<table>
<thead>
<tr>
<th>The geometry of $SU(2)$ recoupling theory</th>
<th>Algebraic background &amp; Askey scheme</th>
<th>Outlook</th>
</tr>
</thead>
</table>

Quantum geometry, quadratic dynamical algebras and discrete polynomials

Annalisa Marzuoli

Dipartimento di Matematica 'F. Casorati' & INFN (Pavia)

Napoli, December 19, 2013
Outline

1. The geometry of $SU(2)$ recoupling theory
   - Binary coupling of angular momenta and 6j symbols
   - Regge symmetries
   - Symmetric coupling: the volume operator
   - Semiclassical Hamiltonian picture

2. Algebraic background & Askey scheme
   - Quadratic Poisson algebras
   - Relation with special function theory
   - Dictionary & eigenfunctions of a quantum of space

3. Outlook

Annalisa Marzuoli
Dipartimento di Matematica 'F. Casorati' & INFN (Pavia)
Quantum geometry, quadratic dynamical algebras and discrete polynomials
Binary coupling of angular momenta and $6j$ symbols

**Standard $SU(2)$ recoupling theory in quantum mechanics**

The coupling theory of (eigenstates of) an ordered triple of $SU(2)$ angular momenta operators $J_1, J_2, J_3$ to states of sharp total angular momentum $J \equiv J_4$ is usually based on **binary couplings**. The admissible schemes are

$J_1 + J_2 = J_{12}; \quad J_{12} + J_3 = J$ and $J_2 + J_3 = J_{23}; \quad J_1 + J_{23} = J$

with complete sets of eigenvectors given by

$|j_{12}> := |(j_1j_2)j_{12}j_3jm>$

$|j_{23}> := |j_1(j_2j_3)j_{23}jm>$

Here (e.g.): $J_i^2 |j_{12}> = j_i (j_i + 1) |j_{12}>$; labels run over **irreps**

$\{0, 1/2, 1, 3/2, \ldots \}$ in $\hbar$ units and $m$ is the eigenvalue of $J_z$ with $-j \leq m \leq j$ in integer steps.
The 6j symbol and the tetrahedron

The Racah–Wigner 6j symbol is the unitary (orthogonal) transformation or 'recoupling coefficient' relating the two sets of binary coupled states ($\Phi \equiv j_1 + j_2 + j_3$)

$$< j_{23} | j_{12} > = (-1)^\Phi [(2j_{12} + 1)(2j_{23} + 1)]^{1/2} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{array} \right\}$$

According to the geometric, semiclassical view of [Wigner, Ponzano & Regge 1968] the 6j is associated with a solid Euclidean tetrahedron bounded by triangular faces $\leftrightarrow$ triads of labels
The geometry of SU(2) recoupling theory

Binary coupling of angular momenta and $6j$ symbols

**Edge $\leftrightarrow j$-label ($J = j + 1/2$)**

Labeling as a recoupling coefficient

$$\{j_1 \ j_2 \ j_{12} \ j_3 \ j_4 \ j_{23}\}$$

Labeling as a [4 parameters + 2 variables] array

$$\begin{pmatrix} a & b & \ell \\ c & d & \tilde{\ell} \end{pmatrix}$$

$$a + b + c + d \equiv j_1 + j_2 + j_3 + j_4: \text{ perimeter of a quadrilateral}$$
Each entry of the $6j$ is equivalent to any other $\Rightarrow$

Tetrahedral symmetry: $T \subset O(3)$ (order 24), isomorphic to the symmetric group $S_4$

Regge symmetries [1959] are 'functional' relations $[s = (a + b + c + d)/2]$

\[
\begin{align*}
\{ a & \ b & \ell \\ c & d & \tilde{\ell} \} = \{ s - a & \ s - b & \ell \\ s - c & s - d & \tilde{\ell} \} & := \{ a' & \ b' & \ell \\ c' & d' & \tilde{\ell} \}
\end{align*}
\]

The number of classical and Regge symmetries is $144 = \text{Order } (S_4 \times S_3)$: the rationale relies on the fact that the $6j$ is a polynomial function (rearranged from the Racah sum rule) $\rightarrow \rightarrow$  

