

Short Note

A constant Q technique for the numerical simulation of attenuation of seismic body waves

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INTRODUCTION

The knowledge of a viscoelastic model is required to simulate the propagation of seismic body waves in attenuating media. For computations in the time domain, the medium is modeled as a superposition of basic elements, each with a different relaxation time (Emmerich and Korn, 1987). Different strategies have been adopted to find relaxation times (Day and Minster, 1984; Emmerich and Korn, 1987; Carcione et al., 1988; Blanch et al., 1995; Xu and McMechan, 1998). Some of these strategies are based on the requirement that the quality factor Q , which is usually formulated as a function of angular frequency (ω), must be nearly constant within an assigned frequency band, according to experimental results (McDonal et al., 1958; Hamilton, 1972; Newman and Worthington, 1982; Gibbs et al., 1994).

We present a model based on the superposition of standard linear solids (SLS). The model is constrained by requiring that, for an assigned value Q_0 of the quality factor, $Q^{-1}(\omega) \simeq Q_0^{-1}$ over a given frequency interval. This condition is sufficient to determine the relaxation times and the other parameters. Numerical 1D tests are performed to assess the reliability of the model.

THE MODEL

We consider the relaxation function of an array of SLS (Liu et al., 1976; Carcione, 1993; Robertsson et al., 1994):

$$\chi(t) = E_R \left[1 - \sum_{j=1}^N \left(1 - \frac{E_{Uj}}{E_{Rj}} \right) e^{-t/\tau_j} \right], \quad (1)$$

where the subscripts R and U refer to relaxed and unrelaxed moduli, respectively, E_R is the relaxed modulus of the system, and τ_j is the relaxation time of the j th SLS. We suppose that $E_{Uj}/E_{Rj} = \text{constant} = K \forall j$. Computing the quality factor

$Q(\omega)$ with this simplification gives

$$\frac{1}{Q(\omega)} = \frac{cB(\omega)}{1 - cA(\omega)}, \quad (2)$$

where

$$A(\omega) = \sum_{j=1}^N \frac{1}{1 + \omega^2 \tau_j^2}, \quad B(\omega) = \sum_{j=1}^N \frac{\omega \tau_j}{1 + \omega^2 \tau_j^2}, \quad (3)$$

and $c = (1 - E_R/E_U)/N$, with $E_U = E_R[1 + N(K - 1)]$. The model will be completely specified if the number N of elements, the value of c , and the relaxation times τ_j are given.

To determine the value of c and the relaxation times, we require that, given a value Q_0 of the quality factor, $Q^{-1}(\omega) \simeq Q_0^{-1}$ over the angular frequency interval $[\omega_0, \omega_1]$. This is translated into two conditions: (1) The mean value of Q over the interval must be equal to Q_0 and (2) The variation of Q around the mean value must be the smallest possible.

Let μ_f and σ_f be the mean value and the standard deviation of the function f over $[\omega_0, \omega_1]$, respectively. Let μ and σ be the mean value and standard deviation of Q^{-1} , respectively. The first constraint requires that $\mu = Q_0^{-1}$. This is accomplished by seeking a suitable expression for the constant c . The value of c depends on the ratio E_R/E_U . Since this ratio decreases as Q_0 decreases, we expect c to be a decreasing function of Q_0 . A trial-and-error analysis shows that the condition on μ is satisfied if

$$c = (\mu_B Q_0 + \mu_A)^{-1}, \quad (4)$$

where

$$\mu_A = \frac{1}{v_1 - v_0} \sum_{j=1}^N v_j \tan^{-1} x \Big|_{v_0/v_j}^{v_1/v_j}, \quad (5)$$

$$\mu_B = \frac{1}{2(v_1 - v_0)} \sum_{j=1}^N v_j \log(1 + x^2) \Big|_{v_0/v_j}^{v_1/v_j}. \quad (6)$$

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Here $v_0 = (2\pi)^{-1}\omega_0$, $v_1 = (2\pi)^{-1}\omega_1$, and $v_j = (2\pi\tau_j)^{-1}$. Using for c the expression of equation (4), the computation of μ leads to

$$\begin{aligned}\mu &= \frac{1}{\omega_1 - \omega_0} \int_{\omega_0}^{\omega_1} \frac{cB(\omega)}{1 - cA(\omega)} d\omega \\ &= \frac{1}{\omega_1 - \omega_0} \int_{\omega_0}^{\omega_1} \frac{B(\omega)}{\mu_B Q_0 + \mu_A - A(\omega)} d\omega \\ &\simeq \frac{1}{\omega_1 - \omega_0} \int_{\omega_0}^{\omega_1} \frac{B(\omega)}{\mu_B Q_0} d\omega = \frac{1}{Q_0},\end{aligned}\quad (7)$$

