Generalized Ginzburg-Landau models for non-conventional superconductors

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Abstract

We review some recent extensions of the Ginzburg-Landau model able to describe several properties of non-conventional superconductors. In the first extension, s-wave superconductors endowed with two different critical temperatures are considered, their main thermodynamical and magnetic properties being calculated and discussed. Instead in the second extension we describe spin-triplet superconductivity (with a single critical temperature), studying in detail the main predicted physical properties. A thorough discussion of the peculiar predictions of our models and their physical consequences is as well performed.

1 Introduction

The Higgs mechanism [1, 2], plays a basic role in the macroscopic theory of gapped BCS superconductors, or in the corresponding Ginzburg-Landau (GL) effective theory, where it accounts for the emergence of short-range electromagnetic interactions mediated by massive-like photons (responsible, for example, of the Meissner effect) [3]. In the GL theory this is achieved by means of a complex order parameter ϕ, which can be interpreted as the wave function of the Cooper pair in its center-of-mass frame. The classical phenomenological GL approach entails a (unique) critical temperature T_C, without assuming a particular temperature-dependence of the coefficient $a(T)$ appearing in the effective free energy function for unit volume, expanded up to the $|\phi|^4$ order:

$$F \simeq F_{\rm n} + a(T)|\phi|^2 + \frac{\lambda}{4}\,|\phi|^4\,. \tag{1}$$

Quantity $F_{\rm n}$ indicates the normal-phase (not superconducting) free energy density; while λ, giving the strength of the Cooper pair binding, is assumed to be approximately constant. Ginzburg and Landau only assumed that the coefficient $a(T)$ is positive above T_C, vanishes when the temperature approaches the critical value and becomes negative for $T < T_C$; around the critical temperature changes very smoothly: $a(T) \simeq \dot{a}(T_C)(T - T_C)$. In the alternative

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quantum field approach analyzed below we instead adopt well-defined analytic expressions for $a(T)$ as a function of the temperature.

Actually, because of the interaction of the charged scalar field ϕ with the electromagnetic field A^μ, the order parameter is usually associated to the Higgs field responsible of the U(1) spontaneous symmetry breaking (SSB) [1, 2, 3] occurring during the normal state-superconducting phase transition. As a consequence of the symmetry breaking, due to a non-vanishing expectation value of the order parameter in the ground state below the critical temperature, the photon acquires a mass (causing the Meissner effect) and the system becomes superconducting. By adopting this approach, we can initially start from a relativistically invariant Lagrangian containing the interaction of a single ϕ with A^μ as well as the λ self-interaction (hereafter $\hbar = c = 1$):

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) + m^2 \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \tag{2}$$

where $m^2 > 0$, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, and $D_\mu \equiv \partial_\mu + 2ieA_\mu$ is the covariant derivative ($2e$ is the electric charge of a Cooper pair). Given the above Lagrangian, the effective free energy density at finite temperature is formally identical to the GL expression given in (1). However, despite the outstanding importance of the GL theory in superconductivity, as well as in other physical systems, it has still not been solved exactly beyond the mean-field approximation. Whilst this was not a serious problem for traditional superconductors, where the Ginzburg temperature interval is small around the critical temperature, the situation has changed especially with the advent of high-T_c superconductors. In fact, for these systems, the Ginzburg temperature interval is large and we may expect strong field fluctuations and critical properties beyond the mean-field approximation. Indeed, in high-T_c superconductors several experiments have observed critical effects in the specific heat [4], although the presence of a magnetic field generally makes the situation more complicated. On a theoretical side, the effect of gauge field fluctuations causes great difficulties in the critical phenomena theory and, unlike the simpler ϕ^4 theory for a neutral superfluid, the exact critical behaviour remains unknown. It is well-known that at the mean-field level the superconductive transition is discontinuous, but it seems that this result is confirmed even when field fluctuations are included [5]. This is also confirmed by numerical simulations of lattice models for small values of the Ginzburg-Landau parameter $\varkappa$, while for large $\varkappa$ the results are consistent with a continuous, second-order phase transition [6]. It is thus a general belief that the standard GL model leads to a first-order transition instead of a continuous transition, but several other studies at one-loop (and even at two-loop) approximation have been carried out in recent years (see, for example, [7, 8, 9, 10] and references therein). Some of them entail runaway solutions of the GL equations (pointing towards first-order transitions), while others find a scaling behaviour with a new stable fixed point in the space of static parameters. Also, again beyond the plain mean-field approximation, Kleinert has shown the existence of a tricritical point in a superconductor, by taking the vortex fluctuations into account.

In the present paper we review some extensions of the GL model, recently proposed [11, 12, 13, 14] in order to describe the properties of non-conventional superconductors. In a first model, discussed in the next section, we describe s-wave superconductors endowed with two (slightly) different critical temperatures, and study in detail both the thermodynamical and the magnetic properties of such systems. Instead in section 3 we report on a straightforward generalization of the standard GL model accounting for spin-triplet superconductivity (with a single critical temperature), again studying the main physical features of the medium considered. Finally, in section 4 we summarize and discuss the relevant results obtained.

2 A GL-like model for s-wave superconductors with two critical temperatures

2.1 The model

Let us start by expanding a complex field ϕ as follows

$$\phi \equiv \frac{1}{\sqrt{2}}(\eta_0 + \eta)\, e^{i\theta/\eta_0}\,, \tag{3}$$

where η_0 is a real constant, η and θ are real fields. Then, if we let the scalar field fluctuate around the minimum of the free energy, a condensation of the field η takes place as a result of the $U(1)$ SSB. In Eq. (3) the constant field $\eta_0/\sqrt{2}$ is defined as the expectation value (the condensation value) of the modulus of the scalar field ϕ. Finite-temperature one-loop quantum corrections to the $T = 0$ expression of the free energy density lead to [15]

$$F_{\rm I} = F_{\rm n} + \frac{1}{2} a_{\rm I}(T)\eta_0^2 + \frac{\lambda}{16}\,\eta_0^4 \tag{4}$$

with

$$a_{\rm I} = -m^2 + \frac{\lambda + 4e^2}{16}\,T^2\,. \tag{5}$$

The parameter $a_{\rm I}$ vanishes when the temperature approaches a critical value given by

