Gravitational collapse and primordial black hole formation

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Outline

- **Gravitational collapse** (perfect fluid, spherical symmetry):
  - Introduction
  - Causal horizons: event/apparent horizon, cosmological horizon
  - Barotropic equation of state: BH formation and critical collapse
  - Non barotropic equation of state: virialized structure
- **Conclusions:** summary & future

Background model

- The most general **spherical symmetric** form of the metric, to describe a deviation from the uniform background, can be written as,

$$ds^2 = -a^2\, dt^2 + b^2\, dr^2 + R^2 d\Omega^2$$

- This defines *a*, *b*, and *R* being functions of the comoving coordinate *r* the often called **cosmic time** *t* and involves a choice of the **time slicing** to keep the metric diagonal (gauge choice). The radius *R* is the **circumferential radial coordinate**.

- The **unperturbed solution,** describing an expanding homogeneous universe**,** is given by the FRW metric: $K = \pm 1, 0$ is the **curvature parameter**, $\tilde{a}(t)$ is the **scale factor**, and $R = \tilde{a}(t)\, r$ is the **circumferential radial coordinate**.

$$ds^2 = -dt^2 + \tilde{a}(t)[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2]$$

- In case of BH formation it is more convenient to use a **null slicing** *(r,u)*:

$$f\, du = a\, dt - b\, dr \qquad\qquad ds^2 = -f^2\, du^2 - 2fb\, dr\, du + R^2 d\Omega$$

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \qquad T_{\mu\nu} = (e+p)u_\mu u_\nu - p g_{\mu\nu}$$

COSMIC TIME

$$D_t \equiv \frac{1}{a}\left(\frac{\partial}{\partial t}\right) \qquad D_r \equiv \frac{1}{b}\left(\frac{\partial}{\partial r}\right)$$

$$U \equiv D_t R \qquad \Gamma \equiv D_r R$$

$$D_t U = -\left[\frac{\Gamma}{(e+p)} D_r p + \frac{M}{R^2} + 4\pi R p\right]$$

$$D_t \rho = -\frac{\rho}{\Gamma R^2} D_r (R^2 U)$$

$$D_t e = \frac{e+p}{\rho} D_t \rho$$

$$D_t M = -4\pi R^2 p U$$

$$D_r a = -\frac{a}{e+p} D_r p$$

$$D_r M = 4\pi R^2 \Gamma e$$

$$\Gamma^2 = 1 + U^2 - \frac{2M}{R}$$

NULL TIME

$$D_t \equiv \frac{1}{f}\left(\frac{\partial}{\partial u}\right) \qquad D_k \equiv D_r + D_t$$

$$D_t U = -\frac{1}{1-c_s^2}\left[\frac{\Gamma}{(e+p)} D_k p + \frac{M}{R^2} + 4\pi R p + \right.$$
$$\left. + c_s^2 \left(D_k U + \frac{2U\Gamma}{R}\right)\right]$$

$$D_t \rho = \frac{\rho}{\Gamma}\left[D_t U - D_k U - \frac{2U\Gamma}{R}\right]$$

$$D_t e = \left(\frac{e+p}{\rho}\right) D_t \rho$$

$$D_t M = -4\pi R^2 p U$$

$$D_k \left[\frac{(\Gamma + U)}{f}\right] = -4\pi R (e+p) f$$

$$D_k M = 4\pi R^2 [e\Gamma - pU],$$

$$\Gamma = D_k R - U = 1 + U^2 - \frac{2M}{R}$$

Equation of State

energy density: $e = \rho(1+\epsilon)$ pressure: $p = (\gamma - 1)\rho\epsilon$

rest mass density

adiabatic index - particle degree of freedom

specific internal energy (velocity dispersion)

- Barotropic fluid (no rest mass density): $p = we$ with $w \in [0,1]$
    - radiation dominated era: $w = 1/3$ RADIATION $(\gamma = 4/3)$
    - matter dominated era: $w = 0$ DUST $(\gamma = 1)$
- Polytropic fluid: $p = K(s)\rho^\gamma$ $(\gamma = 5/3,\ 4/3,\ 2)$
    - If the fluid is adiabatic (no entropy change): $K(s) = K$ (constant)

Radial null ray

- Along a radial null ray

$$ds = d\Omega = 0 \quad \Rightarrow \quad dr = \pm \frac{a}{b} dt$$

with the plus for **outgoing** and the minus for **incoming**.

- For changes in R along a **worldline**: $dR = \frac{\partial R}{\partial t} dt + \frac{\partial R}{\partial r} dr$

$$dR = \left(\frac{\partial R}{\partial t} \pm \frac{a}{b} \frac{\partial R}{\partial r} \right) dt \quad \text{along a radial null ray}$$

- Introducing the radial component of **4-velocity** U and the **Lorentz factor** Γ

$$D_t R \equiv \frac{1}{a} \frac{\partial R}{\partial t} \equiv U \quad \rightarrow \quad \frac{\partial R}{\partial t} = aU$$

$$D_r R \equiv \frac{1}{b} \frac{\partial R}{\partial r} \equiv \Gamma \quad \rightarrow \quad \frac{\partial R}{\partial r} = b\Gamma$$

$$\Rightarrow \quad \boxed{\frac{dR}{dt} = a(U \pm \Gamma)}$$

Calculation of Γ

- For a general perfect fluid, the G_{00} and G_{11} components of the Einstein field equations are

$$(G^0_0) \qquad 4\pi R^2 e R_r = (R + RU^2 - R\Gamma^2)_r/2$$

$$(G^1_1) \qquad 4\pi R^2 a p U = -(R + RU^2 - R\Gamma^2)_t/2$$

- It is convenient to define $M = (R + RU^2 - R\Gamma^2)/2$ to write

$$M = \int 4\pi R^2 e dR$$

$$D_t M = -4\pi R^2 p U$$

and

$$\Gamma^2 = 1 + U^2 - \frac{2M}{R}$$

The last is a **constraint equation** with *M* being the mass inside radius *R*. The second one describes adiabatic expansion or contraction of the fluid.

Horizons within a moving medium

- A black hole **apparent horizon** (event horizon in the static case) is the asymptotic location of the **outermost trapped surface** for outgoing light-rays whereas the **cosmological horizon** is the **innermost trapped surface** for incoming light rays.

- **Apparent horizon :** $$\left(\frac{dR}{dt}\right)_{\rm out} = a(U+\Gamma) = 0 \quad \rightarrow \quad \Gamma = -U$$

- **Cosmological horizon :** $$\left(\frac{dR}{dt}\right)_{\rm in} = a(U-\Gamma) = 0 \quad \rightarrow \quad \Gamma = U$$

$$\Downarrow$$

$$\Gamma^2 = U^2 \qquad \text{inserted into} \qquad \Gamma^2 = 1 + U^2 - \frac{2M}{R}$$

$$\Downarrow$$

The horizon condition is independent of the slicing and holds also within a non-vacuum moving medium

$$R = 2M$$

PBH History

- In the early universe **large amplitude perturbations of the metric** can collapse into **Primordial Black Holes (PBHs)** [**Zeldovich** & **Novikov** (1967); **Hawking** (1971)] characterized by a wide range of masses (from the Planck mass to 10^6 $M_\odot$ for PBHs formed at the Nucleosynthesis).

- Historically the **Hawking evaporation effect** (1974) has been inspired by the idea of PBH formation because PBHs small as particles could form in the Early Universe and will have non negligible quantum effects. PBHs smaller than 10^{15} grams would evaporate by now via Hawking evaporation, becoming **possible sources of Gamma Ray Burst, Cosmic Rays, evaporation remnants as cold dark matter**.

- The threshold amplitude $\delta_c \sim c_s^2$ measured at horizon crossing time **Carr** (1975), tells if a perturbation can collapse into **Primordial Black Holes** or disperse into the surrounding medium. This has been confirmed by **full relativistic numerical simulations** **Nadezin**, **Novikov** & **Polnarev** (1978), and more recently by **Niemeyer** & **Jedamzik** (1998, 1999); **Musco**, **Miller** & **Rezzolla** (2005)] suggesting that **critical collapse** (scaling law) might apply in the early universe, in particular during the radiation dominated era $w = 1/3$

Detection of PBHs

- These PBHs could collide with stars (**Zhilyaev** 2007):
  - So small that there would be hardly any "direct" interaction with stellar matter.
  - Interaction via dynamical friction.
  - potentially observable flashes in X-rays and Υ-rays.
  - for smaller PBHs the signal is too weak while for larger PBH the event rate is too low.

PBHs cosmological seeds of SBHs?

- PBHs ~ 10^5 solar masses during nucleosyinthesis.
- The softening of the equations of state could enhance the collapse (radiation transport of neutrinos)?
- formation mechanism to investigate.

Structure formation

- Non linear gravitational collapse $\Rightarrow$ structure formation
- Barotropic fluid $p = we$ $\Rightarrow$ **Primordial Black Hole** (PBH) formation. Initial conditions: small super horizon perturbations re-entering the horizon in the radiation dominated era.
- In the matter era the universe is dominated by CDM particles where the pressure is characterized by random motions (**velocity dispersion**) that gives **virialization**.
- In a spherically symmetric **Fluid approach** baryoinc matter is characterized by a Mawellian distribution. Non-barotropic equation of state for CDM needs non-Maxwellian velocity dispersion implying non isotropic pressure.
- **Structure formation** is a non relativistic problem that in cosmology is usually studied in 3D using N-body simulations. For relativistic collapse spherical symmetry is very useful. Initial conditions:

   1. Linear cosmological perturbations (early universe).
   2. Perturbations of static solutions.

Initial conditions

- In the early universe we can consider small **cosmological perturbations** of the density as the origin of the future structure observed in the universe coming from **quantum fluctuations** inflated onto **supra-horizon scale**.
- Cosmological perturbations **re-enter the horizon** again as the Universe continues to expand – become **causally connected** – can **collapse**.
- Cosmological perturbations may start as a mixture of **growing and decaying components**, but the decaying components soon become small, leaving just the growing ones.
- In the linear regime, pure growing adiabatic mode can be described as:

$$p = we \qquad \delta f = \epsilon(t)\tilde{f}(r)$$

$$w \neq 0 \Rightarrow \epsilon(t) := \left(\frac{R_H}{R_0}\right)^2 = \epsilon(t_0)\left(\frac{t}{t_0}\right)^{\frac{2(1+3w)}{3(1+w)}} \qquad w \simeq 0 \Rightarrow \epsilon = \frac{K_B T}{(\gamma-1)m}$$

$$\delta e = (e - e_b)/e_b \propto t^{\frac{2(1+3w)}{3(1+w)}} \qquad \delta U = (U - U_b)/U_b \sim \frac{1}{3(1+w)}\delta e$$

BH formation: setting the problem

- Spherical symmetry $$ds^2 = -a^2\,dt^2 + b^2\,dr^2 + R^2 d\Omega^2$$

- Barotropic equation of state: $$p = we$$

- Initial conditions: linear supra-horizon perturbation $(\epsilon << 1)$ of a FRW universe:

$$b = \frac{\partial_r R}{\sqrt{1 - K(r) r^2}}(1 + \epsilon \tilde{b}) \qquad e = e_b(1 + \epsilon \tilde{e})$$

$$\epsilon(t) := \left(\frac{R_H}{R_0}\right)^2 = \epsilon(t_0)\left(\frac{t}{t_0}\right)^{\frac{2(1+3w)}{3(1+w)}} \qquad \tilde{e} = \frac{3(1+w)}{5+3w}\frac{r_0^2}{3r^2}\partial_r\left[K(r) r^3\right]$$

$$K(r) = \exp\left(-\frac{r^2}{2\Delta^2}\right) \qquad \tilde{e} = \frac{3(1+\omega)}{5+3\omega}\Delta^2\left[1 - \left(\frac{R_b}{R_0}\right)^2\right]\exp\left(-\frac{3}{2}\left(\frac{R_b}{R_0}\right)^2\right)$$

$$r_0^2 = 3\Delta^2 \qquad \delta \equiv \left(\frac{4}{3}\pi r_0^3\right)^{-1}\int_0^{r_0} 4\pi \frac{e - e_b}{e_b} r^2 dr = \epsilon(t)\Phi K(r_0) r_0^2$$

Quasi homogeneous solution

- Defining the scale of the cosmological perturbations as R_0 and the **cosmological horizon** scale as $R_H := 1/H_b$. In the linear regime of supra horizon growing modes we can construct a **small parameter** $\epsilon(t) << 1$ as:

$$\epsilon(t) := \left(\frac{R_H}{R_0}\right)^2 = \epsilon(t_0)\left(\frac{t}{t_0}\right)^{\frac{2(1+3w)}{3(1+w)}} \qquad H_b^2 = \frac{8\pi}{3}e_b =: \left(\frac{1}{R_H}\right)^2$$

- **First order** perturbations in ϵ are given by:

$$a = 1 + \epsilon\tilde{a} \qquad e = e_b(1+\epsilon\tilde{e})$$

$$b = \frac{\partial_r R}{\sqrt{1-K(r)r^2}}(1+\epsilon\tilde{b}) \qquad U = H_b R(1+\epsilon\tilde{U})$$

$$R = R_b(1+\epsilon\tilde{R}) \qquad M = \frac{4}{3}\pi e_b R(1+\epsilon\tilde{M})$$

- In the **linear regime**, when $\epsilon << 1$, $K(r)r^2 = 1-\Gamma^2$ is a **time independent curvature profile** because pressure gradients are negligible, and can be used as the only independent source of perturbations.

