Appendix F
Collins–Soper Frame

To prove Eqs. (2.86), we use variables that are manifestly covariant under rotation about the z' axis in the CS frame (the rest frame of the lepton pair: see Figure 2.24). The magnitude of the lepton momenta in their rest frame is given by

$$k = |\boldsymbol{k}| = Q/2\,, \quad Q = \sqrt{(k_1 + k_2)} \tag{F.1}$$

where k_1, k_2 are four momenta of the leptons in the CM frame of the two incoming hadrons. Then, without loss of generality, we can choose the four momenta in the CS frame by

$$\begin{aligned} k'_{10} &= k\,, \quad k'_{1x} = k \sin\theta\,, \quad k'_{1z} = k\cos\theta\,, \\ k'_{20} &= k\,, \quad k'_{2x} = -k\sin\theta\,, \quad k'_{1z} = -k\cos\theta \end{aligned} \tag{F.2}$$

We have fixed the x'–z' plane defined by $\boldsymbol{P}_A$, $\boldsymbol{P}_B$, $\boldsymbol{Q}$. Then, the direction of the y' axis is defined by

$$\hat{\boldsymbol{R}}_T \equiv (0, 0, 1, 0) = \frac{\boldsymbol{P}_A \times \boldsymbol{Q}}{|\boldsymbol{P}_A \times \boldsymbol{Q}|} \tag{F.3}$$

By the boost in the $\boldsymbol{Q}_T$ direction with $\beta\gamma = |\boldsymbol{Q}_T|/Q, \gamma = \sqrt{Q^2 + \boldsymbol{Q}_T^2}/Q$, they become variables in the $*$ frame. We denote the variables with * in the $*$ frame.

$$\begin{aligned} k^*_{10} &= \gamma k(1 + \beta\sin\theta)\,, \quad k^*_{1x} = \gamma k(\ \ \sin\theta + \beta)\,, \quad k^*_{1z} = k\cos\theta \\ k^*_{20} &= \gamma k(1 - \beta\sin\theta)\,, \quad k^*_{2x} = \gamma k(-\sin\theta + \beta)\,, \quad k^*_{2z} = -k\cos\theta \end{aligned} \tag{F.4}$$

$$\begin{aligned} k_1^{+*} &= (k^*_{10} + k^*_{1z})/\sqrt{2} = k[\cos\theta + \gamma(1 + \beta\sin\theta)]/\sqrt{2} \\ k_2^{-*} &= (k^*_{20} - k^*_{2z})/\sqrt{2} = k[\cos\theta + \gamma(1 - \beta\sin\theta)]/\sqrt{2} \\ k_1^{-*} &= (k^*_{10} - k^*_{1z})/\sqrt{2} = k[-\cos\theta + \gamma(1 + \beta\sin\theta)]/\sqrt{2} \\ k_2^{+*} &= (k^*_{20} + k^*_{2z})/\sqrt{2} = k[-\cos\theta + \gamma(1 - \beta\sin\theta)]/\sqrt{2} \end{aligned} \tag{F.5}$$

Elementary Particle Physics, Volume 2: Foundations of the Standard Model, First Edition. Yorikiyo Nagashima.

$$
\begin{aligned}
k_1^{+*}k_2^{-*} - k_1^{-*}k_2^{+*} &= (k^2/2)[\{(\cos\theta + \gamma)^2 - \beta^2\gamma^2\sin^2\theta\} \\
&\quad - \{(-\cos\theta + \gamma)^2 - \beta^2\gamma^2\sin^2\theta\}] \\
&= 2k^2\gamma\cos\theta = 2\left(\frac{Q}{2}\right)^2 \frac{\sqrt{Q^2 + Q_\mathrm{T}^2}}{Q}\cos\theta \\
\therefore \quad \cos\theta &= \frac{2}{Q\sqrt{Q^2 + \boldsymbol{Q}_\mathrm{T}^2}}(k_1^+k_2^- - k_1^-k_2^+)^* \\
&= \frac{2}{Q\sqrt{Q^2 + \boldsymbol{Q}_\mathrm{T}^2}}(k_1^+k_2^- - k_1^-k_2^+) \qquad \text{(F.6)}
\end{aligned}
$$

where z boost invariance was used in deriving the last equality.

$$
\begin{aligned}
\Delta_x^* &= k_{1x}^* - k_{2x}^* = \gamma k(\sin\theta + \beta) - \gamma k(-\sin\theta + \beta) = 2\gamma k\sin\theta \\
\boldsymbol{\Delta}_\mathrm{T}^{2*} &= \Delta_x^{2*} = 4k^2\gamma^2\sin^2\theta = \gamma^2 Q^2\sin\theta^2 \\
(\boldsymbol{\Delta}_\mathrm{T}^* \cdot \boldsymbol{Q}_\mathrm{T}^*)^2 &= (\Delta_x^*)^2(Q_\mathrm{T}^*)^2 = (Q\gamma\sin\theta)^2(Q_\mathrm{T})^2 = Q^2Q_\mathrm{T}^2\gamma^2\sin^2\theta \\
\frac{(\boldsymbol{\Delta}_t^* \cdot \boldsymbol{Q}_\mathrm{T})^2}{Q^2(Q^2 + Q_\mathrm{T}^2)} &= \beta^2\gamma^2\sin^2\theta \\
\therefore \quad \frac{\boldsymbol{\Delta}_\mathrm{T}^{2*}}{Q^2} - \frac{(\boldsymbol{\Delta}_\mathrm{T}^* \cdot \boldsymbol{Q}_\mathrm{T}^*)^2}{Q^2(Q^2 + Q_\mathrm{T}^2)} &= \gamma^2\sin^2\theta - \beta^2\gamma^2\sin^2\theta = \sin^2\theta \qquad \text{(F.7)}
\end{aligned}
$$

Thus far, we have only used covariant variables under rotation around the z' axis. An explicit azimuthal angle dependence of the momentum of the produced lepton pair can be derived using the $\hat{\boldsymbol{R}}_\mathrm{T}$ vector defined in Eq. (F.3).

$$
\begin{aligned}
\boldsymbol{\Delta}_\mathrm{T}^* \cdot \hat{\boldsymbol{R}}_\mathrm{T} &= \Delta_{Ty}^* = Q\sin\theta\sin\phi \\
\boldsymbol{\Delta}_\mathrm{T}^* \cdot \hat{\boldsymbol{Q}}_\mathrm{T}^* &= \Delta_{Tx}^* = \gamma Q\sin\theta\cos\phi \\
\therefore \quad \frac{(Q^2 + \boldsymbol{Q}_\mathrm{T}^2)^{1/2}}{Q}\frac{\boldsymbol{\Delta}_\mathrm{T}^* \cdot \hat{\boldsymbol{R}}_\mathrm{T}}{\boldsymbol{\Delta}_\mathrm{T}^* \cdot \hat{\boldsymbol{Q}}_\mathrm{T}^*} &= \gamma\frac{Q\sin\theta\sin\phi}{\gamma Q\sin\theta\cos\phi} = \tan\phi \qquad \text{(F.8)}
\end{aligned}
$$

Notice that the right-hand side of Eqs. (F.6), (F.7) and (F.8) are all manifestly invariant under z boosts, so that the equations apply equally well using laboratory frame variables as well as those in center of mass frame of the two incoming hadrons.