$$T_1 = 2\sqrt{\frac{4m^2}{\lambda + 4e^2}}\,. \tag{6}$$

Below T_1 the expectation value of η_0^2 which minimizes the free energy function results to be

$$\eta_0^2(T) = -\frac{4a_{\rm I}(T)}{\lambda}\,. \tag{7}$$

Alternatively, we may expand the field ϕ as:

$$\phi \equiv \frac{1}{\sqrt{2}}(\phi_0 + \phi_a + i\phi_b), \tag{8}$$

where ϕ_0 is a real constant, and ϕ_a, ϕ_b are two real scalar fields. Now we assume that a condensation takes place in the field ϕ_a (or, equivalently, in ϕ_b) rather than in the component η. In Eq. (8) the constant field $\phi_0/\sqrt{2}$ is defined as the expectation value of the real part of ϕ. In this case, after such condensation, the effective Helmholtz energy density writes

$$F_{\rm II} = F_{\rm n} + \frac{1}{2} a_{\rm II}(T)\phi_0^2 + \frac{\lambda}{16}\,\phi_0^4 \tag{9}$$

with [2]

$$a_{\rm II} = -m^2 + \frac{\lambda + 3e^2}{12}\,T^2\,. \tag{10}$$

From the vanishing of $a_{\rm II}$ we now derive a *different* critical temperature

$$T_2 = 2\sqrt{\frac{3m^2}{\lambda + 3e^2}}\,. \tag{11}$$

Since $\infty > \lambda > 0$, we correspondingly have $\dfrac{\sqrt{3}}{2} T_1 < T_2 < T_1$. Accordingly, for very large self-interaction, $\lambda/e^2 \to \infty$, *we predict a maximum difference of* 15% *between the two critical temperatures* [11].

Below T_2 the expectation value for ϕ_0^2 which minimizes the free energy function is given by

$$\phi_0^2(T) = -\frac{4a_{\rm II}(T)}{\lambda}\,. \tag{12}$$

We understand the appearing of a new lower critical temperature when expanding the exponential in Eq. (3) in θ/η_0 and comparing with Eq. (8):

$$\begin{aligned}\phi_0 &\sim \eta_0 \\ \phi_a &\sim \eta - \frac{\theta}{2}\left(\frac{\theta}{\eta_0}\right) + \cdots \\ \phi_b &\sim \theta + \eta\left(\frac{\theta}{\eta_0}\right) + \cdots\,.\end{aligned} \tag{13}$$

The degrees of freedom carried out by the real scalar fields ϕ_a, ϕ_b are different from those corresponding to η, θ, and tend to coincide only in the limit $\eta_0 \to \infty$. Actually, in Eqs. (13) the higher orders in η_0^{-1} contribute at the denominator of the expression (6) as an additional $\lambda/3$ term; that is an increased effective self-interaction of the Cooper pairs arises ($\lambda \to \lambda_{\rm eff} = 4\lambda/3$) [11].

Since, as we have seen, two different condensations are allowed to occur inside the same system, we do not a priori exclude any of them. Hence we are led to introduce two order parameters, that is two scalar charged fields: the first one related to the condensation of the modulus of $\phi_{\rm I}$ (the corresponding phase will be hereafter denominated as "phase I"); while the second one related to the condensation of the real part of $\phi_{\rm II}$ ("phase II").

Neglecting possible interactions between the two scalar fields, the total Lagrangian now writes:

$$\mathcal{L} = (D_\mu\phi_{\rm I})^\dagger\,(D^\mu\phi_{\rm I}) + m^2\phi_{\rm I}^\dagger\phi_{\rm I} - \frac{\lambda}{4}(\phi_{\rm I}^\dagger\phi_{\rm I})^2 + (D_\mu\phi_{\rm II})^\dagger\,(D^\mu\phi_{\rm II}) + m^2\phi_{\rm II}^\dagger\phi_{\rm II} - \frac{\lambda}{4}(\phi_{\rm II}^\dagger\phi_{\rm II})^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \tag{14}$$

As a matter of fact, starting from high values and then lowering the temperature we meet a first SSB at the critical temperature T_1: the medium becomes superconducting. Since the II-phase term $a_{\rm II}(T){\phi_0}^2 + \lambda\,{\phi_0}^4$ in the free energy density is negative for $T < T_2$, by further lowering the temperature at $T = T_2$ the condensation involving the second order-parameter is energetically favored and a new (second-order) phase transition starts. Below T_2 the system is "more" superconducting with respect to the GL standard case since, in addition to the phase-I Cooper pairs, we should observe also the formation of phase-II Cooper pairs. Such two superconducting phases correspond to different condensations of electrons in Cooper pairs which exhibit different self-interaction, and are described by different scalar fields. The realization of one of the two regimes is ruled by the relative strength of the Cooper pair self-interaction (λ) with respect to the electromagnetic interaction (e).

Correspondingly, the total free energy density, being an additive quantity, results as the sum of contributions from normal-conducting electrons, phase-I superconducting Cooper pairs, and phase-II superconducting Cooper pairs:

$$F = F_{\rm n} \quad \text{for} \quad T > T_1\,, \tag{15}$$

$$F = F_{\rm n} + \frac{1}{2} a_{\rm I}(T) {\eta_0}^2 + \frac{\lambda}{16}\, {\eta_0}^4 \quad \text{for} \quad T_2 < T < T_1\,, \tag{16}$$

$$F = F_{\rm n} + \frac{1}{2} a_{\rm I}(T) {\eta_0}^2 + \frac{\lambda}{16}\, {\eta_0}^4 + \frac{1}{2} a_{\rm II}(T) {\phi_0}^2 + \frac{\lambda}{16}\, {\phi_0}^4 \quad \text{for} \quad T < T_2\,, \tag{17}$$

(η_0 indicates the expectation value of $|\phi_{\rm I}|$; ϕ_0 indicates the expectation value of $\mathrm{Re}\{\phi_{\rm II}\}$).

