_0$ and $\mathbf{H}_0$.

The first term in (80.2), with $\mathbf{E}$ written in complex form, is

$$\tfrac{1}{2}(\mathbf{E} + \mathbf{E}^*) \cdot \tfrac{1}{2}(\dot{\mathbf{D}} + \dot{\mathbf{D}}^*)/4\pi,$$

and similarly for the second term. The products $\mathbf{E} \cdot \dot{\mathbf{D}}$ and $\mathbf{E}^* \cdot \dot{\mathbf{D}}^*$ vanish when averaged

† This statement applies to bodies which, in the absence of the variable field, are in thermodynamic equilibrium; we assume this condition to hold. If the body is not in thermal equilibrium, then Q may in principle be negative. The second law of thermodynamics requires only a net increase in entropy as a result of the effects of the variable electromagnetic field and of the absence of thermodynamic equilibrium, the latter effect being independent of the presence of the field. An example of such a body is one in which all the atoms have been excited artificially (i.e. not by spontaneous thermal excitation, but by an external "pumping" field).

over time, and can therefore be ignored, leaving

$$\frac{1}{16\pi}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}^*}{\partial t}+\mathbf{E}^*\cdot\frac{\partial\mathbf{D}}{\partial t}\right). \tag{80.9}$$

We write the derivative $\partial\mathbf{D}/\partial t$ as $\hat{f}\mathbf{E}$, where $\hat{f}$ is the operator $\partial\hat{\varepsilon}/\partial t$, and ascertain the effect of this operator on a function of the form (80.8). If $\mathbf{E}_0$ were a constant, we should have simply $\hat{f}\mathbf{E}=f(\omega)\mathbf{E}$, where $f(\omega)=-i\omega\varepsilon(\omega)$. We expand the function $\mathbf{E}_0(t)$ as a series of Fourier components $\mathbf{E}_{0\alpha}e^{-i\alpha t}$, with constant $\mathbf{E}_{0\alpha}$. Since $\mathbf{E}_0(t)$ varies only slowly, this series will include only components with $\alpha \ll \omega_0$. We can therefore put

$$\begin{aligned}\hat{f}\mathbf{E}_{0\alpha}e^{-i(\omega_0+\alpha)t} &= f(\alpha+\omega_0)\mathbf{E}_{0\alpha}e^{-i(\omega_0+\alpha)t}\\ &\cong f(\omega_0)\mathbf{E}_{0\alpha}e^{-i(\omega_0+\alpha)t}+\alpha\frac{\mathrm{d}f(\omega_0)}{\mathrm{d}\omega_0}\mathbf{E}_{0\alpha}e^{-i(\omega_0+\alpha)t}.\end{aligned}$$

Summing the Fourier components, we have

$$\hat{f}\mathbf{E}_0(t)e^{-i\omega_0 t}=f(\omega_0)\mathbf{E}_0e^{-i\omega_0 t}+i\frac{\mathrm{d}f(\omega_0)}{\mathrm{d}\omega_0}\frac{\partial\mathbf{E}_0}{\partial t}e^{-i\omega_0 t}.$$

Omitting henceforward the suffix 0 to ω, we thus obtain

$$\frac{\partial\mathbf{D}}{\partial t}=-i\omega\varepsilon(\omega)\mathbf{E}+\frac{\mathrm{d}(\omega\varepsilon)}{\mathrm{d}\omega}\frac{\partial\mathbf{E}_0}{\partial t}e^{-i\omega t}. \tag{80.10}$$

Substituting this expression in (80.9) and neglecting the imaginary part of $\varepsilon(\omega)$ gives

$$\frac{1}{16\pi}\frac{\mathrm{d}(\omega\varepsilon)}{\mathrm{d}\omega}\left(\mathbf{E}_0{}^*\cdot\frac{\partial\mathbf{E}_0}{\partial t}+\mathbf{E}_0\cdot\frac{\partial\mathbf{E}_0{}^*}{\partial t}\right)=\frac{1}{16\pi}\frac{\mathrm{d}(\omega\varepsilon)}{\mathrm{d}\omega}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}^*),$$

since $\mathbf{E}\cdot\mathbf{E}^*=\mathbf{E}_0\cdot\mathbf{E}_0{}^*$. Adding a similar expression involving the magnetic field, we conclude that the steady rate of change of the energy in unit volume is given by $\mathrm{d}\bar{U}/\mathrm{d}t$, where

$$\bar{U}=\frac{1}{16\pi}\left[\frac{\mathrm{d}(\omega\varepsilon)}{\mathrm{d}\omega}\mathbf{E}\cdot\mathbf{E}^*+\frac{\mathrm{d}(\omega\mu)}{\mathrm{d}\omega}\mathbf{H}\cdot\mathbf{H}^*\right]. \tag{80.11}$$

In terms of the real fields $\mathbf{E}$ and $\mathbf{H}$ this expression can be written

$$\bar{U}=\frac{1}{8\pi}\left[\frac{\mathrm{d}(\omega\varepsilon)}{\mathrm{d}\omega}\overline{E^2}+\frac{\mathrm{d}(\omega\mu)}{\mathrm{d}\omega}\overline{H^2}\right] \tag{80.12}$$

(L. Brillouin, 1921).