which holds provided that $\mu_B Q_0 \gg \mu_A - A(\omega) \forall \omega \in [\omega_0, \omega_1]$. Under the same condition we obtain

$$\begin{aligned}\frac{\sigma^2}{\mu^2} &= \frac{\mu_{Q_0}^{-2}}{\mu^2} - 1 \\ &= \frac{Q_0^2}{\omega_1 - \omega_0} \int_{\omega_0}^{\omega_1} \frac{B^2(\omega)}{(\mu_B Q_0 + \mu_A - A(\omega))^2} d\omega - 1 \\ &\simeq \frac{Q_0^2}{\omega_1 - \omega_0} \int_{\omega_0}^{\omega_1} \frac{B^2(\omega)}{(\mu_B Q_0)^2} d\omega - 1 \\ &= \frac{\mu_B^2}{\mu_B^2} - 1 = \frac{\sigma_B^2}{\mu_B^2}.\end{aligned}\quad (8)$$

In Figure 1a the ratio σ/μ and μ are plotted as functions of σ_B/μ_B . The ratio σ/μ was computed using the definitions of μ and σ . The ratio σ_B/μ_B has a closed expression. In fact, we have $\sigma_B^2/\mu_B^2 = \mu_B^2/\mu_B^2 - 1$, where μ_B is given in equation (6) and

$$\begin{aligned}\mu_B^2 &= \frac{1}{v_1 - v_0} \left\{ \frac{1}{2} \sum_{j=1}^N v_j \left[\tan^{-1} x - \frac{x}{1+x^2} \right]_{v_0/v_j}^{v_1/v_j} \right. \\ &\quad + 2 \sum_{j=1}^{N-1} \sum_{k=j+1}^N \frac{v_j v_k}{v_j^2 - v_k^2} \left(v_j \tan^{-1} x \Big|_{v_0/v_j}^{v_1/v_j} \right. \\ &\quad \left. \left. - v_k \tan^{-1} x \Big|_{v_0/v_k}^{v_1/v_k} \right) \right\}.\end{aligned}\quad (9)$$

For the computation we chose $Q_0 = 100$, $v_0 = 0.1$ Hz, $v_1 = 15$ Hz, and $N = 3$. The relaxation frequencies were extracted randomly in sets of three in the range 10^{-3} – 10^2 Hz. The results were ordered by increasing values of σ_B/μ_B . The linear trend expected according to equation (8) is clearly seen, so the condition $\mu_B Q_0 \gg \mu_A - A(\omega)$ is fulfilled. This is confirmed by the fact that $\mu = Q_0^{-1}$ for all the extractions.

The general validity of the condition $\mu_B Q_0 \gg \mu_A - A(\omega)$ is very difficult to assess analytically. However, we can show that

$$\begin{aligned}\frac{\mu_A - A(\omega)}{\mu_B} &< \frac{v_1(v_1 - v_0)^{-1} [2 \tan^{-1}(v_1/v_0) - \pi/2] - 1/2}{(v_1/v_0)/[1 + (v_1/v_0)^2]} \\ &= U.\end{aligned}\quad (10)$$

If Q_0 is smaller than U , then an analysis like the one described above should be carried out. Otherwise, we expect the condition $\mu_B Q_0 \gg \mu_A - A(\omega)$ to hold. For example, for $Q_0 = 100$, $v_0 = 2$ Hz, and $v_1 = 60$ Hz, we have $U = 32$; so we expect the linear relationship of equation (8) to be verified. For $v_0 = 0.1$ Hz and $v_1 = 15$ Hz, we have $U = 160$; but equation (8) holds, as shown in Figure 1a.

The second constraint we impose on the model is that the relative variation σ/μ of Q around μ has to be the smallest possible. Equation (8) implies that this constraint can be equivalently imposed on σ_B/μ_B . Thus, we have to find the relaxation times that minimize this ratio over the frequency interval $[\omega_0, \omega_1]$. To find the optimal values of the relaxation times, we used a downhill simplex algorithm (Press et al., 1992). We performed the calculation for different values of N . The results are summarized in Table 1.