From Eqs.(6) and (11) we are able to put the two free parameters of our theory, i.e. the "mass squared" m^2 and the self-interaction coupling constant λ as functions of the two critical temperatures:

$$m^2 = \frac{e^2 T_1^2 T_2^2}{4(4T_2^2 - 3T_1^2)}\,, \tag{18}$$

$$\lambda = \frac{12 e^2 (T_1^2 - T_2^2)}{4T_2^2 - 3T_1^2}\,. \tag{19}$$

Therefore experimental measurements of T_1 and T_2 could yield an estimate of the dynamical parameters ruling the SSB and the electron binding in Cooper pairs. Notice that such a goal is not possible in the framework of the standard GL, theory where the parameters in (1) are not explicitly determined.

2.2 Thermodynamical properties

Inserting the expressions obtained above in (5), (7), (10), and (12), also the expectation values of the two scalar fields can be expressed in terms of T_1 and T_2:

$$\eta_0^2(T) = \frac{T_2^2(T_1^2 - T^2)}{12(T_1^2 - T_2^2)}\,, \tag{20}$$

$$\phi_0^2(T) = \frac{T_1^2(T_2^2 - T^2)}{12(T_1^2 - T_2^2)}\,. \tag{21}$$

By inserting in (16) and (17) we may compare, for $T < T_2$, the behavior of the free energy in the GL case, where (16) holds also for $T < T_2$, and in the case of two-phases superconductors for which, instead, (17) applies. The free energy difference results to be

$$\Delta F \equiv F_{\rm GL} - F_{\rm 2ph} = \frac{e^2 T_1^4 (T_2^2 - T^2)^2}{48(4T_2^2 - 3T_1^2)(T_1^2 - T_2^2)}\,. \tag{22}$$

We see that such a difference increases by lowering the temperature and reaches its maximum for $T = 0$.

The pressure is given by $P = - \left.\frac{\partial \mathcal{F}}{\partial V}\right|_T$, where $\mathcal{F} = FV$ is the free energy. Since that the superconductive part of the free energy density is independent of the volume, we have

$$\Delta P \equiv P_{\rm GL} - P_{\rm 2ph} = -\Delta F < 0\,. \tag{23}$$

Hence the pressure is expected to be larger for two-phases superconductors. Thus the differences in the free energy and in the pressure become more sensible far from T_2, near to absolute zero.

From $S = - \left.\frac{\partial F}{\partial T}\right|_V$, for $T < T_2$, we get the difference in the entropy density:

$$\Delta S \equiv S_{\rm GL} - S_{\rm 2ph} = \frac{e^2 T_1^4 T (T_2^2 - T^2)}{12(4T_2^2 - 3T_1^2)(T_1^2 - T_2^2)}\,. \tag{24}$$

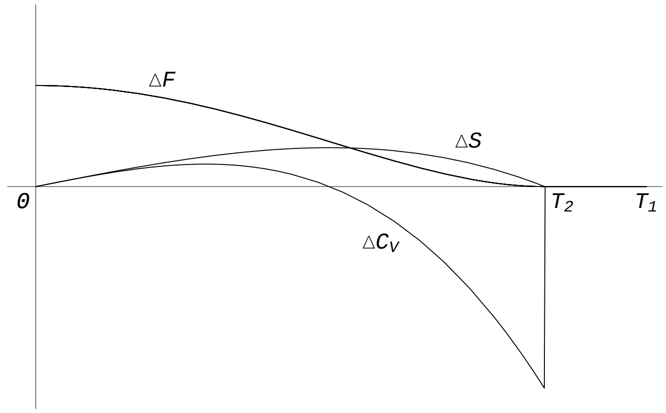

Figure 1: Differences between GL and two-phases superconductors

Being $\Delta S > 0$, we can say that the two-phases superconductors are in a sense more "ordered" than the GL ones, the maximum difference for the entropy being reached at $T = T_2/\sqrt{3}$.

We can compare as well the latent heat absorbed during the formation of the superconducting phase in GL and two-phase superconductors at a given temperature $T < T_2$ (S_0 indicates the entropy of the normal-phase)

$$\lambda_{\rm GL}(T) = T(S_0 - S_{\rm GL}) = \frac{e^2 T_2^4 T^2 (T_1^2 - T^2)}{12(4T_2^2 - 3T_1^2)(T_1^2 - T_2^2)} \; ; \tag{25}$$

$$\lambda_{\rm 2ph}(T) = T(S_0 - S_{\rm 2ph}) = \frac{e^2 T^2 [T_2^4 (T_1^2 - T^2) + T_1^4 (T_2^2 - T^2)]}{12(4T_2^2 - 3T_1^2)(T_1^2 - T_2^2)} \, . \tag{26}$$

The difference between $\lambda_{\rm GL}$ and $\lambda_{\rm 2ph}$ reaches its maximum at $T = T_2/2$.

Finally, applying the well-known formula for the specific heat at constant volume

$$C_V = T \left. \frac{\partial S}{\partial T} \right|_V \, , \tag{27}$$

we obtain the difference in C_V ($T < T_2$)

$$\Delta C_V \equiv C_{V_{\rm GL}} - C_{V_{\rm 2ph}} = \frac{e^2 T_1^4 T (T_2^2 - 3T^2)}{12(4T_2^2 - 3T_1^2)(T_1^2 - T_2^2)} \, , \tag{28}$$

which is positive for $0 < T < T_2/\sqrt{3}$, negative for $T_2/\sqrt{3} < T < T_2$, and vanishes at $T = T_2/\sqrt{3}$. As it is seen in the figure, whilst $\Delta F(T_2)$ and $\Delta S(T_2)$ vanish, so that F and S are continuous in T_2, quantity $\Delta C_V(T_2)$ is not zero:

$$\Delta C_V(T_2) = -\frac{e^2 T_1^4 T_2^3}{6(4T_2^2 - 3T_1^2)(T_1^2 - T_2^2)} < 0 \, . \tag{29}$$