Annalisa Marzuoli
Dipartimento di Matematica 'F. Casorati' & INFN (Pavia)
Quantum geometry, quadratic dynamical algebras and discrete polynomials
The Racah polynomial can be written in terms of the $4F_3$ hypergeometric function evaluated for the argument $z = 1$

$$\left\{\begin{array}{ccc} a & b & d \\ c & f & e \end{array}\right\} = \Delta(abe) \Delta(cde) \Delta(acf) \Delta(bdf) \left(-\right)^{\beta_1} (\beta_1 + 1)!$$

$$\times 4F_3 \left(\begin{array}{cccc} \alpha_1-\beta_1 & \alpha_2-\beta_1 & \alpha_3-\beta_1 & \alpha_4-\beta_1 \\ -\beta_1-1 & \beta_2-\beta_1+1 & \beta_3-\beta_1+1 \\ 1 \end{array}\right)$$

$$\times \frac{\Delta(ab)(\beta_2-\beta_1)(\beta_3-\beta_1)(\beta_1-\alpha_1)(\beta_1-\alpha_2)(\beta_1-\beta_3)(\beta_1-\alpha_4)!}{(\beta_2-\beta_1)! (\beta_3-\beta_1)! (\beta_1-\alpha_1)! (\beta_1-\alpha_2)! (\beta_1-\beta_3)! (\beta_1-\alpha_4)!}$$

$\beta_1 = \min(a + b + c + d; a + d + e + f; b + c + e + f)$; $\beta_2, \beta_3$ are identified in either way with the pair remaining in the 3–tuple $(a + b + c + d; a + d + e + f; b + c + e + f)$; $\alpha$’s may be identified with any permutation of $(a + b + e; c + d + e; a + c + f; b + d + f)$; $\Delta (abc) = \left[\left(\left((a + b - c)! (a - b + c)! (-a + b + c)!\right)/(a + b + c + 1)!\right)^{1/2}\right.$
Symmetry Properties of Racah's Coefficients.

T. Regge

Istituto Nazionale di Fisica Nucleare, Sezione di Torino - Torino

(ricettuto il 9 Ottobre 1968)

We have shown in a previous letter (1) that the true symmetry of Clebsch-Gordan coefficients is much higher than was before believed. A similar result has been now obtained for Racah's coefficients. Although no direct connection has been established between these wider symmetries it seems very probable that it will be found in the future. We shall merely state here the results which can be checked very easily with the help of the well known Racah's formula:
The geometry of $SU(2)$ recoupling theory

Symmetric coupling: the volume operator

The volume operator

Alternatively to the binary coupling of angular momenta, the volume operator

\[ K := J_1 \cdot J_2 \times J_3 \]

acts 'democratically' on vectors $J_1$, $J_2$ and $J_3$ plus a fourth one, $J_4$,\(^1\) which close a (not necessarily planar) quadrilateral vector diagram

\[ J_1 + J_2 + J_3 + J_4 = 0 \]

**Generalized recoupling coefficients** between a binary–coupled state, say $|j_{12} > \equiv |\ell >$ and a ket $|j_1, j_2, j_3, j_4; k > \equiv |k >$ where $K|k > = k|k >$ (eigenvalues $k$ come in pairs $\pm k$).

Their symmetrized form is denoted in short $\Phi^{(k)}_{\ell}$ (eigenfunctions of the volume operator expanded in the $\ell$-representation)

\(^1\) i.e. on either a composite system of 4 objects with vanishing total angular momentum, or a system of 3 objects with total angular momentum $J_4$
The eigenfunctions $\Phi^{(k)}_\ell$ satisfy a three-terms recursion relation\(^2\) which can be turned into a a real, finite-difference Schrödinger-like equation \(^3\)

\[
\alpha_{\ell+1} \Phi^{(k)}_{\ell+1} + \alpha_{\ell} \Phi^{(k)}_{\ell-1} = k \Phi^{(k)}_{\ell}
\]

The matrix elements $\alpha_\ell$ are given in terms of geometric quantities

\[
\alpha_\ell = \frac{F(\ell;j_1 + 1/2;j_2 + 1/2)F(\ell;j_3 + 1/2;j_4 + 1/2)}{\sqrt{(2\ell + 1)(2\ell - 1)}}
\]

\[
F(A, B, C) = \frac{1}{4}[(A + B + C) (-A + B + C) (A - B + C) (A + B - C)]^{1/2}
\]
is the area of a triangle with side lengths $A, B$ and $C$