- Solution of Einstein equations to the first order in ϵ (**Polnarev** & **Musco** 2007):

$$\tilde{e} = \Phi \frac{r_0^2}{3r^2} \partial_r \left[K(r) r^3 \right]$$

$$\Phi = \frac{3(1+w)}{5+3w}$$

$$\tilde{U} = \frac{1}{2} \left[\Phi - 1 \right] K(r) r_0^2$$

$$\Psi_1 = \frac{3w}{(1+3w)(5+3w)}$$

$$\tilde{M} = \Phi K(r) r_0^2$$

$$\tilde{a} = -\Phi \frac{\omega}{1+\omega} \frac{r_0^2}{3r^2} \partial_r \left[K(r) r^3 \right]$$

$$\Psi_2 = \frac{2}{(1+3w)(5+3w)}$$

$$\tilde{b} = \Psi_1 r \partial_r \left[\frac{r_0^2}{3r^2} \partial_r \left(K(r) r^3 \right) \right]$$

$$\tilde{R} = -\Psi_1 \frac{r_0^2}{3r^2} \partial_r \left[K(r) r^3 \right] + \Psi_2 \frac{K(r)}{2} r_0^2$$

- Constraints: $\lim_{r \to \infty} r^3 K(r) = 0$ and $K(r_0) + \frac{r_0}{3} \partial_r K(r_0) = 0$

- The **perturbation amplitude** δ can be measured by the mass excess inside the over dense region.

$$\delta \equiv \left(\frac{4}{3}\pi r_0^3\right)^{-1} \int_0^{r_0} 4\pi \frac{e - e_b}{e_b} r^2 dr = \epsilon(t)\Phi K(r_0) r_0^2$$

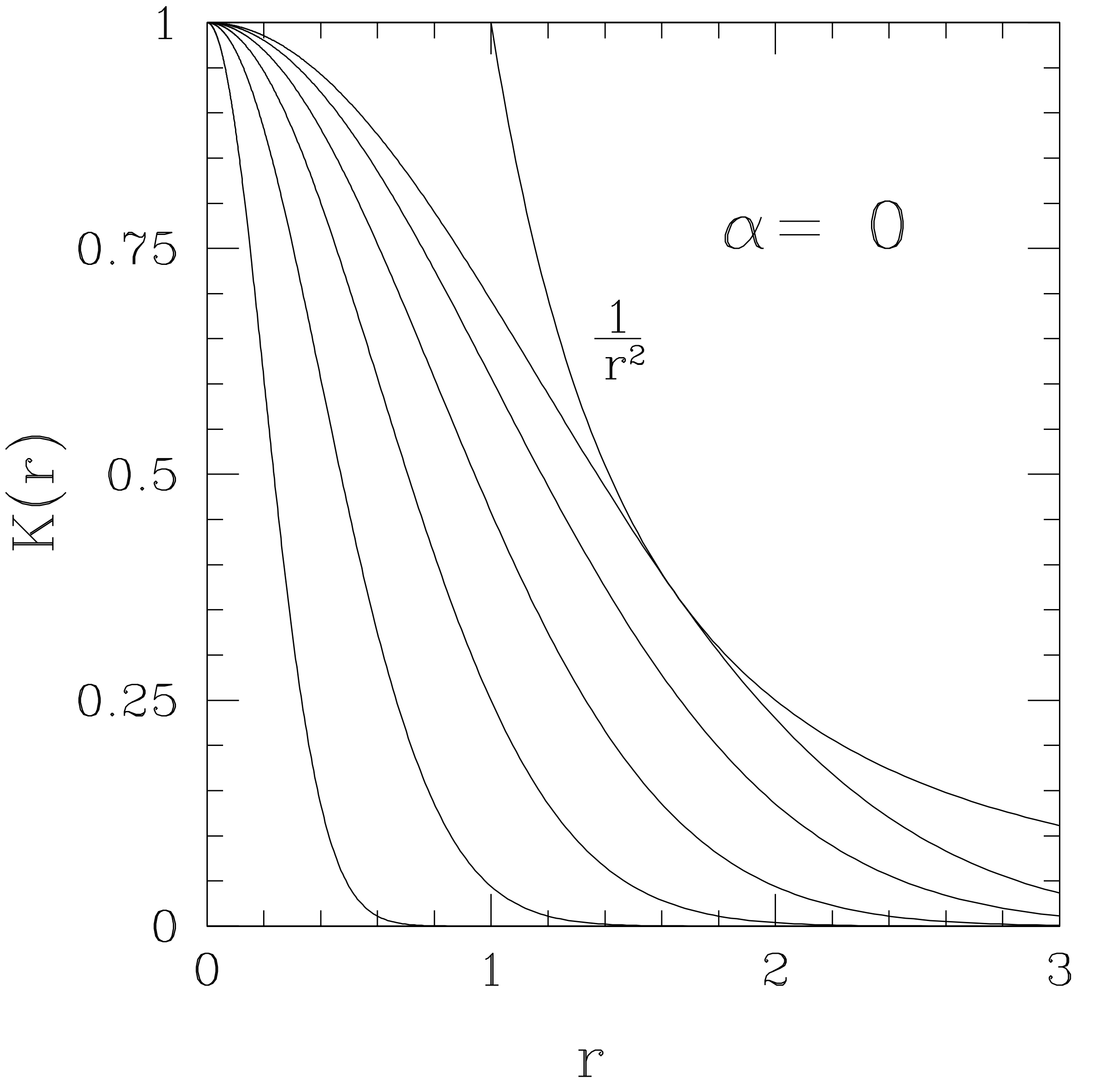
- The simplest curvature profile is given by a **Gaussian profile** of *K(r)* which gives a **Mexican Hat profile** for the energy density *e*:

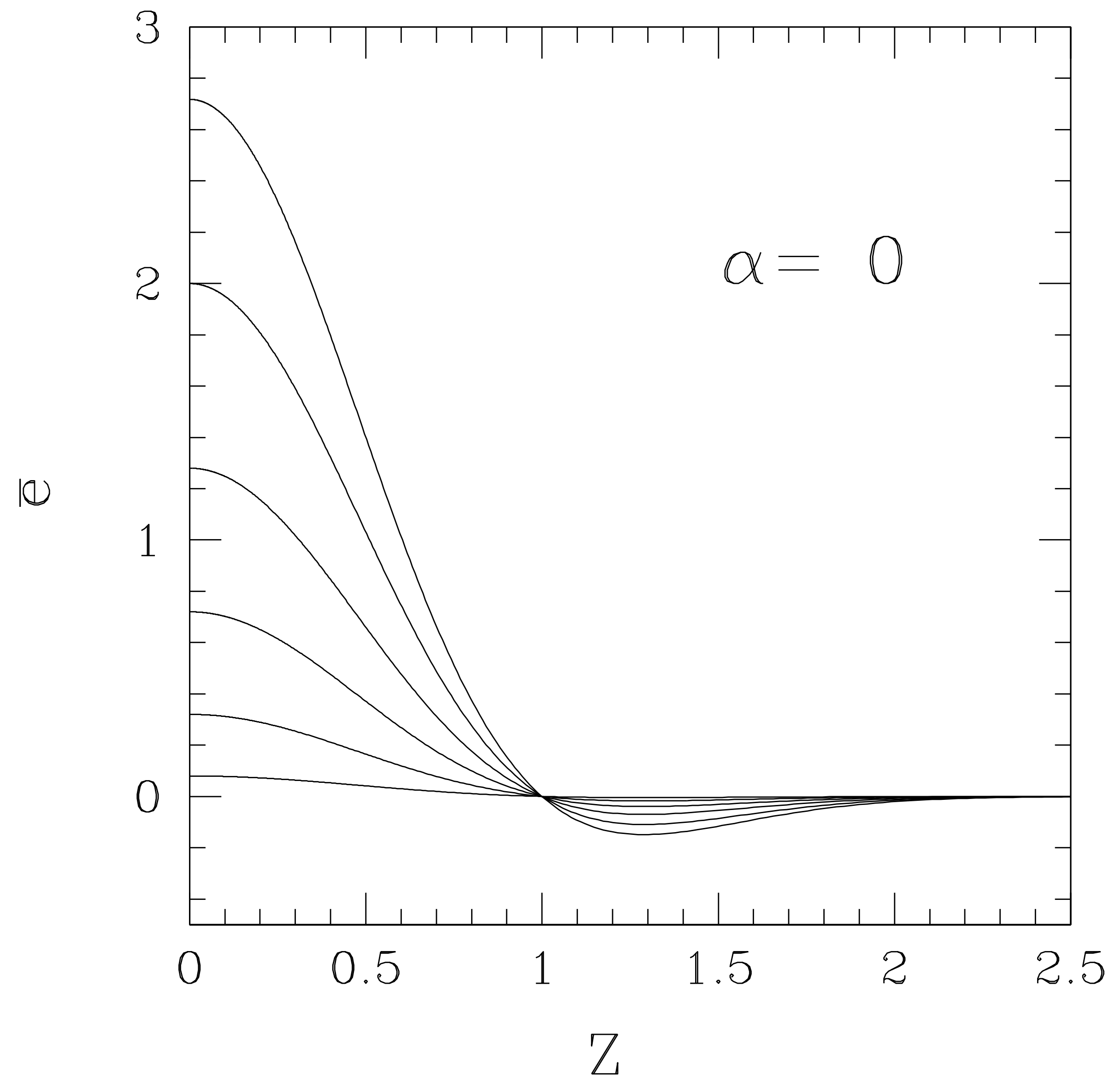
$$K(r) = \exp\left(-\frac{r^2}{2\Delta^2}\right) \qquad \Rightarrow \qquad r_0^2 = 3\Delta^2$$

$$\tilde{e} = \frac{3(1+\omega)}{5+3\omega}\Delta^2 \left[1 - \left(\frac{R_b}{R_0}\right)^2\right] \exp\left(-\frac{3}{2}\left(\frac{R_b}{R_0}\right)^2\right)$$

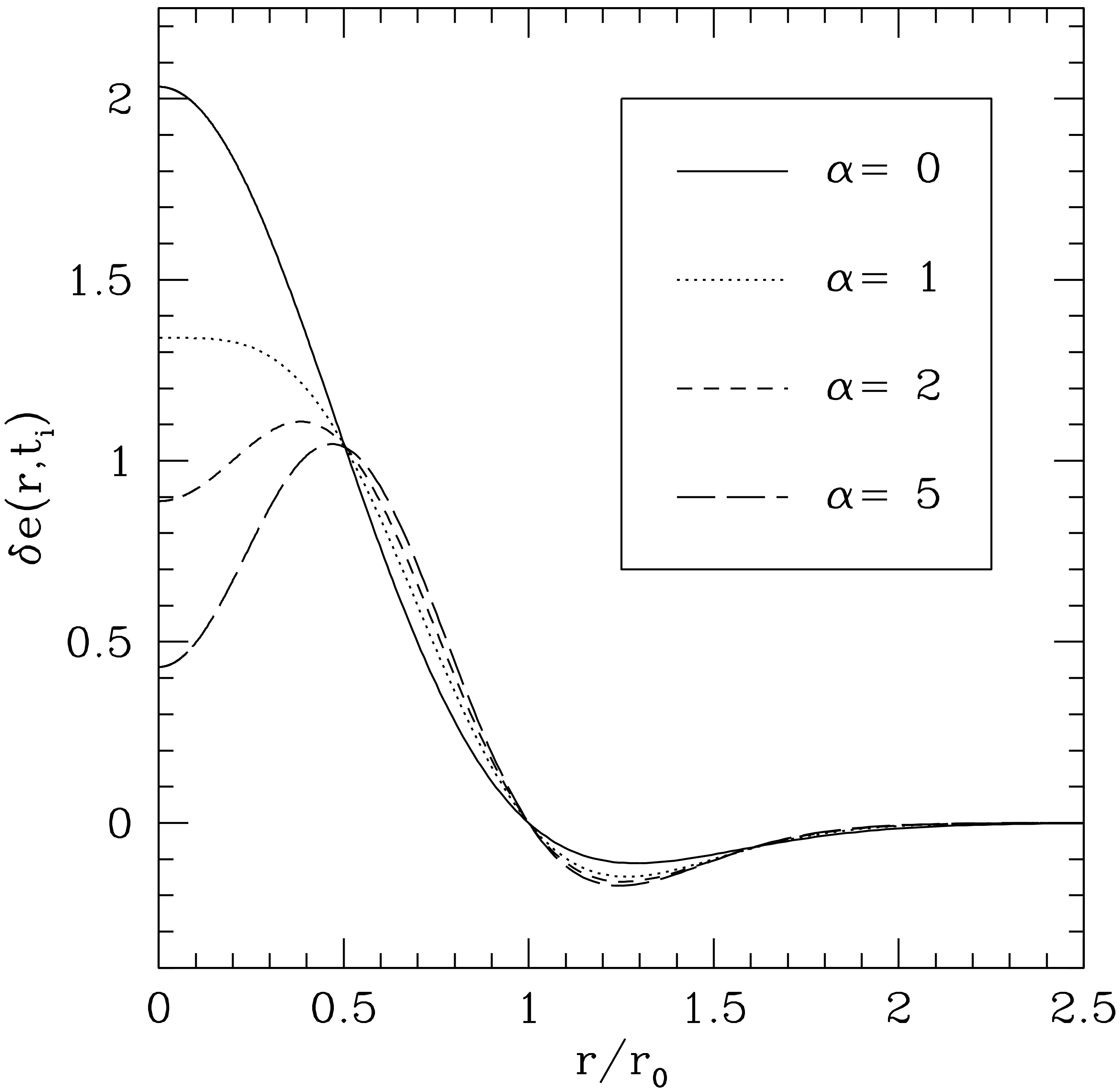
$$\delta = \epsilon(t)\tfrac{9(1+\omega)}{5+3\omega}\Delta^2 \exp\left(-\tfrac{3}{2}\right)$$

- **Gaussian Curvature - Mexican hat energy density perturbation**: the amplitude Δ of the Gaussian profile of $K(r)$ gives a measure of the central peak of the Mexican hat energy density profile $\bar{e}(r)$ that integrated on the 3D spherical volume gives the perturbation amplitude δ.