We then observe a finite jump in the specific heat also in the transition from the superconducting first phase (I) to the second one (II), while in GL superconducting media only one discontinuity is expected (for $T = T_1$, when the system changes from the normal to the superconducting regime). This sudden change in the heat capacity is a distinguishing characteristic of a first order phase transition. Since the jump of the specific heat in $T = T_1$ results to be

$$\Delta C_V(T_1) = -\frac{e^2 T_2^4 T_1^3}{6(4T_2^2 - 3T_1^2)(T_1^2 - T_2^2)} \, , \tag{30}$$

the ratio between the two discontinuities can be written as

$$\frac{\Delta C_V(T_2)}{\Delta C_V(T_1)} = \frac{T_1}{T_2} \,. \tag{31}$$

Being the above ratio larger than 1 (and smaller than $\sqrt{4/3}$), we expect the two jumps to be comparable. Thus also the second jump at the lower temperature —just a novel effect because it happens between two superconducting phases— could be experimentally investigated and measured. Notice also that, as expected, both discontinuities increase indefinitely in the large Cooper pairs self-interaction limit, $\lambda/e^2 \to \infty$, $(T_1/T_2)^2 \to 4/3$.

2.3 Magnetic properties

Let us start with the *Meissner effect*, namely the rapid decaying to zero of magnetic fields in the bulk of a superconductor. The distance from the surface beyond which the magnetic field vanishes is known as the *London penetration depth*, and can be written in terms of the effective (after SSB and Higgs mechanism [16, 3, 1]) photon mass

$$\delta = \frac{1}{m_A} = \sqrt{\frac{1}{8e^2|\phi_{\min}(T)|^2}} \,,$$

where $\phi_{\min}(T)$ is the expectation value of the field in the minimum energy state at a given temperature. Exploiting Lagrangian (14), in the phase-I ($T_2 < T < T_1$) we therefore have

$$\delta_{\mathrm{I}} = \sqrt{\frac{1}{8e^2|\eta_0(T)|^2}} \,, \tag{32}$$

whilst in the phase-II ($T < T_2$) we now have two contributions to the photon mass

$$\delta_{\mathrm{II}} = \sqrt{\frac{1}{8e^2(|\eta_0(T)|^2 + |\chi_0(T)|^2)}} \,. \tag{33}$$

The expectation values $\eta_0(T)$ and $\chi_0(T)$ can be expressed [12] as functions of the critical temperatures

$$\eta_0^2(T) = -\frac{2\,a_{\mathrm{w}}(T)}{\lambda} = \frac{T_2^2(T_1^2 - T^2)}{24(T_1^2 - T_2^2)} \,, \tag{34}$$

$$\chi_0^2(T) = -\frac{2\,a_{\mathrm{s}}(T)}{\lambda} = \frac{T_1^2(T_2^2 - T^2)}{24(T_1^2 - T_2^2)} \,. \tag{35}$$

As a consequence we can write the London penetration lengths as follows:

$$\delta_{\mathrm{I}} = \left[\frac{e^2 T_2^2 T_1^2}{3(T_1^2 - T_2^2)}\left(1 - \frac{T^2}{T_1^2}\right)\right]^{-\frac{1}{2}} \tag{36}$$

for the phase-I; and

$$\delta_{\mathrm{II}} = \left\{\frac{e^2 T_2^2 T_1^2}{3(T_1^2 - T_2^2)}\left[\left(1 - \frac{T^2}{T_1^2}\right) + \left(1 - \frac{T^2}{T_2^2}\right)\right]\right\}^{-\frac{1}{2}} \tag{37}$$

for the phase-II. Let us stress that, below the second critical temperature T_2, the penetration length of the magnetic field is smaller with respect to the GL one-phase superconductors.

The *coherence length*

$$\xi = \frac{1}{m_\phi(T)}$$

measures the distance over which the scalar field varies sensitively and is related to the mean binding length of the electrons in a Cooper pair. The coherence length can be expressed (via Higgs mechanism [16, 1, 3]) as a function of the temperature-dependent effective mass of the scalar field which results from a quantum field calculation including one-loop radiative correction: namely

$$m_{\phi_{\rm w}}(T) = \sqrt{-a_{\rm w}(T)} \tag{38}$$

for the weak field ($T < T_1$), and

$$m_{\phi_{\rm s}}(T) = \sqrt{-a_{\rm s}(T)} \tag{39}$$

for the strong field ($T < T_2$). As expected, for $T \neq 0$,

$$m^2_{\phi_{\rm w}} = m^2 - \frac{\lambda + 4e^2}{16} T^2 \; > \; m^2_{\phi_{\rm s}} = m^2 - \frac{\lambda + 3e^2}{12} T^2$$

since the (negative) binding energy between the electrons is larger for the strongly-coupled Cooper pairs.

By applying the above definition we get *two different coherence lengths* for the two fields

$$\xi_{\rm w}(T) = \frac{1}{m_{\phi_{\rm w}}(T)}\,, \tag{40}$$

$$\xi_s(T) = \frac{1}{m_{\phi_{\rm s}}(T)}\,. \tag{41}$$

At absolute zero the renormalized masses are equal to the bare mass m and the two coherence lengths reduce to the common value:

$$\xi_0 = \frac{1}{m} = 2\frac{\sqrt{(4T_2^2 - 3T_1^2)}}{eT_1T_2}\,. \tag{42}$$

Let us write explicitly the temperature dependence of the coherence lengths for the two types of Cooper pairs :

$$\xi_{\rm w}(T) = \frac{1}{m_{\phi_{\rm w}}(T)} = \frac{\xi_0}{\sqrt{1 - \left(\frac{T}{T_1}\right)^2}}, \qquad \text{for } T < T_1\,, \tag{43}$$

$$\xi_{\rm s}(T) = \frac{1}{m_{\phi_{\rm s}}(T)} = \frac{\xi_0}{\sqrt{1 - \left(\frac{T}{T_2}\right)^2}}, \qquad \text{for } T < T_2\,. \tag{44}$$