---

\(^2\)Levy-Leblond (1965) up to [Carbone, Carfora, Marzuoli (2002)]

The role of Regge symmetry

\[ J_1 = j_1 + \frac{1}{2}, \quad J_2 = j_2 + \frac{1}{2}, \quad J_3 = j_3 + \frac{1}{2}, \quad J_4 = j_4 + \frac{1}{2} \]

are parameters labeling a quadrilateral with vertices (1,2,3,4), which, together with its Regge-conjugate [vertices (1',2',3',4')]

\[ J'_1 = j'_1 + \frac{1}{2}, \quad J'_2 = j'_2 + \frac{1}{2}, \quad J'_3 = j'_3 + \frac{1}{2}, \quad J'_4 = j'_4 + \frac{1}{2} \]

ccharacte\textit{rize the analysis of the discrete, Regge–invariant Schrödinger Eq for a quantum of space} 'bounded' by two confocal ellipses

\textbf{NB} The parameter \( u \) in the figures comes out once a reparametrization is performed: then the (1,2,3,4) quadrilateral \( \leftrightarrow \) a quaternion \( Q \), while (1',2',3',4') \( \leftrightarrow \tilde{Q} \), its conjugate (whose common real part is \( s \), the semi–perimeter in Regge’s formula)
Symmetric coupling: the volume operator

Annalisa Marzuoli

Dipartimento di Matematica 'F. Casorati' & INFN (Pavia)

Quantum geometry, quadratic dynamical algebras and discrete polynomials
The geometry of $SU(2)$ recoupling theory

Algebraic background & Askey scheme

Outlook

Symmetric coupling: the volume operator

Symmetric coupling: the volume operator

$\pi/2 + \varphi$

Annalisa Marzuoli
Dipartimento di Matematica 'F. Casorati' & INFN (Pavia)

Quantum geometry, quadratic dynamical algebras and discrete polynomials
In terms of the shift operator $e^{\pm i \varphi} \Phi^{(k)}_{\ell} = \Phi^{(k)}_{\ell \pm 1}$, the Hamiltonian operator is

$$\hat{H} = (\alpha_{\ell} e^{-i \varphi} + \alpha_{\ell+1} e^{i \varphi})$$

with $\varphi = -i \frac{\partial}{\partial \ell}$

representing the variable canonically conjugate to $\ell$.

The semiclassical behavior of the discrete SE is studied by resorting to a discrete WKB approach [Braun 1993]: the two-dimensional phase space $(\ell, \varphi)$ supports the classical Hamiltonian

$$H = 2 \alpha_{\ell + \frac{1}{2}} \cos \varphi$$

Geometrically, the two Regge–conjugate ‘fluttery’ quadrilaterals fold along the common diagonal $\ell$ with $\varphi$ perceived as a torsion angle $\rightarrow$ (figure above)
Analytical and numerical results

- Analytical expressions for (lower) eigenvalues and explicit form of \( \Phi^{(k)}_{\ell} \) [Carbone et al 2002] (complicated)

- Characterization of \( \{ \Phi^{(k)}_{\ell} \} \) as a family of orthogonal polynomials of the discrete variable \( k \) (not evenly-spaced) with degree given by \( \ell \) (see below)

- Characterization of the 'dual' family \( \{ \Psi^{(\ell)}_{k} \} \) (see below)

- Spectrum of the volume operator (numerically): the horizontal lines represent the eigenvalues \( k \), the curves are the caustics (the turning points of the semiclassical analysis), which limit the classically allowed region:
  - in red \( U^+_\ell = 2\alpha_\ell \)
  - in blue \( U^-_\ell = -2\alpha_\ell \)
Numerical evaluation

Left: parameters $j_1, j_2, j_3, j_4 = 8.5, 10.5, 13.5, 14.5$ or $s, u, r, v = 23.5, -4.5, 1.5, 0.5$.
Right: all four parameters are doubled.
Quadratic Poisson algebras

Quantum mechanical dynamical algebras on 3 generators

\( K_1, K_2, K_3 \): generators of quadratic operator algebras\(^4\)
(if \( K_{1,2} \) are Hermitian then \( K_3 \) is anti–Hermitian).