Different shapes: $K(r) = \left(1 + \alpha \frac{r^2}{2\Delta^2}\right) \exp\left(-\frac{r^2}{2\Delta^2}\right)$



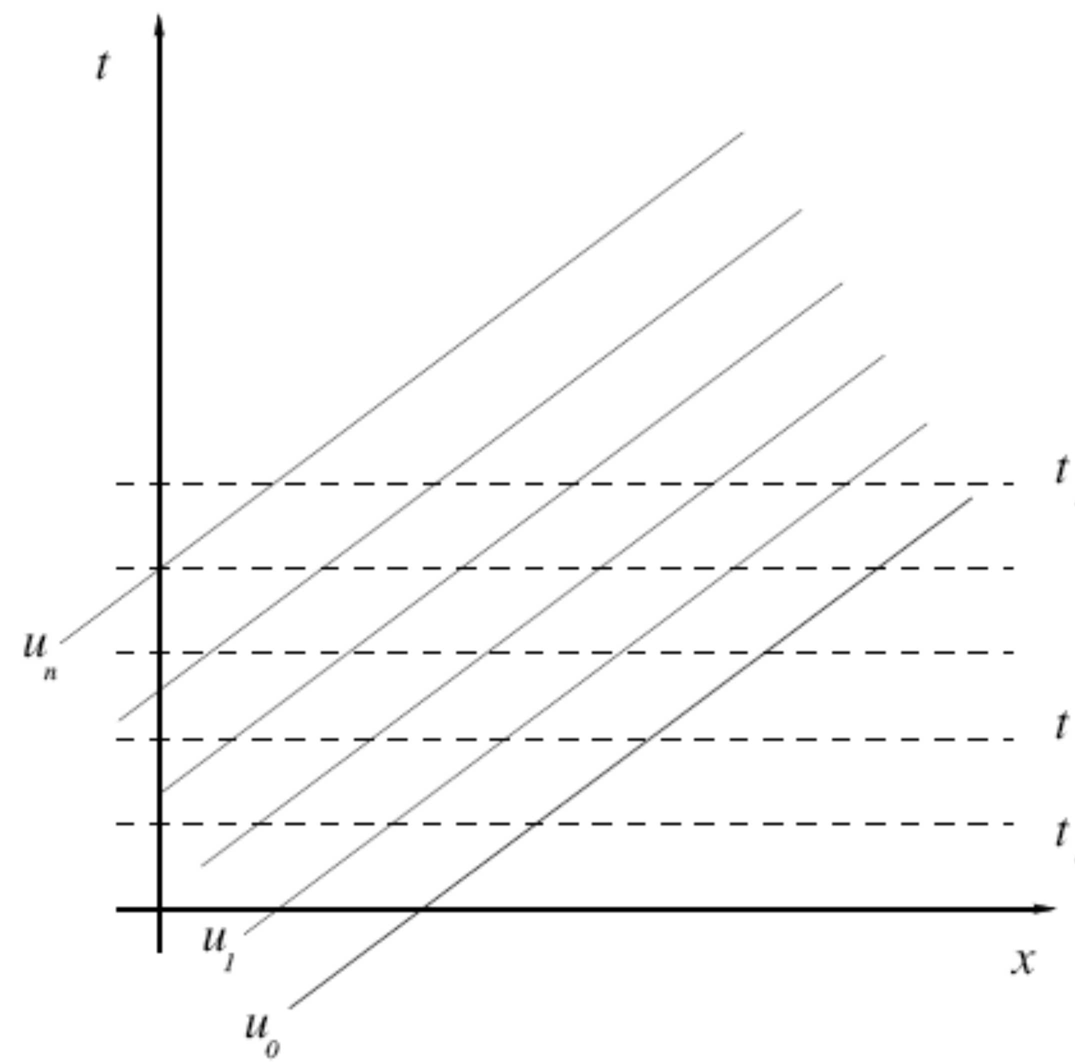
Numerical Results: the method

- Simulations are performed using a **Lagrangian spherically symmetric GR hydro code with an adaptive grid** (AMR).

- We set initial conditions using a **cosmic time coordinate** t.

- We transfer those onto a **null foliation** of the space time, then evolved using an **observer time coordinate** u.

- The **formation of a PBH is seen by a distant external observer** (the singularity is hidden by the asymptotic formation of the apparent horizon).



Critical Collapse

- **Universality with respect to initial data:** generic initial data parameterized by one parameter δ have only two possible **end states:**

    1. Give rise to a black hole formation if *($\delta > \delta_c$)*.
    2. Dispersion to infinity of the mass energy if *($\delta < \delta_c$)*.

- **Scale invariance of the critical solution** *($\delta=\delta_c$)***:** it is possible to rescale the solution (**self similarity**)

$$g_{\mu\nu}(\tau, x^i) = e^{-2\tau}\tilde{g}_{\mu\nu}(x^i)$$

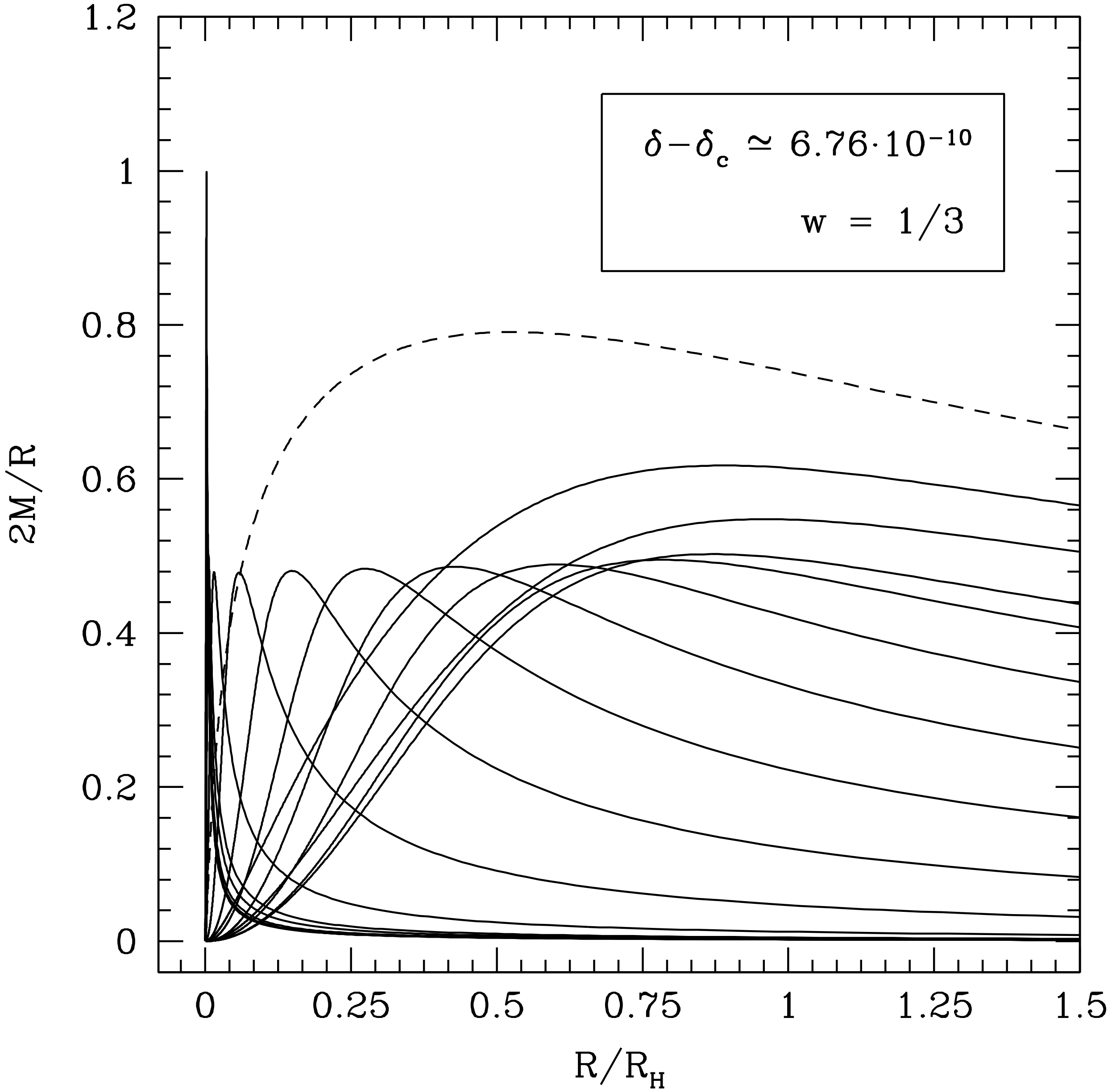
- **Black Hole mass scaling:** near the threshold, black holes with arbitrarily small masses can be created, and the black hole mass scales as

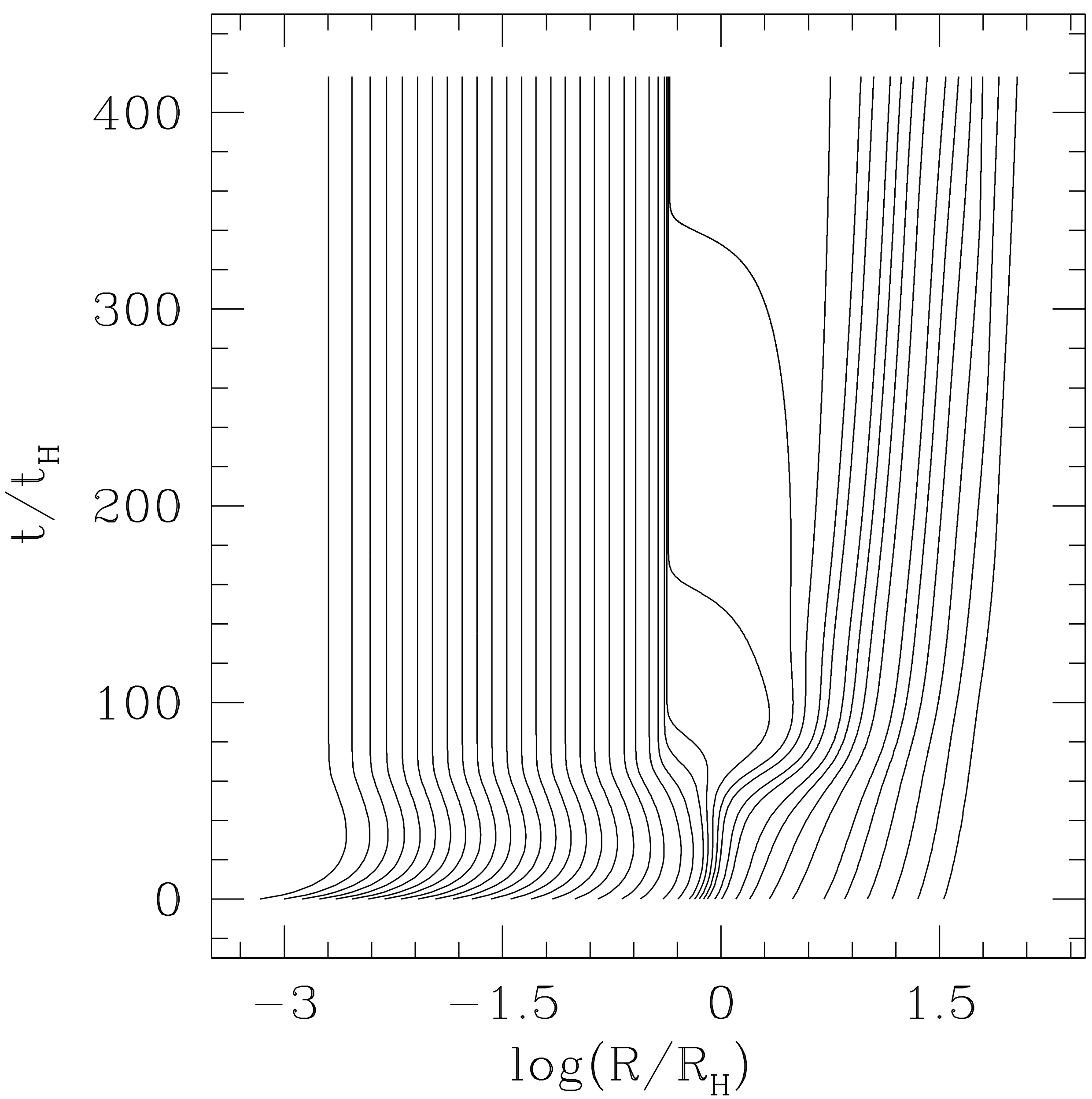
$$M_{BH} \propto (\delta - \delta_c)^\gamma$$

- The **critical exponent** γ is universal with respect to initial data, independent of the particular 1-parameter family, depends only on the **equation of state**.