The optimal number N of components depends on Q_0 . This is shown in Figure 1b, in which σ/μ is plotted against σ_B/μ_B for $Q_0 = 20$ and $N = 3$ and for relaxation frequencies randomly extracted as for the computations in Figure 1a. We notice a spread of σ_B/μ_B around the linear trend. The fit quality significantly improves if we introduce one more component. The same behavior is observed for μ . In general, the rule of thumb for choosing the number of SLS is to have one component per decade in the frequency range $[v_0, v_1]$ (Blanch et al., 1995). If the quality factor Q_0 is low, then a few more components may be used to increase the accuracy of our method.

NUMERICAL TEST

To assess the reliability of our model, we compare synthetic displacements computed in a 1D homogeneous medium using our model and that of Kjartansson (1979). The latter provides a constant Q and is based on the relaxation function

$$\chi(t) = \frac{M_0}{\Gamma(1 - 2\gamma)} \left(\frac{t}{t_0} \right)^{-2\gamma}, \quad (11)$$

where t_0 is an arbitrary reference time, M_0 is an arbitrary reference modulus, and $\gamma = \tan^{-1}(Q_0^{-1})/\pi$. For the computations performed using our model, we assume the phase velocity at infinite frequency to equal 1 km/s. To compute synthetic displacements, we use a solution of the equation of motion based on the correspondence principle (Bland, 1960).

The first case we consider is a viscoelastic medium with $Q_0 = 100$ and $N = 3$. In Figure 2a we show the quality factor and the phase and group velocity of our model (solid lines). For $\omega_0 \ll \omega \ll \omega_1$, both phase and group velocity are nearly linear with $\log v$. The quality factor has some peaks within

Table 1. Optimized values of the relaxation frequencies for different values of the number N of components.

N	v_1	v_2	v_3	v_4	v_5
2	0.5932654	9.092981			
3	0.1297524	1.517497	14.25790		
4	0.1424290	0.7890241	5.947443	33.43636	
5	$8.434136 \cdot 10^{-2}$	0.3453467	2.036172	11.07403	53.35315

the chosen frequency range, the greatest of which occurs at 1.5 Hz. A Ricker wavelet of unit amplitude and central frequency 1.5 Hz is chosen as source function with the aim of testing the effect on the waveforms of that peak. The amplitude spectrum of the source is shown in Figure 2a (long dashed line). With this choice of the central frequency, the characteristic wavelength of the source is about 660 m. The values M_0 and

t_0 are chosen so that the phase velocity at the central frequency of the source is the same for both models.

In Figures 2b–2d we show the displacements recorded at three regularly spaced offsets. These offsets were chosen considering that Q_0 is the number of wavelengths that a wave with a given frequency can travel before its amplitude drops by a factor $e^{-\pi}$ (Aki and Richards, 1980). The greatest offset

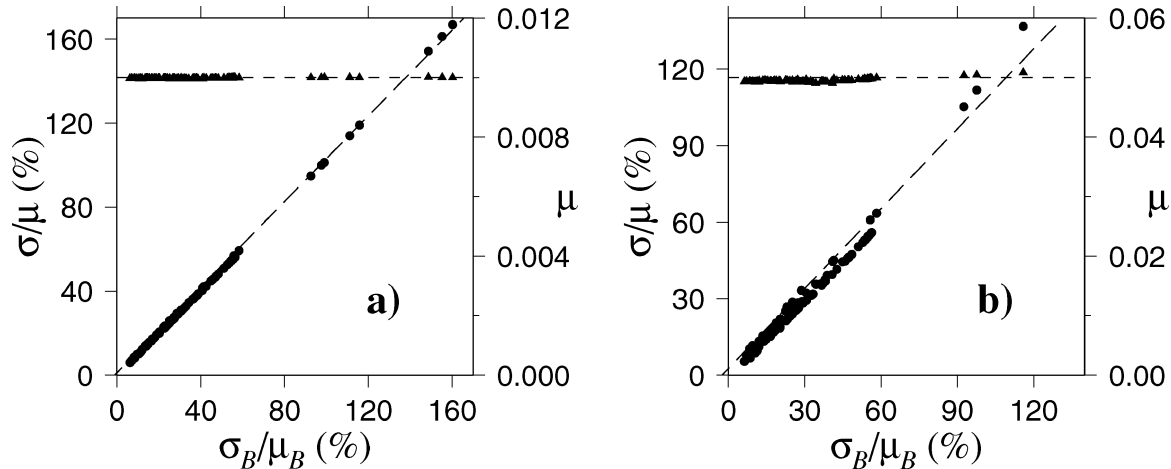


FIG. 1. (a) Mean relative variation of Q (circles) and mean value of Q (triangles) as a function of the mean relative variation of B for $Q_0 = 100$ and $N = 3$. (b) Same as (a) for $Q_0 = 20$ and $N = 3$.