This is the required result: $\bar{U}$ is the mean value of the electromagnetic part of the internal energy per unit volume of a transparent medium. If there is no dispersion, ε and μ are constants, and (80.12) becomes the mean value of (80.3), as it should.

If the external supply of electromagnetic energy to the body is cut off, the absorption which is always present (even though very small) ultimately converts the energy $\bar{U}$ entirely into heat. Since, by the law of increase of entropy, there must be evolution and not absorption of heat, we must have $\bar{U}>0$. It therefore follows, by (80.11), that the inequalities

$$\mathrm{d}(\omega\varepsilon)/\mathrm{d}\omega>0,\quad \mathrm{d}(\omega\mu)/\mathrm{d}\omega>0 \tag{80.13}$$

must hold. In reality, these conditions are necessarily fulfilled, by virtue of more stringent inequalities always satisfied by the functions $\varepsilon(\omega)$ and $\mu(\omega)$ in transparency ranges (see the footnote to §84).

It should be emphasized once more that the expression (80.12) has been derived in the first approximation with respect to the frequencies α with which the amplitude $\mathbf{E}_0(t)$ varies. It is therefore valid only for fields whose amplitude varies sufficiently slowly with time. This comment applies also to the calculation of the stress tensor in §81.

§81. The stress tensor in dispersive media

Another topic of considerable interest is the determination of the (time) averaged stress tensor giving the forces on matter in a variable electric field. We shall show that, even when dispersion is present (but absorption is again absent), the expression for this tensor contains no derivatives with respect to the frequency, unlike the energy equation (80.12). In particular, for a transparent isotropic dispersive fluid in a monochromatic electric field, the mean value $\bar{\sigma}_{ik}$ is derived from (15.9) by simply replacing ε by $\varepsilon(\omega)$ and the products E_iE_k and $\mathbf{E}^2$ by their mean values $\overline{E_iE_k}$ and $\overline{\mathbf{E}^2}$ (L. P. Pitaevskiĭ, 1960).

To prove this, we return to the proof given in §15 and reformulate it to some extent. We there considered a dielectric-filled plane capacitor and determined the stress tensor from the equality between the work done by the ponderomotive forces in a movement of the plates and the change in the appropriate thermodynamic potential. We now write this condition for total quantities (not those per unit area), in the form

$$A\sigma_{ik}\xi_i n_k = (\delta\,\mathscr{U})_{\mathscr{S},e}, \tag{81.1}$$

where A is the area of the capacitor plates. The potential $\tilde{\mathscr{F}}$ is here replaced by the ordinary energy $\mathscr{U}$, the change in which is taken for fixed values of the entropy $\mathscr{S}$ of the dielectric and of the total charges $\pm e$ on the capacitor plates (instead of a fixed potential ϕ), and the result $(\delta\tilde{\mathscr{F}})_{T,\phi} = (\delta\,\mathscr{U})_{\mathscr{S},e}$ given by the theorem of small increments is used.

In the form (81.1), this condition has a particularly simple significance: a thermally insulated capacitor with fixed charges on the plates is an electrically closed system. If mechanical work is done on it (to move the plates) by an external source, all the work goes to increase the energy of the capacitor. The latter is

$$\mathscr{U} = \mathscr{U}_0 + e^2/2C, \tag{81.2}$$

where $\mathscr{U}_0$ is the energy of the dielectric in the absence of the field (for the same value of the entropy $\mathscr{S}$) and C is the capacitance. For a plane capacitor, $C = \varepsilon A/4\pi h$, where h is the distance between the plates. Hence

$$(\delta\,\mathscr{U})_{\mathscr{S},e} = (\delta\,\mathscr{U}_0)_{\mathscr{S}} - \frac{e^2}{2C^2}(\delta C)_{\mathscr{S}}. \tag{81.3}$$

On expressing δC in terms of the displacement ξ of the plates (taking into account the dependence of ε on the density of the dielectric, which is changed by the displacement), we immediately find (15.9)†; since the result is obvious, it will not be given here.

† It is expressed in terms of different variables, with the adiabatic instead of the isothermal derivative $\partial\varepsilon/\partial\rho$ and function P_0. The two forms are, of course, equivalent.