$$\gamma = 1/\lambda$$

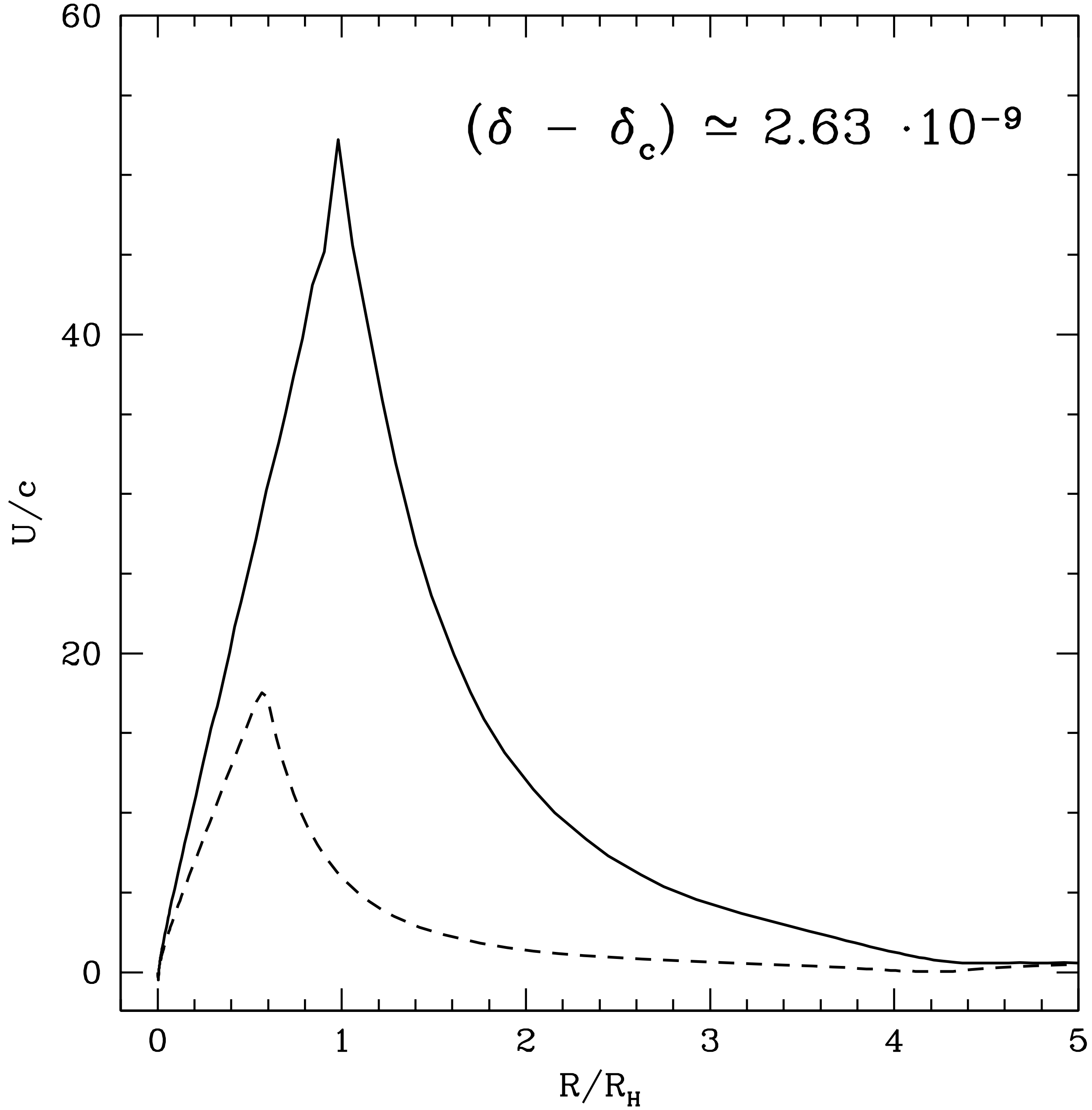
Numerical Results: BH formation

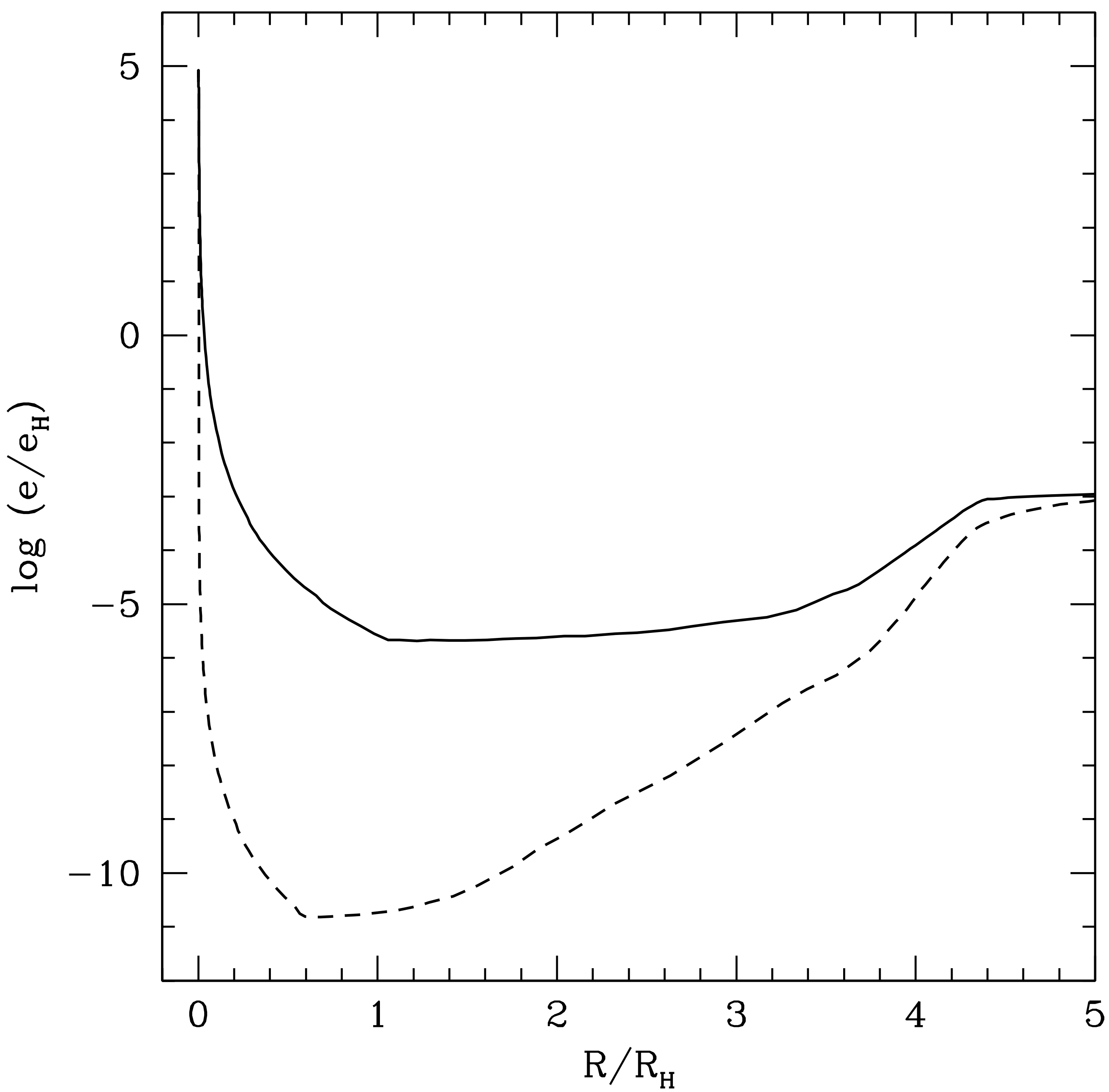




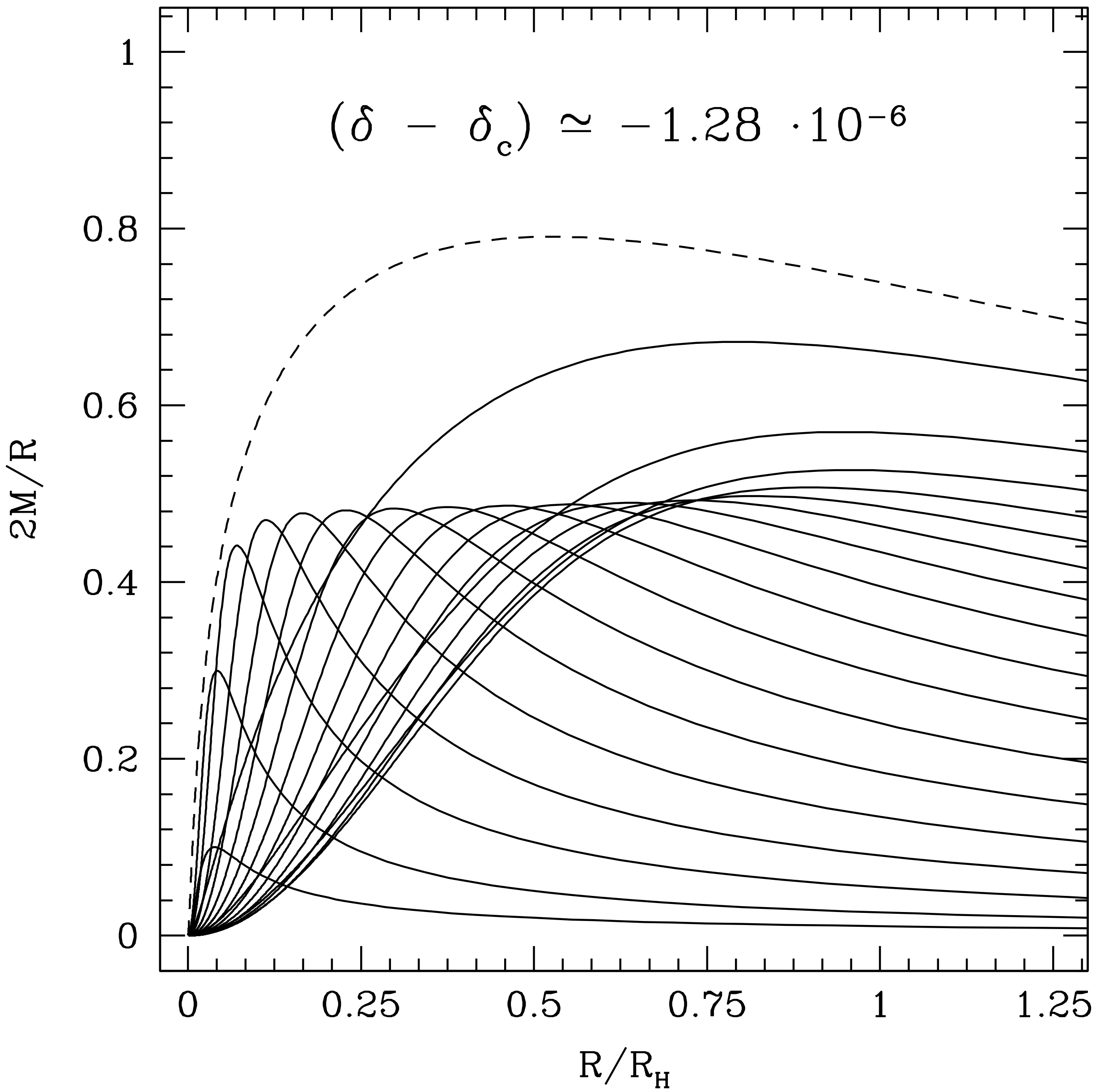
Numerical Results: relativistic wind

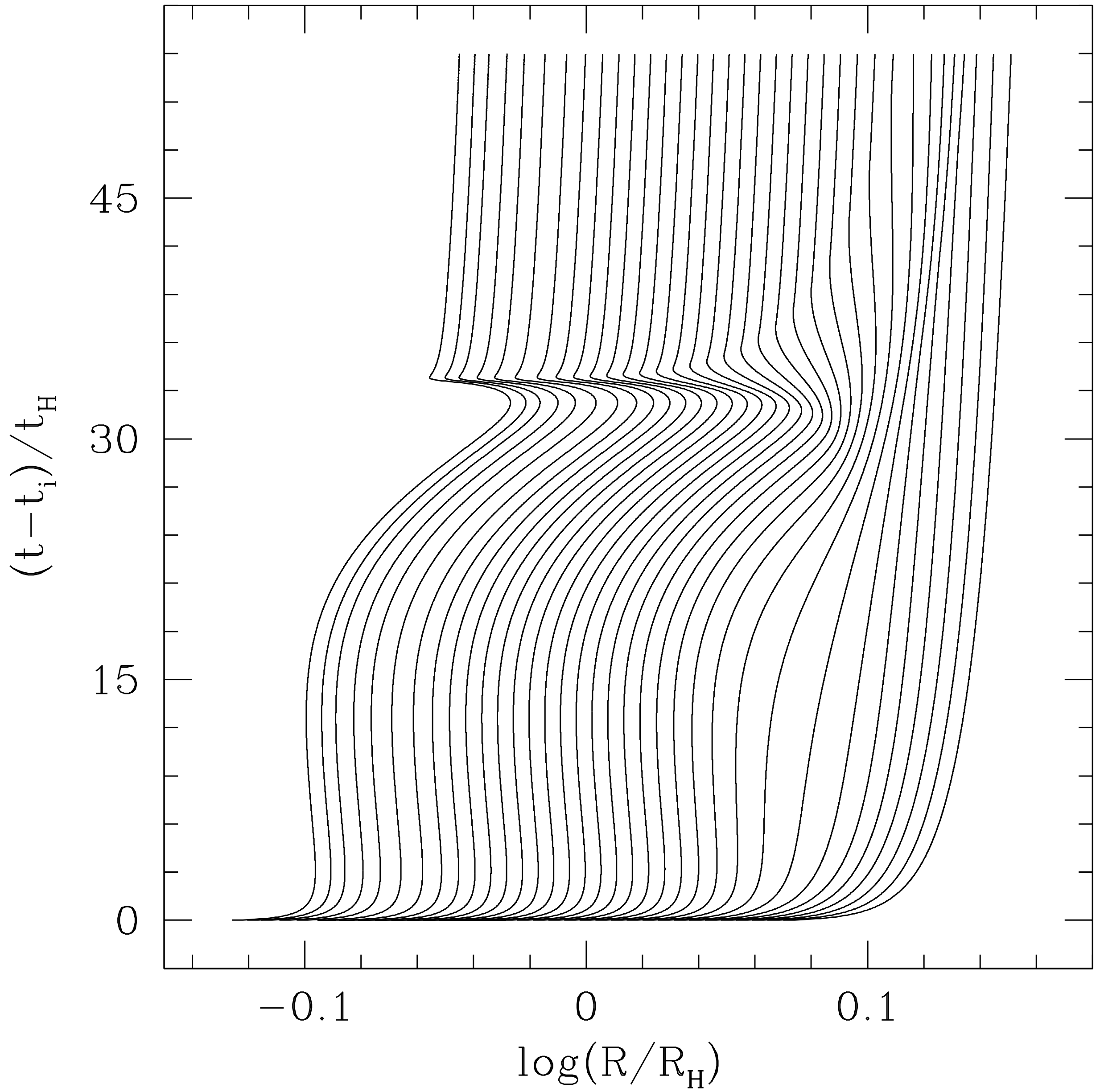
The collapse shows the presence of a very strong wind during the collapse that blows out a lot of material (*solid line*) which creates a deep evacuated region around the BH (*dashed line*). This region is then refilled very gently after a large number of dynamical time scales.