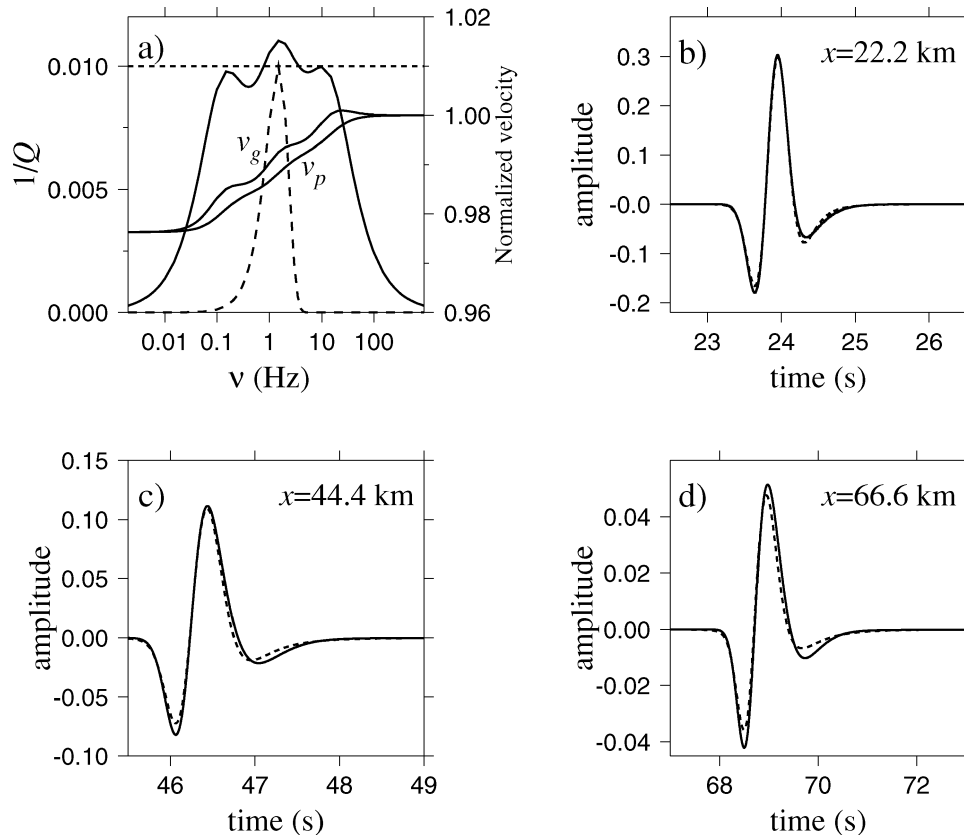


FIG. 2. Comparison between synthetic displacements collected at different offsets x for a viscoelastic medium with Q given by our model (solid line) and by the model of Kjartansson (1979) (dashed line). The source function is a Ricker wavelet of unit amplitude whose amplitude spectrum is shown in (a) (long dashed line). For this simulation $Q_0 = 100$ and $N = 3$. (a) Plot of the quality factor Q , group velocity v_g , and phase velocity v_p . Velocities are normalized to the value of phase velocity at infinite frequency. (b–d) Synthetic displacements.

corresponds to the attenuation of the central frequency of the source.

The general agreement between the solution computed using our model (solid line) and the solution computed using the model of Kjartansson (1979) is satisfactory up to 44.4 km. The maximum difference between the solutions is about 10%, which is the difference between the value of Q^{-1} and Q_0^{-1} at 1.5 Hz. This maximum difference increases to 17% at 66.6 km, but the amplitude of the signal is only 4% of the original value.

We obtain consistent results for the case $Q_0 = 20$ and $N = 4$ (Figure 3). The difference of quality factor at 1.5 Hz between the two models is now very small (see Figure 3a), so we obtain smaller differences between the displacements with respect to the previous case. These differences are less than 4% for the curves in Figures 3b and 3c. Again, the difference increases as we consider offsets at which the signal is strongly attenuated (Figure 3d). Interestingly, in this case the maximum difference is about the same as the difference between the quality factors at low frequency.

DISCUSSION AND CONCLUSION

We have presented a model for the attenuation of seismic body waves in a linear viscoelastic medium based on the superposition of SLS. The parameters of the model are the constant c and the relaxation times τ_j . We determine the relaxation times by minimizing σ_B/μ_B , which has a closed expression. The constant c can then be calculated analytically for any value of the quality factor Q_0 . Numerical test shows

that our model reproduces the waveforms of a constant Q model within the accuracy of the variation of $Q(\omega)$ around Q_0 . The number N of SLS can be chosen according to the rule of thumb of one component per decade in the frequency range considered, increasing the number of components of a few units if a greater accuracy is required.