By increasing the intensity of the magnetic field entering type-I superconductors, when H reaches a critical value $\mathcal{H}_{\rm c}$, perfect diamagnetism and superconductivity are suddenly destroyed through a first-order phase transition. The critical magnetic field measures the "condensation energy", given by the difference between the free energies of the normal and superconducting states

$$F - F_{\rm n} = -\frac{1}{2}\mu_0 {\mathcal{H}_{\rm c}}^2 \, .$$

Exploiting the above equation we obtain the critical field in the phase-I, for $T_2 < T < T_1$:

$$\mathcal{H}_{\rm c}{}^{\rm I} = \sqrt{\frac{2}{\mu_0 \lambda}}\, |a_{\rm w}| = \frac{e T_2^2 (T_1^2 - T^2)}{2\sqrt{6\mu_0 (4T_2^2 - 3T_1^2)(T_1^2 - T_2^2)}} \, , \tag{45}$$

while in the phase-II, for $T < T_2$, we have

$$\mathcal{H}_{\rm c}{}^{\rm II} = \sqrt{\frac{2}{\mu_0 \lambda}\, (a_{\rm w}^2 + a_{\rm s}^2)} = \mathcal{H}_{\rm c}{}^{\rm I} \sqrt{1 + \frac{T_1^4 (T_2^2 - T^2)^2}{T_2^4 (T_1^2 - T^2)^2}} \, . \tag{46}$$

Let us notice that there is no discontinuity at T_2: $\mathcal{H}_{\rm c}{}^{\rm I}(T_2) = \mathcal{H}_{\rm c}{}^{\rm II}(T_2)$; but, while for $T \lesssim T_1$ the critical field decreases linearly, for $T \lesssim T_2$ we have instead a quadratic behavior

$$\mathcal{H}_{\rm c}{}^{\rm II} \simeq \mathcal{H}_{\rm c}{}^{\rm I}(T_2) \left[1 + \frac{2T_1^4}{T_2^2 (T_1^2 - T^2)^2} (T_2 - T)^2 \right] \, . \tag{47}$$

For type-II superconductors there exist two different critical fields, $\mathcal{H}_{\rm c1}$, the *lower critical field*, and $\mathcal{H}_{\rm c2}$ the *upper critical field*. We observe perfect diamagnetism only applying a field $H < \mathcal{H}_{\rm c1}$ whilst, when $\mathcal{H}_{\rm c1} < H < \mathcal{H}_{\rm c2}$, non superconducting vortices can arise in the bulk of the medium. Abrikosov [17, 18] showed that a vortex consists of regions of circulating supercurrent around a small central normal-metal core: the magnetic field is able to penetrate through the sample inside the vortex cores, and the circulating currents serve to screen out the magnetic field from the rest of the superconductor outside the vortex.

In the present model there are actually two different coherence lengths for two different types of Cooper pairs. Correspondingly, in the phase-II we shall have *different upper and lower critical fields* in "domains" of the sample occupied by Cooper pairs of the same type, either weakly-coupled or strongly-coupled. As a consequence, in a given domain we can have (or not) Abrikosov vortices depending on the type of scalar field condensed in that domain. Such an inhomogeneity of the spatial distribution of the vortices in two-phase superconductors should result in a net, detectable difference with respect to GL superconductors: in principle, for $T < T_2$, a section of the material should show vortical and non-vortical sectors, by contrast to the homogeneous distribution of vortex cores in ordinary superconductors.

Now, by starting from the London equation we can easily obtain the explicit expression of the lower critical field [19, 17]

$$\mathcal{H}_{\rm c1} = \frac{\Phi_0}{4\pi\mu_0 \delta^2} \ln\left(\frac{\delta}{\xi}\right)$$

where $\Phi_0 \equiv \dfrac{\pi}{e}$ is the so-called quantum magnetic flux unit. Thus, in the phase-I, the lower critical field writes:

$$\mathcal{H}_{\rm c1}^{\rm I} = \frac{\Phi_0}{4\pi\mu_0 \delta_{\rm I}^2} \ln\left(\frac{\delta_{\rm I}}{\xi_{\rm w}}\right) = h_1 \left(1 - \frac{T^2}{T_1^2}\right) \tag{48}$$

where

$$h_1 \equiv \frac{\Phi_0}{4\pi\mu_0} \frac{e^2 T_1^2 T_2^2}{3(T_1^2 - T_2^2)} \ln\left[\sqrt{\frac{6(T_1^2 - T_2^2)}{4T_2^2 - 3T_1^2}}\right] , . \tag{49}$$

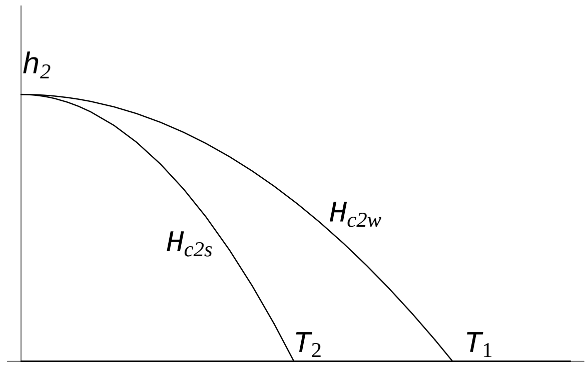

Figure 2: Upper critical fields vs. temperature for weak-field and strong-field domains

As said above, in the phase-II we have two distinct $\mathcal{H}_{\mathrm{c}1}^{\mathrm{II}}$:

$$\mathcal{H}_{\mathrm{c}1_{\mathrm{w}}}^{\mathrm{II}} = \frac{\Phi_0}{4\pi\mu_0\delta_{\mathrm{II}}^2}\ln\left(\frac{\delta_{\mathrm{II}}}{\xi_{\mathrm{w}}}\right) = h_1\left[\left(1-\frac{T^2}{T_1^2}\right)+\left(1-\frac{T^2}{T_2^2}\right)\right]\left\{1-\frac{\ln\left[1+\frac{T_1^2(T_2^2-T^2)}{T_2^2(T_1^2-T^2)}\right]}{\ln\left[\frac{6(T_1^2-T_2^2)}{4T_2^2-3T_1^2}\right]}\right\} \tag{50}$$

in the weak-field domains; and

$$\mathcal{H}_{\mathrm{c}1_{\mathrm{s}}}^{\mathrm{II}} = \frac{\Phi_0}{4\pi\mu_0\delta_{\mathrm{II}}^2}\ln\left(\frac{\delta_{\mathrm{II}}}{\xi_{\mathrm{s}}}\right) = h_1\left[\left(1-\frac{T^2}{T_1^2}\right)+\left(1-\frac{T^2}{T_2^2}\right)\right]\left\{1-\frac{\ln\left[1+\frac{T_2^2(T_1^2-T^2)}{T_1^2(T_2^2-T^2)}\right]}{\ln\left[\frac{6(T_1^2-T_2^2)}{4T_2^2-3T_1^2}\right]}\right\} \tag{51}$$

in the strong-field domains.