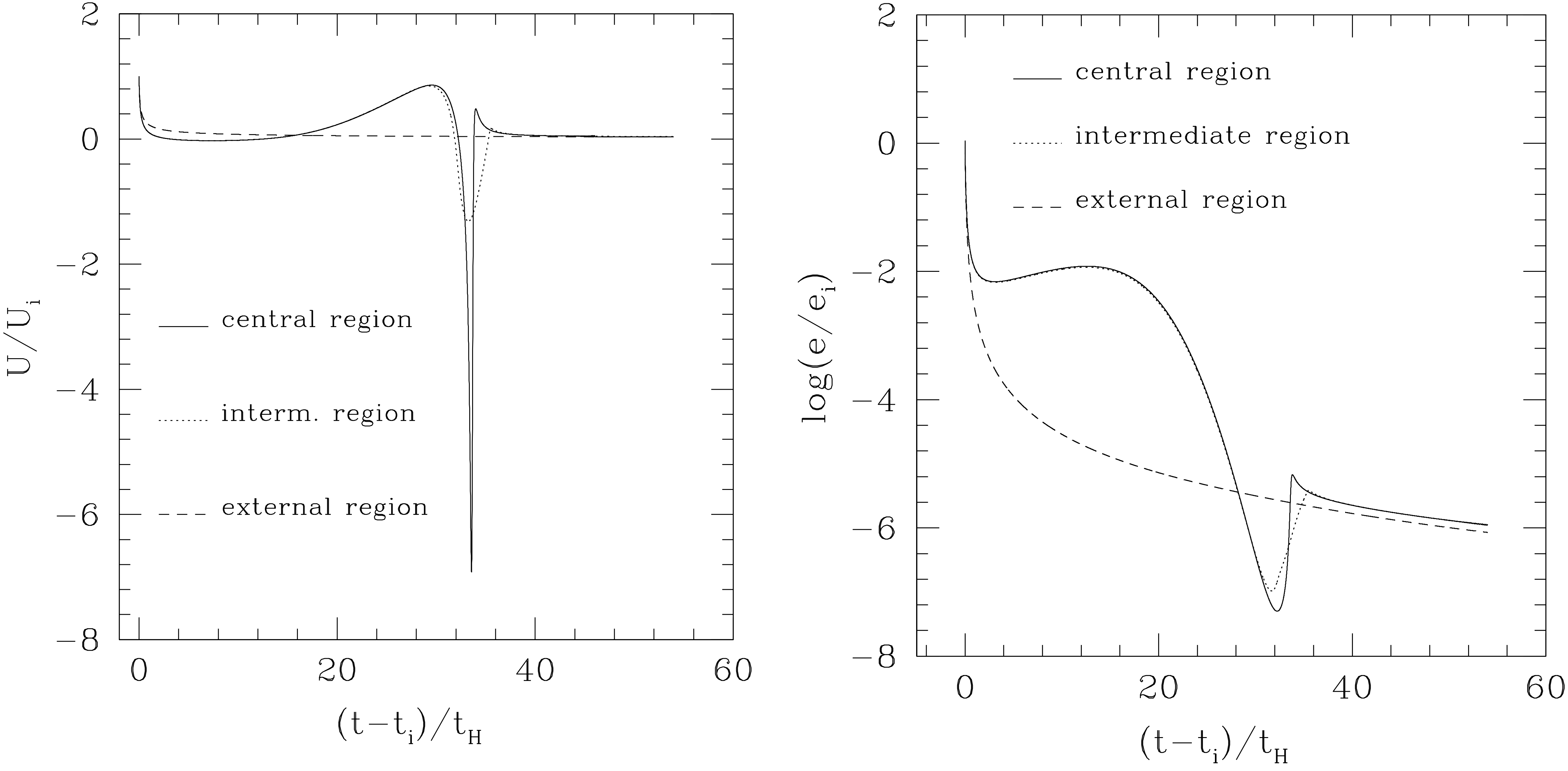


Numerical Results: no-BH formation

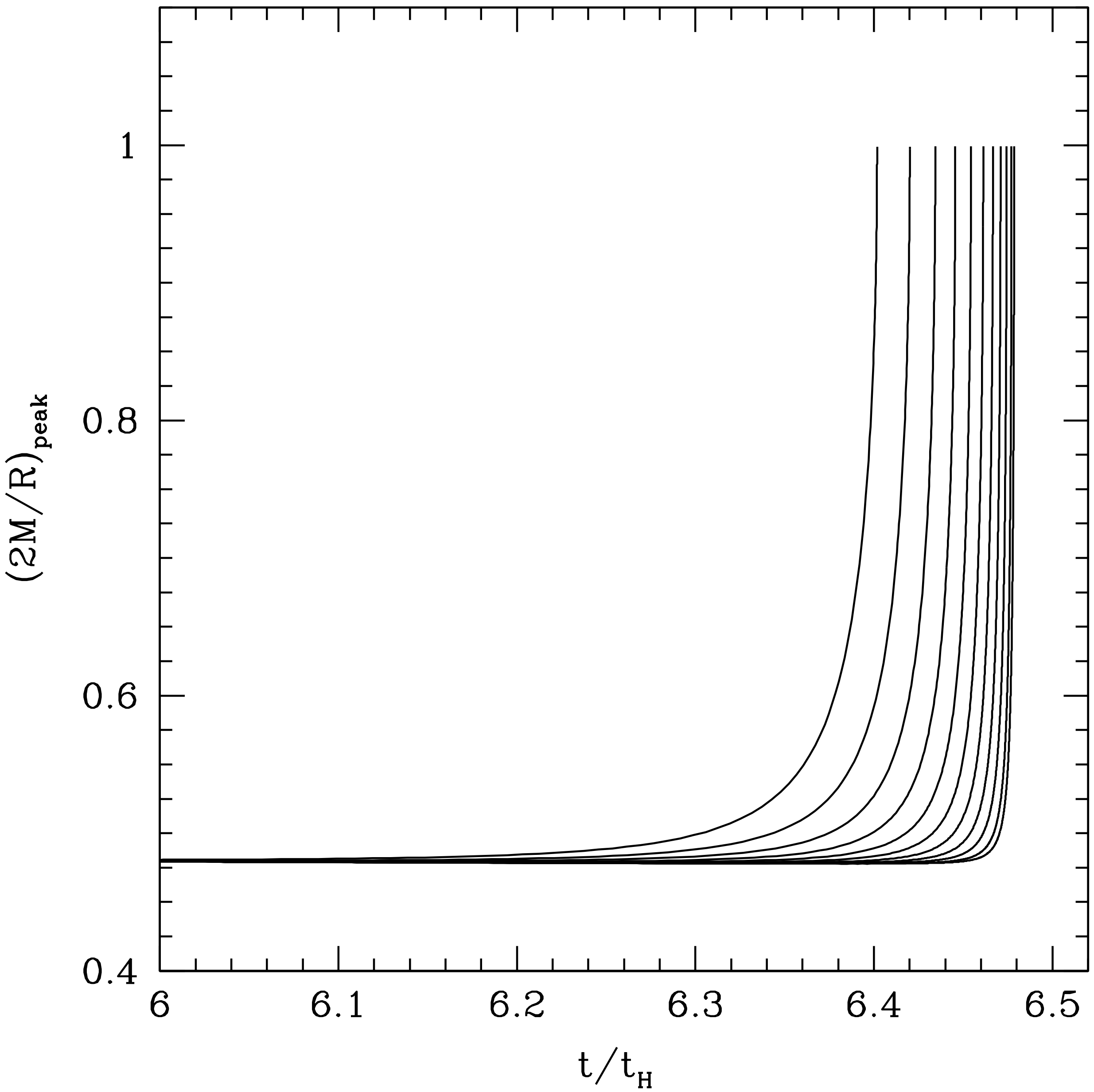


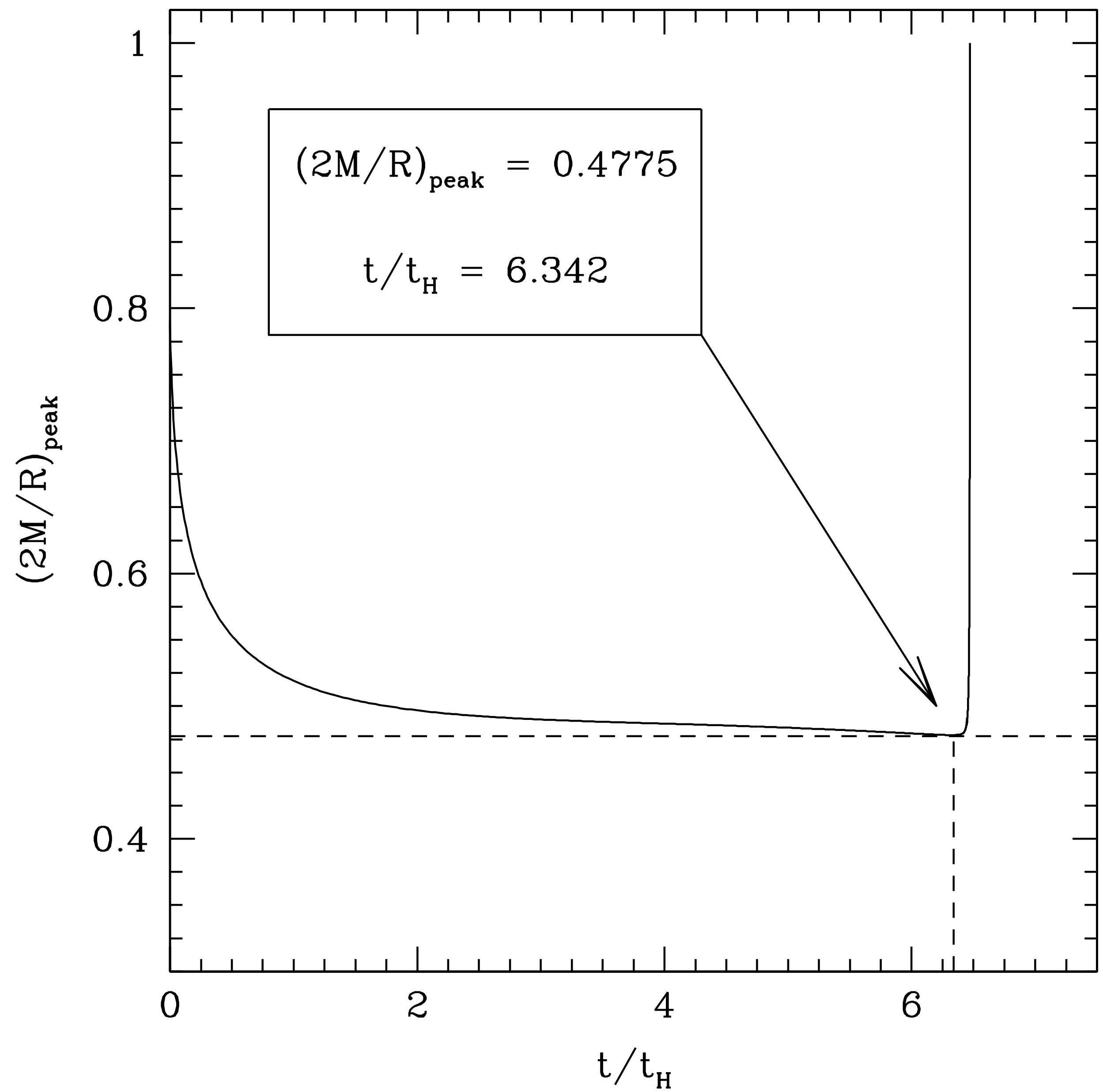


Numerical Results: perturbation bounce



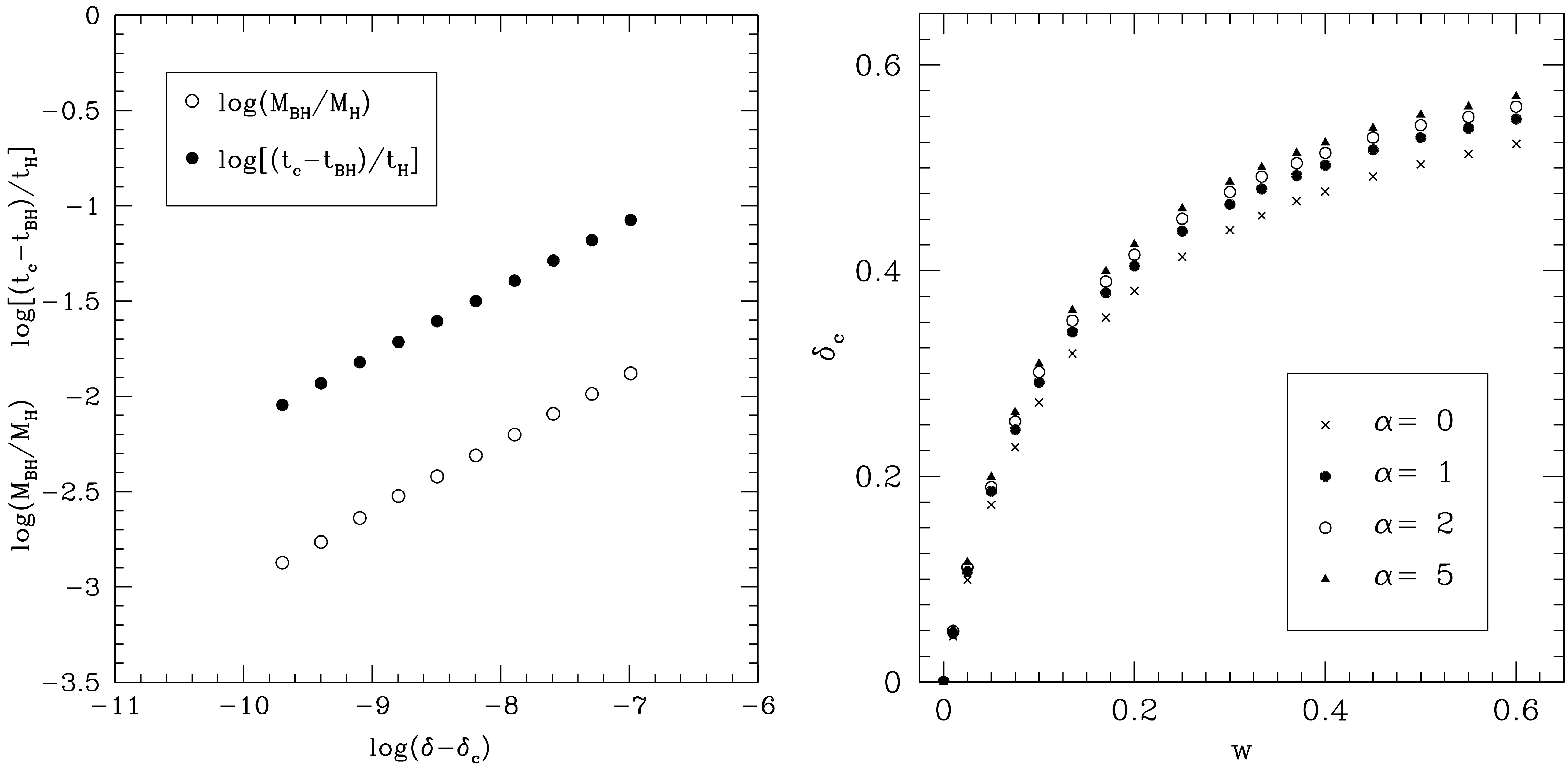
Numerical Results: intermediate state





Numerical Results: scaling law and threshold

$$M_{BH} = K(\delta - \delta_c)^{\gamma} M_H$$



Self similar equations

The Einstein + fluid equations can be written in a self similar form, using the self similar variables $(U,\ \Omega = 4\pi R^2 e,\ \Phi = M/R,\ \Gamma,\ f)$, function of $\xi = R/(\pm t)$