In the literature another model based on the superposition of N SLSs, with N odd, is proposed by Carcione (1992). The authors find a simplified form of Q based on the knowledge of N relaxation times τ_{0i} . The relaxation frequencies are assumed regularly distributed in the frequency band of interest, and the quality factor at the center of the band is set equal to Q_0 . Our model is not based on a given distribution of relaxation frequencies but allows for the computation of optimal values for them.

Another model based on the superposition of SLS, the tau model, has been proposed by Blanch et al. (1995). They use a simplified form of the relaxation function in equation (1). The simplified relaxation function depends on the parameter τ . The product $Q_0\tau$ is determined by a least-squares minimization of the function $Q(\omega)$ so that τ can be determined easily for any value of Q_0 , as the constant c in our case. The advantage of our model is that it allows one to determine the relaxation times, while the tau model does not. We verified that the tau model and ours give identical waveforms when we choose the parameters so that both models have the same phase velocity at zero and infinite frequency and our relaxation times are used.

Xu and McMechan (1998) also propose a model based on the superposition of N SLS. Their relaxation function does

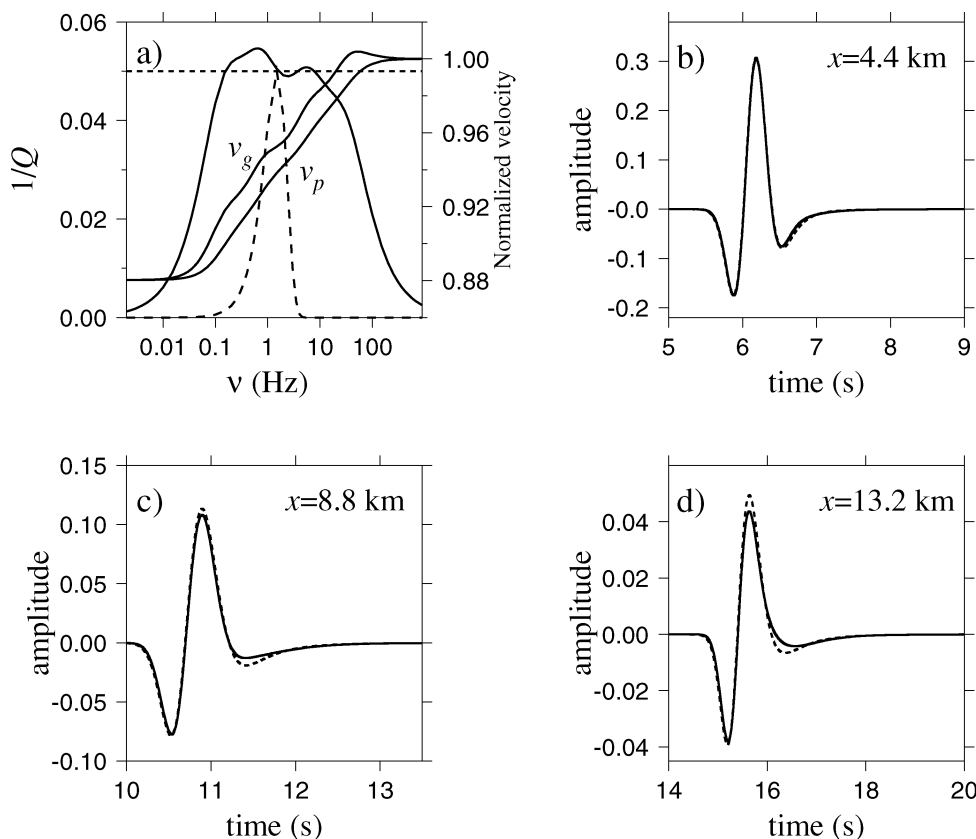


FIG. 3. Same as Figure 2 but for $Q_0 = 20$ and $N = 4$.

not rely on any simplification. Their method aims at finding the best combination of the weights of the SLS instead of the relaxation times, which are given a priori. The numerical minimization they adopt to find the weights depends on the value of Q_p , Q_s , and Poisson's ratio. Although we use a less general relaxation function, our approach is less time consuming when heterogeneous Q media are considered. In fact, the relaxation times are found independently of Q_0 . Once they are known, it is easy to compute c for any value of the quality factor.

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