Let us now pass to the upper critical field which can be expressed as follows [19, 17]

$$\mathcal{H}_{\mathrm{c}2} = \frac{\Phi_0}{2\pi\mu_0}\frac{1}{\xi^2}\,.$$

In the weak-field ($T < T_1$) and strong-field ($T < T_2$) domains the above equation writes, respectively:

$$\mathcal{H}_{\mathrm{c}2_{\mathrm{w}}} = \frac{\Phi_0}{2\pi\mu_0}\frac{1}{\xi_{\mathrm{w}}^2} = h_2\left(1-\frac{T^2}{T_1^2}\right), \qquad \text{for } T < T_1\,, \tag{52}$$

$$\mathcal{H}_{\mathrm{c}2_{\mathrm{s}}} = \frac{\Phi_0}{2\pi\mu_0}\frac{1}{\xi_{\mathrm{s}}^2} = h_2\left(1-\frac{T^2}{T_2^2}\right), \qquad \text{for } T < T_2\,, \tag{53}$$

$$h_2 \equiv \frac{\Phi_0}{\pi\mu_0}\frac{e^2T_1^2T_2^2}{(4T_2^2-3T_1^2)}\,. \tag{54}$$

For $T \to 0$ we have:

$$\mathcal{H}_{\mathrm{c}2_{\mathrm{w}}} = \mathcal{H}_{\mathrm{c}2_{\mathrm{s}}} = h_2\,, \tag{55}$$

while, for $T \to T_2$:

$$\mathcal{H}_{\mathrm{c}2_{\mathrm{w}}} \neq \mathcal{H}_{\mathrm{c}2_{\mathrm{s}}} = 0. \tag{56}$$

The well-known dimensionless *Ginzburg-Landau parameter*

$$\kappa \equiv \frac{\delta}{\xi}$$

determines whether a medium is a type-I or type-II superconductor: for $\kappa < 1/\sqrt{2}$ it is a type-I superconductor; whilst for $\kappa > 1/\sqrt{2}$ it is a type-II superconductor.

In the phase-I we find:

$$\kappa_{\mathrm{I}} = \sqrt{\frac{6(T_1^2 - T_2^2)}{4T_2^2 - 3T_1^2}} \tag{57}$$

which is independent of temperature in agree with the GL theory. On imposing the condition $\kappa_{\mathrm{I}} < 1/\sqrt{2}$ we infer that the material, for $T_2 < T < T_1$, is a type-I superconductor if $1 \leq \left(\frac{T_1}{T_2}\right)^2 \leq \frac{16}{15}$; and a type-II superconductor if $\frac{16}{15} \leq \left(\frac{T_1}{T_2}\right)^2 \leq \frac{4}{3}$.

In the phase-II we easily get:

$$\kappa_{\mathrm{IIw}} = \left[1 + \frac{T_1^2(T_2^2 - T^2)}{T_2^2(T_1^2 - T^2)}\right]^{-\frac{1}{2}} \kappa_{\mathrm{I}} \tag{58}$$

for the weak-field domains; and

$$\kappa_{\mathrm{IIs}} = \left[1 + \frac{T_2^2(T_1^2 - T^2)}{T_1^2(T_2^2 - T^2)}\right]^{-\frac{1}{2}} \kappa_{\mathrm{I}} \tag{59}$$

for the strong-field domains.

The dependence of κ_{II} on temperature could be observed as a net deviation from the temperature-independent behavior of GL superconductors. Notice that $\kappa_{\mathrm{II}} < \kappa_{\mathrm{I}}$: therefore a type-II superconductor for temperatures in the region $T_2 < T < T_1$ could become a type-I superconductor for $T < T_2$, if κ decreases below $1/\sqrt{2}$. Quantities κ_{IIw} and κ_{IIs} turn out to be equal for $T \sim 0$:

$$\kappa_{\mathrm{IIw}} = \kappa_{\mathrm{IIs}} \simeq \frac{\kappa_{\mathrm{I}}}{\sqrt{2}}\,. \tag{60}$$

3 Generalized GL model for p-wave superconductors

If we introduce two *mutually interacting* order parameters, we will not in general describe two-phases superconductors but, actually, *spinning* Cooper pairs and rotational degrees of freedom in superconductivity. In the Bardeen, Cooper, and Schrieffer (BCS) theory for conventional superconductors, the electrons are paired into a zero total angular momentum state, with zero spin and zero orbital angular momentum: $J = L = S = 0$. As a matter of fact, in BCS superconductors the s-wave is shown to correspond to the minimum energy state with maximum attraction between the electrons in a Cooper pair. Indeed, soon after the BCS theory was advanced, Kohn and Luttinger [20] predicted that, if the mutual interaction is *repulsive* in all partial wave channels, the Cooper pairs result to be bonded by a weak residual attraction (out of the Coulomb repulsion) in higher angular momentum channels: this is the so-called *Kohn-Luttinger effect*. On the other hand it exists also a p-wave Cooper-pairing in superfluid ^{3}He (which is, as said above, the liquid counterpart of GL superconductors). Actually, we can meet p-wave superconductivity in certain "heavy-electron" compounds (*heavy fermion systems* as, e.g., UPt_3) and in special materials recently discovered as, e.g., Sr_2RuO_4 [21] which is the only known metal oxide displaying p-wave superconductivity. Let us recall that the p-wave Cooper pairs are always spin-triplets (S=1) because of Pauli's exclusion principle applied to systems composed of a pair of particles endowed with odd (L=1) total orbital quantum number. Taking into account this property, we shall now put forward a simple GL-like model just for spin-triplet superconductors.