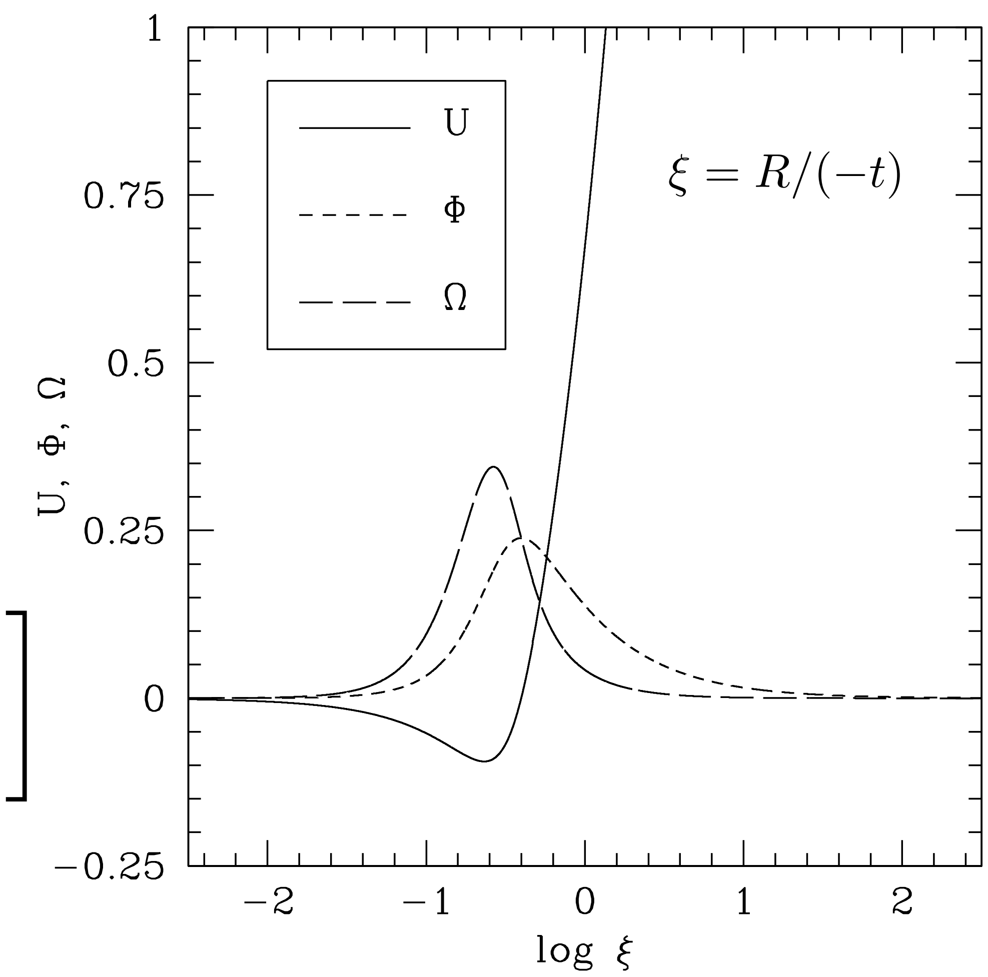
$$\frac{d\ln U}{d\ln\xi} = \left[\frac{(\Phi + w\Omega)^2 - 2w\Gamma^2\Phi}{U^2(\Phi + w\Omega)^2 - w\Gamma^2(\Omega - \Phi)^2}\right]\left[(\Omega - \Phi) - \frac{(1+w)\Omega U}{(\Gamma + U)}\right]$$

$$\frac{d\ln\Omega}{d\ln\xi} = \frac{(1+w)(\Omega - \Phi)}{(\Phi + w\Omega)}\frac{d\ln U}{d\ln\xi} + \frac{2w}{(\Phi + w\Omega)}\left[(\Omega - \Phi) - \frac{(1+w)\Omega U}{(\Gamma + U)}\right]$$

$$\frac{d\ln\Phi}{d\ln\xi} = \frac{1}{\Phi}\left[(\Omega - \Phi) - \frac{(1+w)\Omega U}{(\Gamma + U)}\right]$$

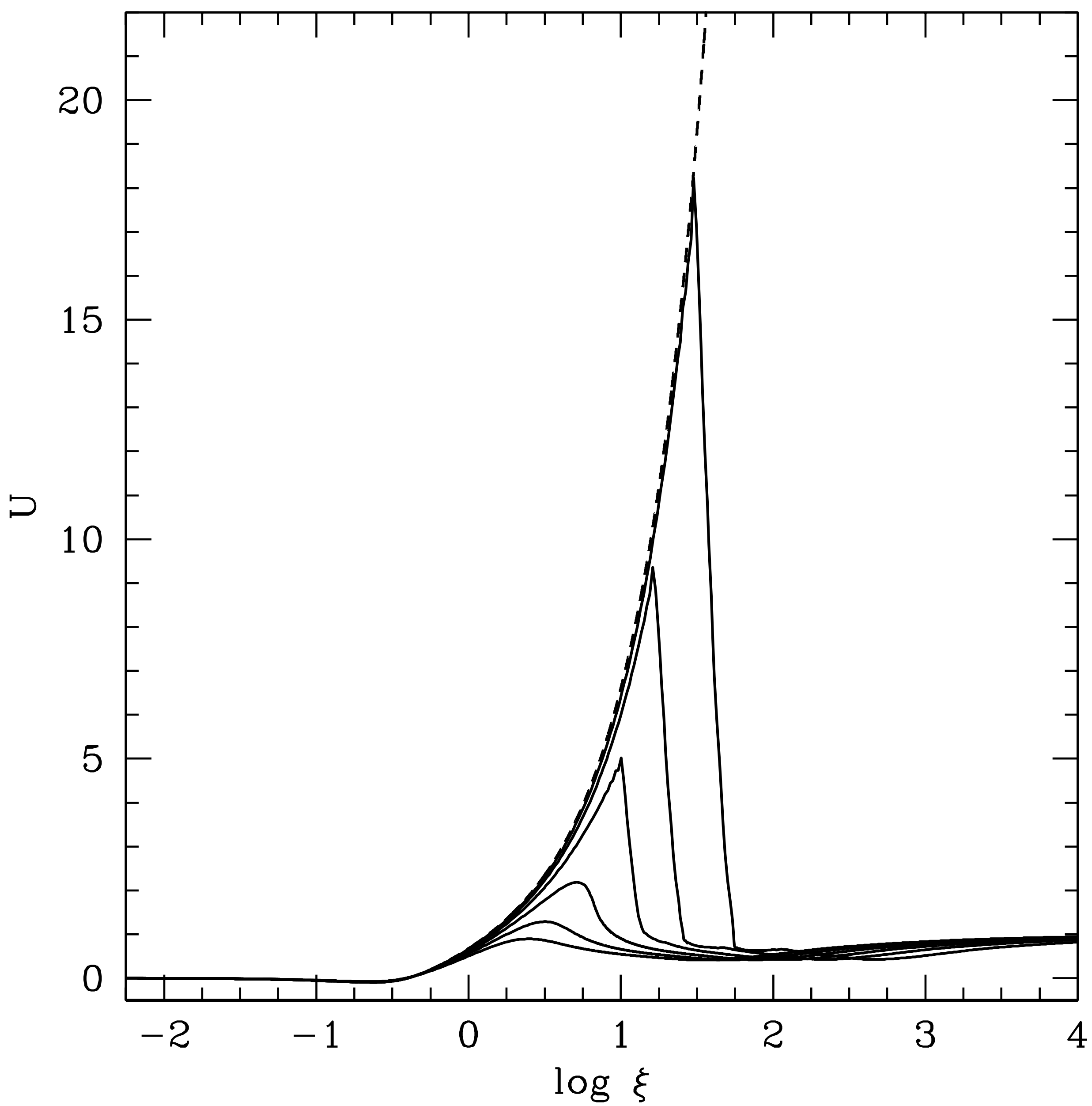
$$\Gamma = 1 + U^2 - 2\Phi$$

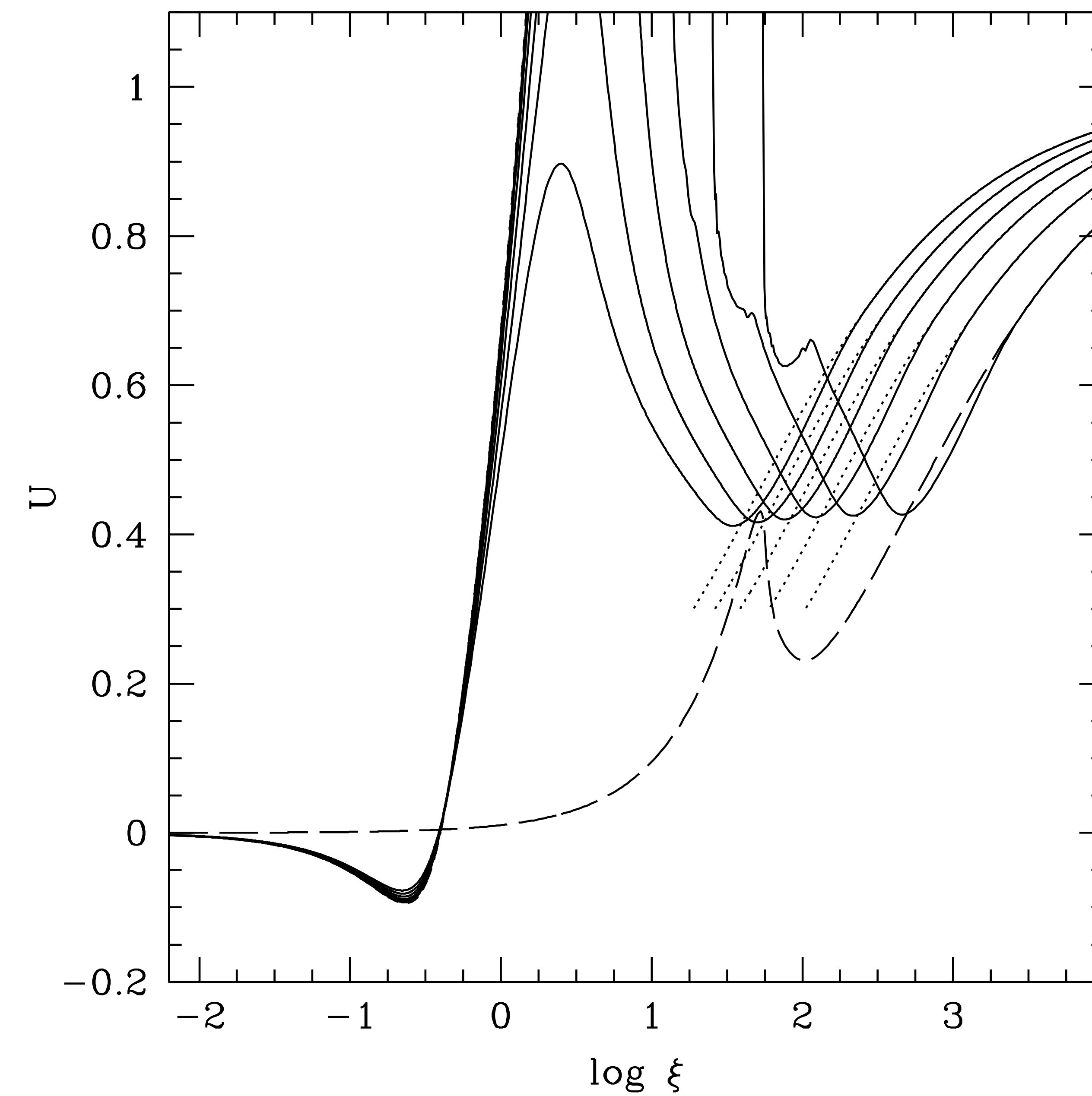
$$f = \pm\frac{\xi}{(1+w)\Omega U}\left[(\Omega - \Phi) - \frac{U}{\Gamma}(\Phi + w\Omega)\right]$$