3.1 The model

Let us consider the physical system described by one doublet of complex scalar fields ϕ_1, ϕ_2 through the following Lagrangian density:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi_1)^\star\,(D^\mu\phi_1) + (D_\mu\phi_2)^\star\,(D^\mu\phi_2) - \lambda(|\phi_1|^2 - \frac{1}{2}\phi_0^2)^2 \\ &\quad -\lambda(|\phi_2|^2 - \frac{1}{2}\phi_0^2)^2 + V(\phi_1,\phi_2).\end{aligned} \tag{61}$$

Here the covariant derivative $D_\mu \equiv \partial_\mu + ieA_\mu$ describes the minimal electromagnetic interaction of the two scalar fields, while the first term in the Lagrangian (with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$) accounts for the kinetic energy of the free electromagnetic field A_μ. The complete potential term for the interaction of the two scalar fields is composed of three different terms, $V = V_{\phi A} + V_{\rm self} + V(\phi_1,\phi_2)$, the first two of them describing the usual interaction between the electromagnetic field and the charged scalar field (coming from the covariant derivative), and $V_{\rm self}$ ruling the self interaction of the scalar fields: $V_{\rm self} \equiv \lambda|\phi_1|^4 + \mu|\phi_2|^4$. For the interaction between the two scalar fields we instead adopt the following nonlinear term:

$$V(\phi_1,\phi_2) \equiv -\frac{\lambda\phi_0^4}{8}\ln^2\frac{\phi_1}{\phi_1^\star}\frac{\phi_2^\star}{\phi_2}. \tag{62}$$

Let us study the small fluctuations of the two scalar fields around the minimum of the energy corresponding to $\phi_1 = \phi_2 = \phi_0/\sqrt{2}$ by expanding both scalar fields as follows:

$$\phi_1 \equiv \frac{1}{\sqrt{2}}(\phi_0 + \eta_1)\,{\rm e}^{i\theta_1/\phi_0}, \tag{63}$$

$$\phi_2 \equiv \frac{1}{\sqrt{2}}(\phi_0 + \eta_2)\,{\rm e}^{i\theta_2/\phi_0}, \tag{64}$$

where η_1, η_2, θ_1, θ_2 are real fields. From these definitions, the above interaction term can be written more simply as follows

$$V(\phi_1,\phi_2) = \frac{\lambda\phi_0^2}{2}(\theta_1 - \theta_2)^2\,. \tag{65}$$

Notice that $V(\phi_1,\phi_2)$ is positive-definite, then describing a *repulsion* between the two fields with strength $\lambda\phi_0^2$ equal to the mass squared m_W^2 (see below). Note also that $V(\phi_1,\phi_2)$ corresponds to the main term of the expansion for small phase differences [22] of the Legget interaction $\gamma(\phi_1^\star\phi_2 + \phi_1\phi_2^\star)$.

By inserting Eqs. (63,64) into the Lagrangian density (61) and performing the gauge transformation: $A_\mu \to A_\mu + \partial_\mu\Lambda$ with

$$\Lambda \equiv -\frac{1}{2e\phi_0}(\theta_1 + \theta_2)\,, \tag{66}$$

we obtain the following Lagrangian, up to quadratic terms in the fields:

$$\begin{aligned}\mathcal{L} &\simeq -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2\phi_0^2 A_\mu A^\mu + \frac{1}{2}\partial_\mu\eta_1\partial^\mu\eta_1 + \frac{1}{2}\partial_\mu\eta_2\partial^\mu\eta_2 + \frac{1}{2}\partial_\mu(\theta_1-\theta_2)\partial^\mu(\theta_1-\theta_2) \\ &\quad +\lambda\phi_0^2\eta_1^2 + \lambda\phi_0^2\eta_2^2 + \frac{\lambda\phi_0^2}{2}(\theta_1-\theta_2)^2.\end{aligned} \tag{67}$$

Let us set

$$\eta_3 \equiv \frac{1}{\sqrt{2}}(\theta_1 - \theta_2) \tag{68}$$

and define the triplet field $W_a \equiv (\eta_1, \eta_2, \eta_3)$. The Lagrangian describing our physical system now becomes:

$$\mathcal{L} \simeq -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m_A^2 A_\mu A^\mu + \frac{1}{2}(\partial_\mu W_a)(\partial^\mu W_a) + m_W^2 W_a W_a \tag{69}$$

with

$$m_A^2 = e^2\phi_0^2, \qquad m_W^2 = \lambda\phi_0^2. \tag{70}$$

As a result, only one of the original four degrees of freedom embedded into two charged (complex) scalar fields is disappeared, by giving rise to a massive photon as in the standard GL model. By virtue of the interaction potential in Eq. (62), the remaining three degrees of freedom all have the same mass, and can thus be combined to form a triplet field W (i.e. a triplet spinor representation of SU(2)), suitable to describe a so-called p-wave superconductor. We stress that, notwithstanding the simultaneous condensation of two real degrees of freedom, the key point in our model is the particular interaction term we have introduced, which prevents a gauge transformation to re-absorb one more degree of freedom (only the *sum* of the phases of the complex fields turns out to be "eaten", but not even the *difference*). Such a very peculiar interaction breaks the isotropy of the original medium and allows pairs of electrons to arrange into possible $S = 1$ (instead of $S = 0$) Cooper pairs. As a matter of fact, the emergence of a triplet field is a signal of the occurred "anisotropization" of the system, which can no more be described by a singlet scalar field.