Numerical Results: self similarity (U)

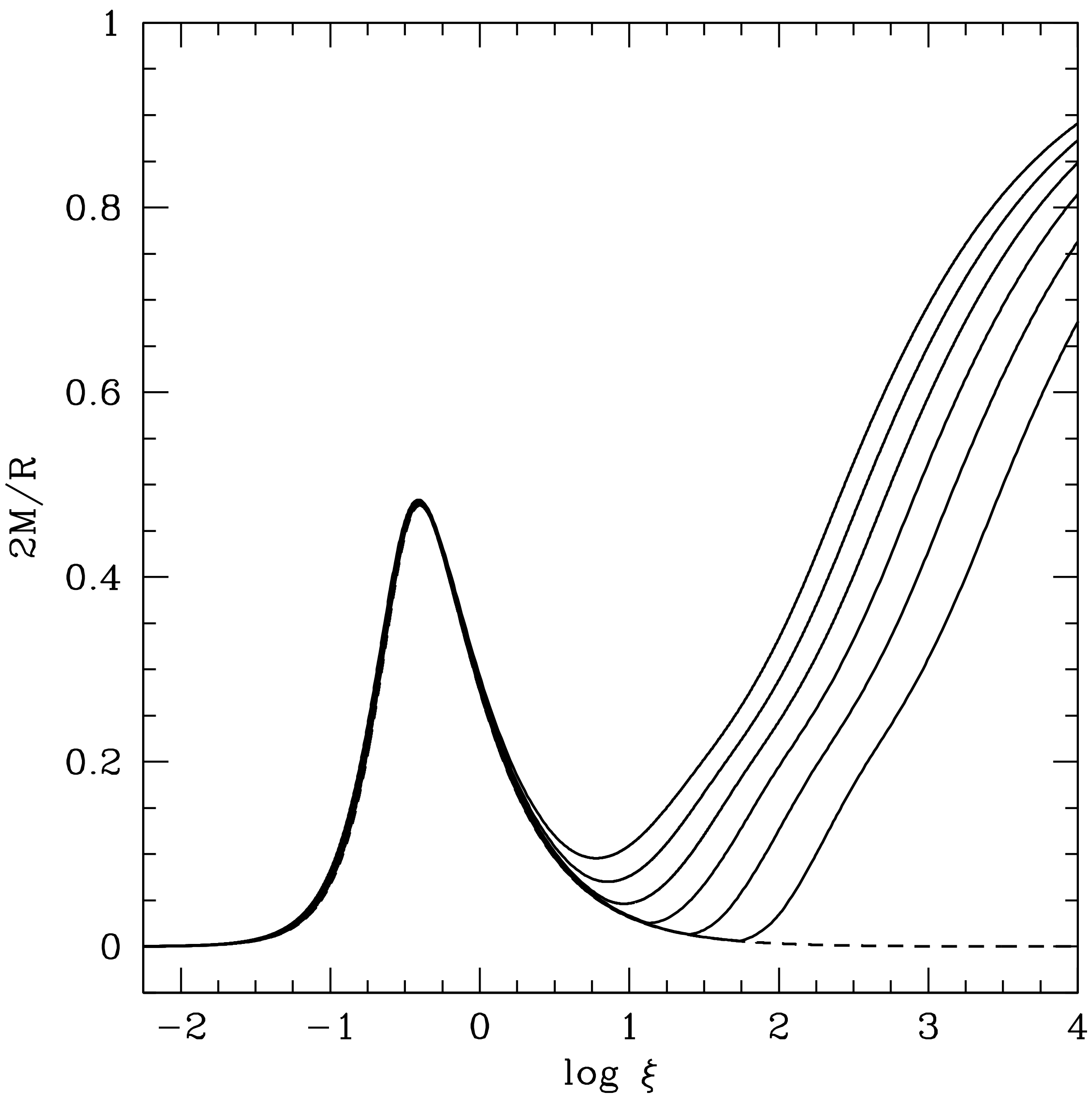
$$\xi = R/(t_c - t)$$

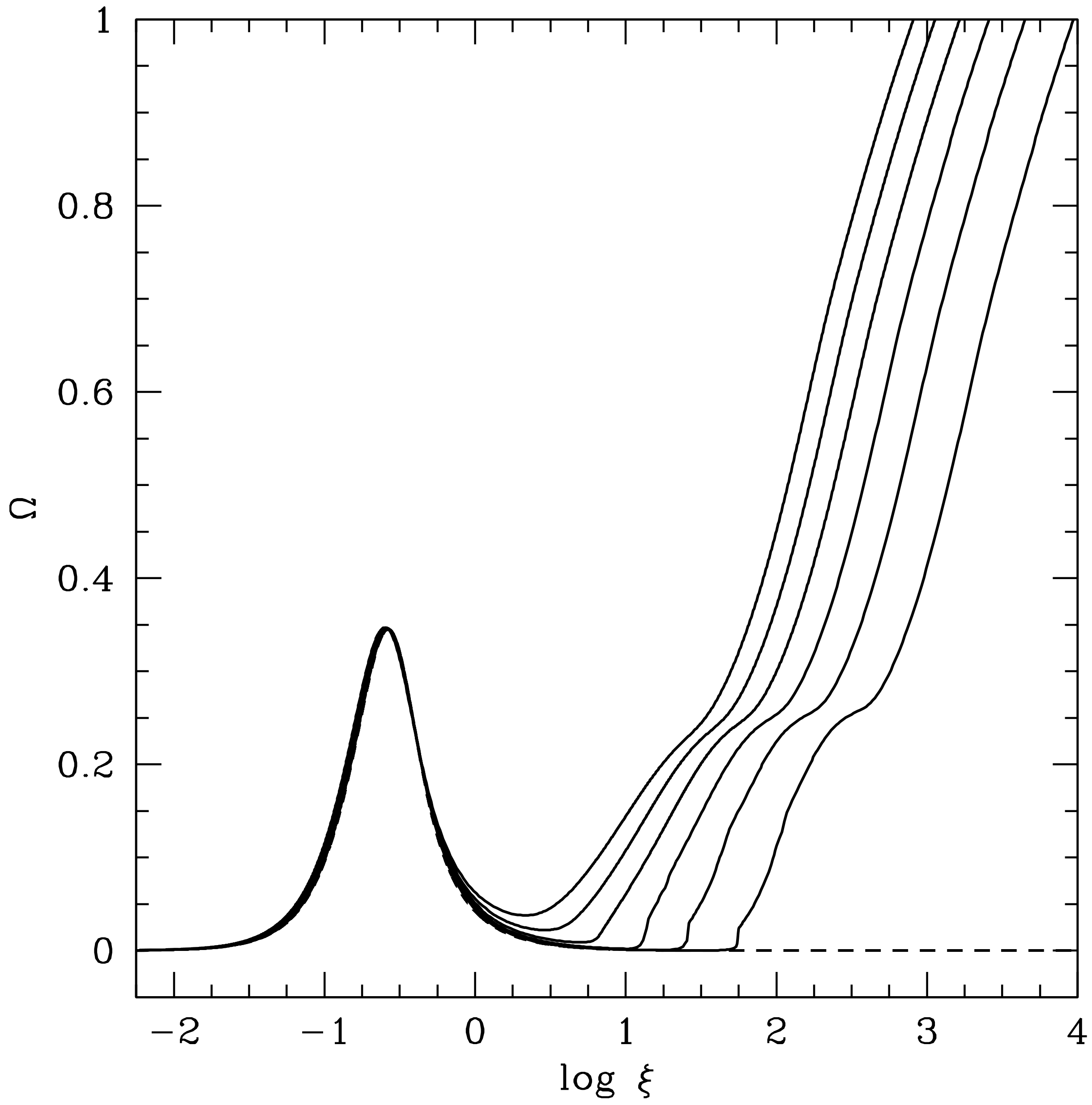




Numerical Results: self similarity (Φ,Ω)