3.2 Predicted properties for triplet superconductors

The order parameter describing p-wave superconductors may be associated in our model to the above triplet Higgs field W_a which is responsible of the U(1) spontaneous symmetry breaking occurring during the normal state-superconducting-phase transition. Therefore, from the Lagrangian (61), the effective free energy density at finite temperature T, resulting from the quantum fields calculation, including one-loop radiative corrections [2, 15], is given by

$$F(T) = F_{\mathrm{n}}(T) + a(T)|\phi_1|^2 + a(T)|\phi_2|^2 + \lambda|\phi_1|^4 + \lambda|\phi_2|^4 + a(T)\frac{\phi_0^2}{8}\left|\ln\frac{\phi_1}{\phi_1^\star}\frac{\phi_2^\star}{\phi_2}\right|^2, \tag{71}$$

where

$$a(T) = -m_W^2 + \frac{\lambda + e^2}{4}T^2\,, \tag{72}$$

label n referring to the normal (non superconducting) phase. The coefficient a vanishes when the temperature approaches a critical value given by

$$T_c = \sqrt{\frac{4m_W^2}{\lambda + e^2}}\,. \tag{73}$$

Below T_c the expectation values of the scalar fields ϕ_1 and ϕ_2 which minimize the free energy function results to be

$$|\phi_1(T)| = |\phi_2(T)| = \sqrt{-\frac{a(T)}{2\lambda}}\,, \tag{74}$$

while the third degree of freedom defined in Eq. (68) fluctuates around the zero expectation value, corresponding to $\theta_1 = \theta_2$. This last occurrence directly comes from the fact that the non-linear characteristic potential term in Eq. (62) is non-negative definite, so that the minimum of the free energy is reached when it vanishes. In this case, our model practically reduces to a "simple" doubling of the standard GL theory making recourse to two scalar order parameters. As a consequence, it is very easy to re-obtain the usual main properties for p-wave superconductors considered here.

The London penetration length of the magnetic field inside the superconductor arises due to the presence of a massive photon, that is:

$$\delta = \frac{1}{m_A} = \frac{1}{e\phi_0}\,, \tag{75}$$

while the coherence length of the Cooper pairs described by the triplet scalar field is given by:

$$\xi = \frac{1}{m_W} = \frac{1}{\phi_0\sqrt{\lambda}} = \frac{\xi_0}{\sqrt{1-\frac{T^2}{T_c^2}}}\,. \tag{76}$$

The *critical magnetic field* H_c, measuring the condensation energy $F(T) - F_{\rm n}(T) = -\mu_0 H_c^2/2$ of the superconductor system can be obtained as follows:

$$H_c^2 = \frac{1}{\mu_0}\,\frac{a^2(T)}{\lambda} = H_{c0}^2\left(1-\frac{T^2}{T_c^2}\right)^2\,. \tag{77}$$

By taking the derivative of the free energy function with respect to temperature, we easily get the entropy gain with respect to the normal phase:

$$S - S_n = \frac{\partial}{\partial T}\left(-\frac{a^2(T)}{2\lambda}\right) = S_0\left(1-\frac{T^2}{T_c^2}\right)\frac{T}{T_c}\,. \tag{78}$$

Finally, we can write down the expected discontinuity of the specific heat at the critical point:

$$\Delta C_V = T\,\frac{\partial}{\partial T}\left(S - S_n\right) = S_0\left(1-3\,\frac{T^2}{T_c^2}\right)\frac{T}{T_c}. \tag{79}$$

4 Conclusion

In the framework of the GL theory, we have developed some models in order to describe different physical systems experiencing two $S = 0$ superconducting phases, whose critical temperatures T_1, T_2 differ at most of 15%, or, alternatively, p-wave superconductors by means of two mutually interacting order parameters which condensate simultaneously at a same critical temperature.

In the first model, two different condensation regimes of two scalar fields with equal (bare) mass and self-interaction strength arise, describing Cooper pairs formed by differently interacting electrons, such different interaction arising from quantum loop corrections. We have calculated the thermodynamical properties of such two-phase superconductors, the most peculiar one being a second discontinuity in the specific heat, and considered the main magnetic properties as well. Below the second critical temperature, the penetration length of the magnetic field is smaller with respect to the usual one-phase superconductors (even of about 70%), depending on temperature. This is easily explained by the emergence of a second kind of Cooper pairs, in which electrons are more bonded than in the first kind of pairs, leading also to two distinct behaviors of the critical magnetic fields in the two superconducting phases. As a consequence, two different coherence lengths for the electrons in the different Cooper pairs exist, this resulting in peculiar superconductive properties. For instance, even if in the region between T_1 and T_2 the system is a type-II superconductor, depending on the ratio of the critical temperatures $[16/15 \leq (T_1/T_2)^2 \leq 4/3]$, below T_2 it instead could behave as a type-I superconductor if κ decreases sufficiently (κ becoming $< 1/\sqrt{2}$). Moreover, for $T < T_2$, the GL parameter κ is not constant, and exhibits a characteristic dependence on the temperature, a result strongly deviating from the predictions of the GL theory. Perhaps all these effects have not yet been observed in any material, due to the very small difference (no more than 15%) between the two critical temperatures, but this seems to be not a really difficult task for dedicated experiments, since those effects are very peculiar. The experimental investigation on the novel kind of superconductors discussed here should then consist mainly in the careful search for materials which may exhibit the particular properties of the two kinds of electron pairs considered. The major merit of this model, with respect to the existing theories for (usual and) unusual superconductors is its full predictability, since all the basic physical properties are expressed in terms of the two measurable critical temperatures, rather than in terms of unknown quantities such as self-interaction coupling constants, quasi-particle effective masses, or mean distance between Cooper electrons. Let us also remark that attractive interactions ("Cooper-effect") and "gapped" energy spectrum characterize both quantum theory of *fermionic* superfluids (as e.g. ^{3}He) and BCS theory of superconductivity. Consequently we might expect that the basic properties for two-phase superconductors could analogously apply to a sort of "two-phase Fermi superfluids" (endowed with two critical temperatures) as well.

In the second model, the standard GL theory has been generalized in order to describe p-wave superconductors by means of two mutually interacting order parameters which condensate simultaneously at the same critical temperature (since the $\lambda\phi^4$ self-interaction is the same for both fields). After the condensation we remain with three massive degrees of freedom (in addition to a massive photon, related to the Meissner-effect) which can be put in correspondence to the three components of a $S = 1$ triplet mean-field describing spinning p-wave Cooper pairs. In this model the main magnetic and thermodynamical properties (including the discontinuity in the specific heat) of p-wave superconductors turn out to be essentially the same as for conventional s-wave superconductors discussed above.

It is very intriguing that completely different systems may exhibit similar physical properties, thus encouraging further theoretical and experimental studies in this direction.

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