$$\xi = R/(t_c - t)$$

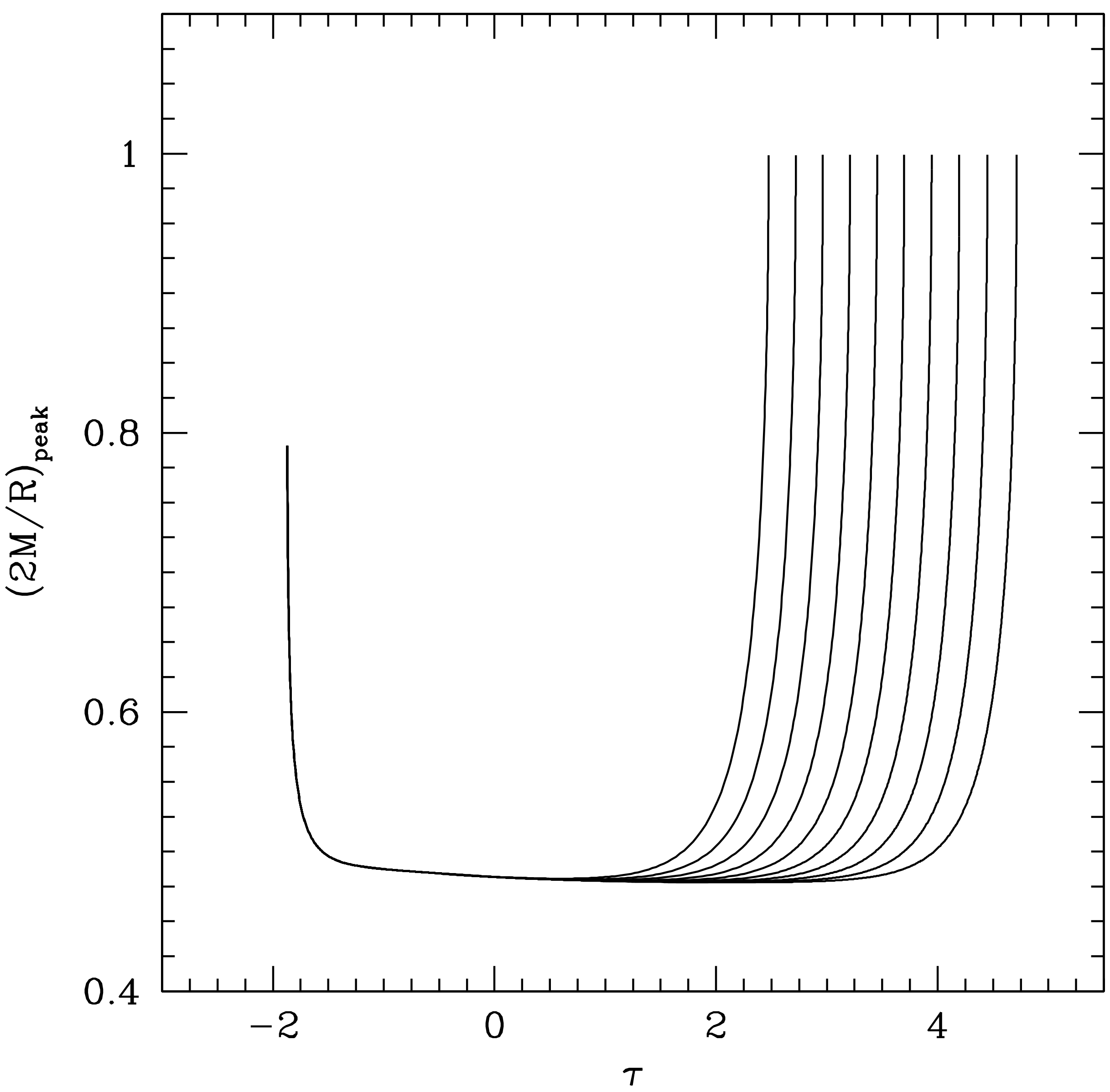


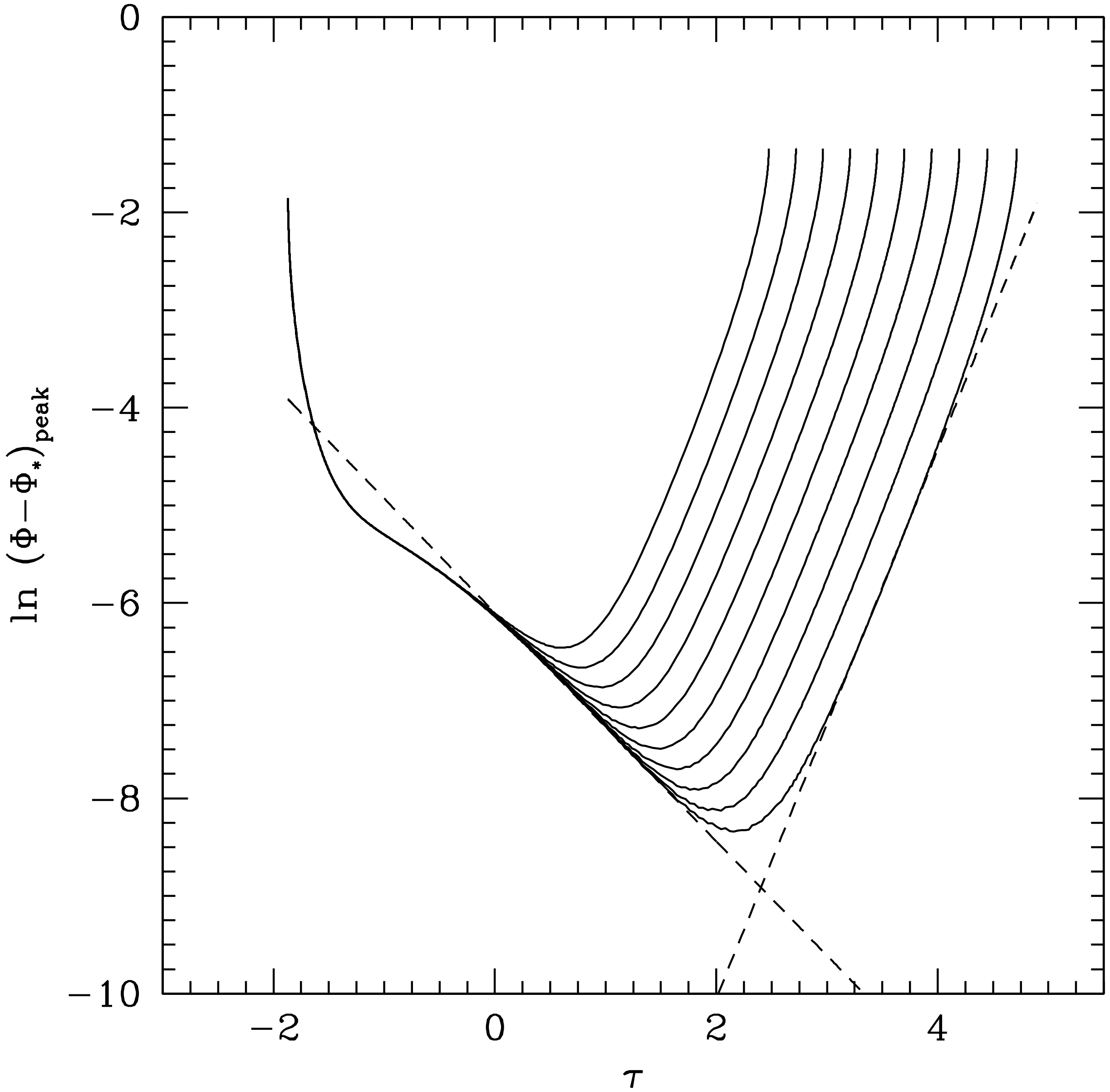


Numerical Results: growing and decaying modes

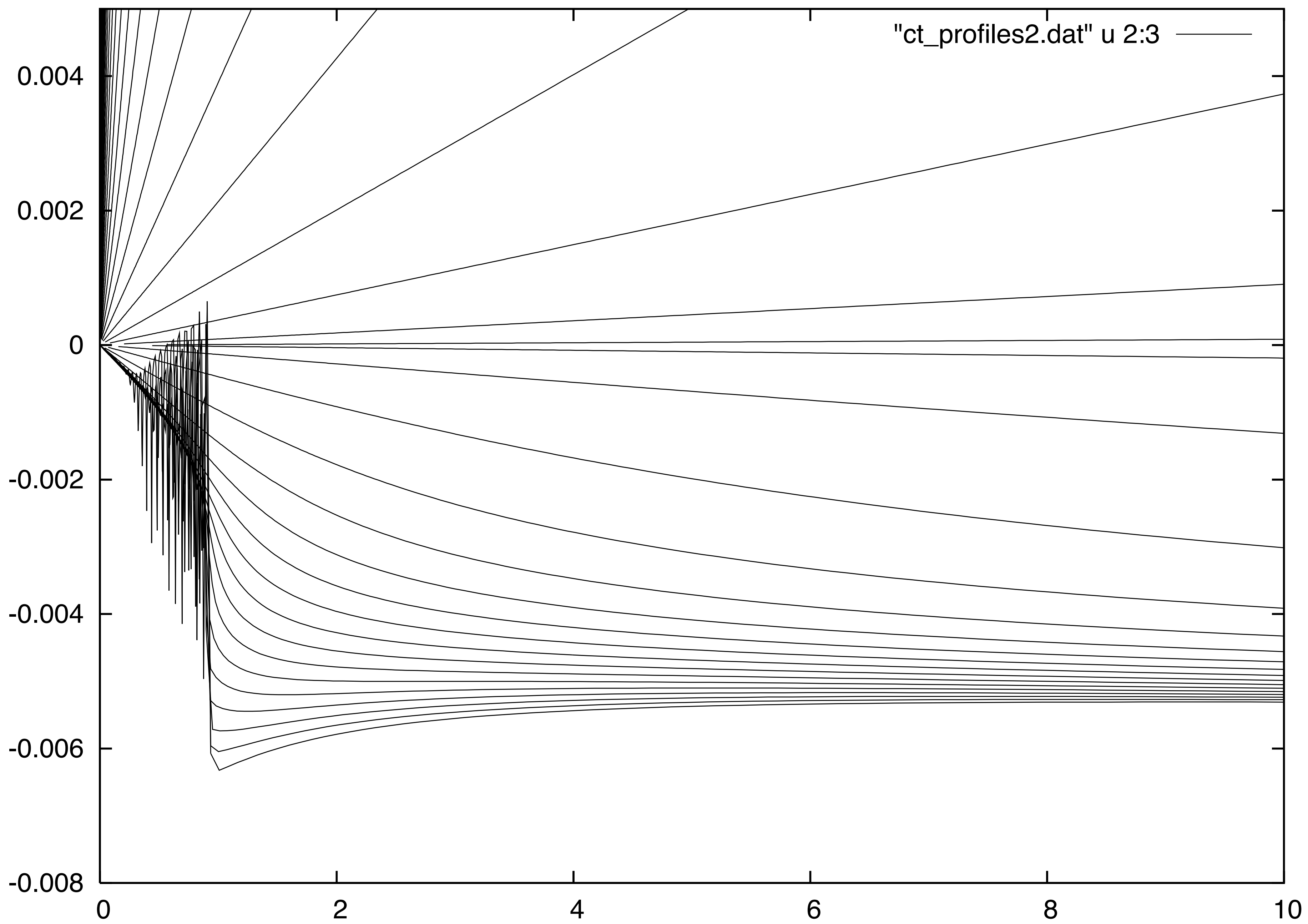
$$\Phi_+(\xi,\tau) - \Phi_\star(\xi) \propto (\delta - \delta_c) e^{\lambda_0 \tau} \psi_0(\xi) \qquad \tau = -\ln[(t_c - t)/t_H]$$

$$\Phi_-(\xi,\tau) - \Phi_\star(\xi) \propto e^{\lambda_1 \tau} \psi_1(\xi) \qquad 1/\lambda_0 = 0.356 = \gamma$$

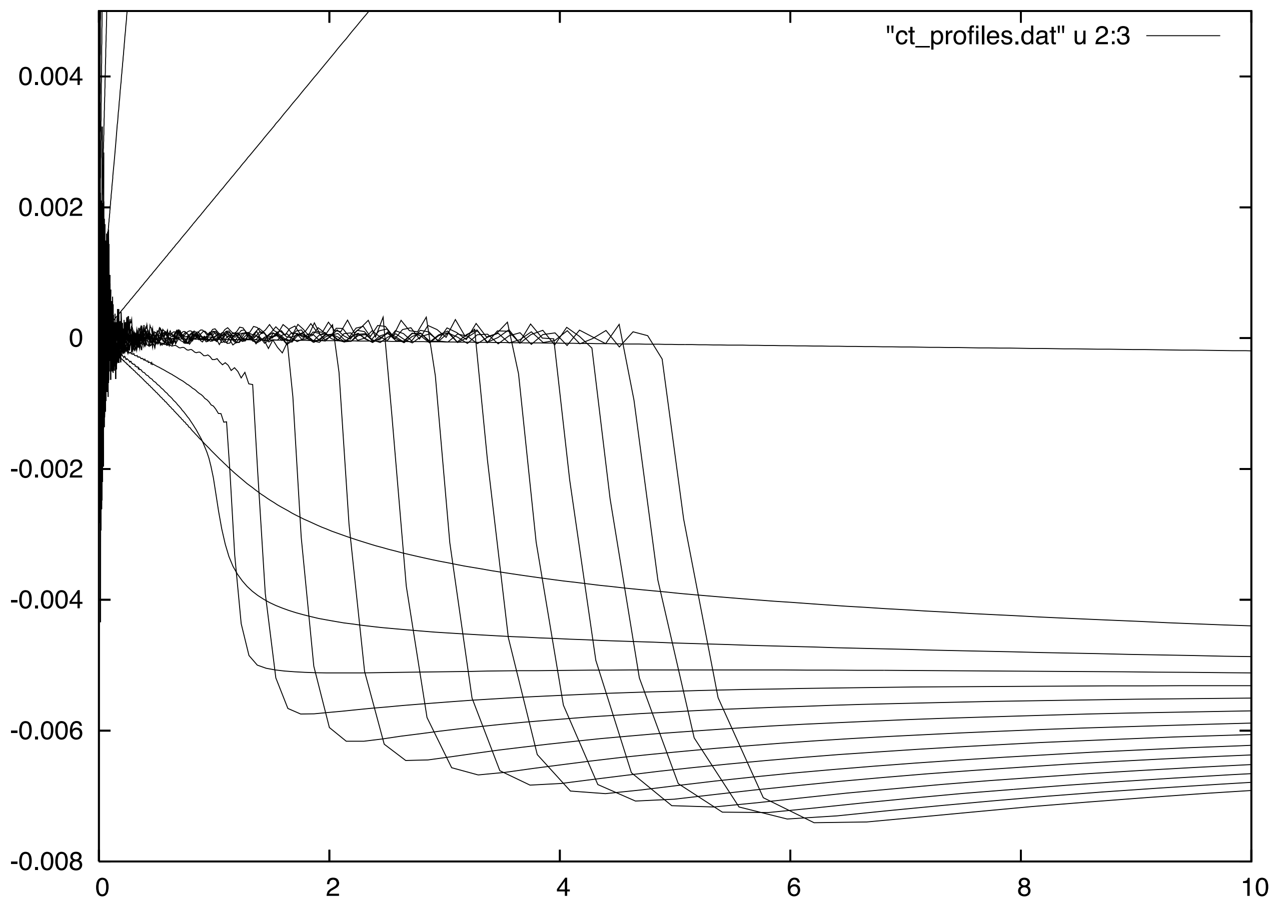


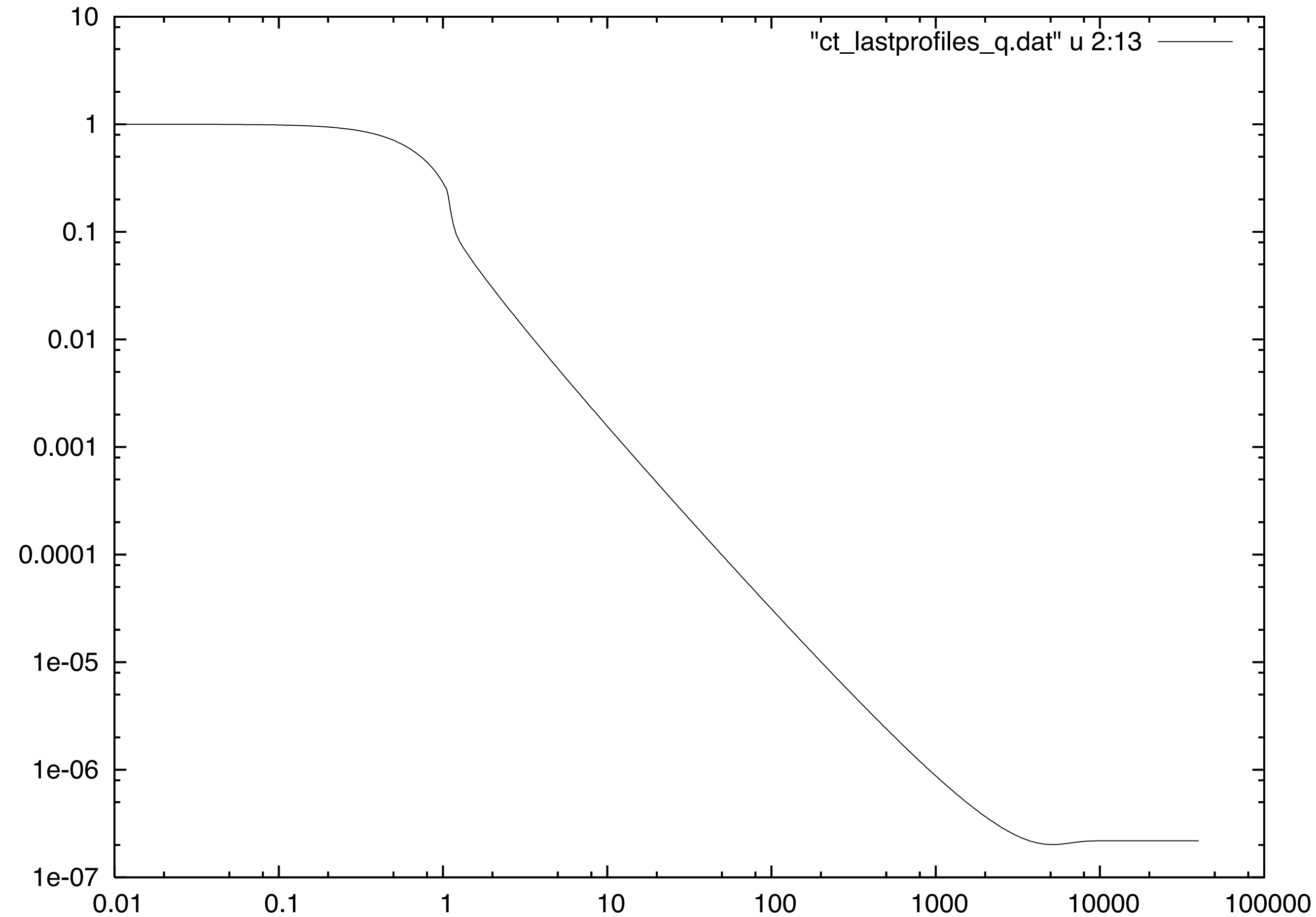


Numerical Results: virialization (no shock treatment)



Numerical Results: virialization (artificial viscosity)





Conclusions

- Starting with self similar solutions (FRW universe)...

$$\xi = R/t \qquad U = HR = \frac{2\xi}{3(1+w)} \qquad \Omega = 3\Phi = \frac{3}{2}U^2$$

- ... introducing a linear perturbation that is able to collapse, at the end of the non linear evolution, a collapsing self similar solution arises.

CRITICAL SOLUTION FOR PBH FORMATION

$$\xi = R/(-t)$$

REFERENCE: I Musco & J.C. Miller, CQG 30 (2013) 145009