Commutation relations, where \( \{ \, , \} \) is the anticommutator:

\[
\begin{align*}
[K_1, K_2] &= K_3 \\
[K_2, K_3] &= 2R K_2 K_1 K_2 + A_1 \{K_1, K_2\} + A_2 K_2^2 + C_1 K_1 + D k_2 + G_1 \\
[K_3, K_1] &= 2R K_1 K_2 K_1 + A_1 K_1^2 + A_2 \{K_1, K_2\} + C_2 K_2 + D K_1 + G_2
\end{align*}
\]

\(^4\)Classically: Poisson algebras on three dynamical variables \( k_1, k_2, k_3 \). The further property of ‘mutual integrability’ of \( k_{1,2} \) or \( K_1, K_2 \) is required.
Classification of quadratic algebras [1986 → Zhedanov et al]

<table>
<thead>
<tr>
<th></th>
<th>$R$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$C$ &amp; $D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{AW}(3)$ (Askey–Wilson)</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$\mathbb{R}(3)$ (Racah)</td>
<td>0</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$\mathbb{H}(3)$ (Hahn)</td>
<td>0</td>
<td>0</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$\mathbb{J}(3)$ (Jacobi)</td>
<td>0</td>
<td>0</td>
<td>*</td>
<td>0</td>
</tr>
<tr>
<td>Lie algebras:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$su(2), su(1,1), h(1)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>*</td>
</tr>
</tbody>
</table>

The case $R = 0$ corresponds to the Racah algebra and the explicit construction of the relevant Hilbert (representation) spaces can be done in particular when $K_1, K_2$ have discrete spectra.
Ladder representations and eigenvalue problems

Starting from the eigenvalue problem for $K_1$

$$K_1 \psi_p = \lambda_p \psi_p, \quad p = 0, 1, 2, \ldots; \{\lambda_p\} \text{ evenly-spaced}$$

the commutation relations imply that the operator $K_2$ is tridiagonal in this basis

$$K_2 \psi_p = \alpha_{p+1} \psi_{p+1} + \alpha_p \psi_{p-1} + \beta_p \psi_p$$

and similarly

$$K_2 \phi_s = \mu_s \phi_s \quad (s = 0, 1, 2, \ldots) \Rightarrow$$

$$K_1 \phi_s = \gamma_{s+1} \phi_{s+1} + \gamma_s \phi_{s-1} + \delta_s \phi_s$$

(The matrix coefficients $\alpha, \beta, \gamma$ are evaluated from the specific commutation relations)
The overlap functions of the two (normalized) bases $\psi_p, \phi_s$

\[ < \phi_s | \psi_p > \equiv < s | p > \quad \text{and} \quad < \psi_p | \phi_s > \equiv < p | s > \]

are hypergeometric orthogonal polynomials of one discrete variable, here ($R = 0$ and eigenvalues of both $K_1$ and $K_2$ varying uniformly) Racah polynomials $\rightarrow$ on the top of the Askey scheme $\rightarrow$

**Duality property of the algebra $R(3)$**

- The exchange of generators $K_1 \leftrightarrow K_2, K_3 \rightarrow -K_3$ represents an automorphism of the Racah algebra $R(3)$
- Under this automorphism: the (discrete) variables of the two families of overlap functions and their degrees as polynomials are $\leftrightarrow$

**NB** All other functions in the lower levels of the Askey (Wilson) scheme are obtained by suitable limiting procedures on parameters (dropped in the present notation) and/or on spectral parameters.
The geometry of $SU(2)$ recoupling theory

Algebraic background & Askey scheme

Outlook

Relation with special function theory

Annalisa Marzuoli
Dipartimento di Matematica 'F. Casorati' & INFN (Pavia)

Quantum geometry, quadratic dynamical algebras and discrete polynomials
The operator $K_3$ and associated overlap functions

In the $K_1$–eigenbasis $K_3$ satisfies

$$K_3 \psi_p = (\lambda_{p+1} - \lambda_p) \alpha_{p+1} \psi_{p+1} - (\lambda_p - \lambda_{p-1}) \alpha_p \psi_{p-1}$$

($\alpha, \lambda$ as before)

$K_3$ has a discrete, but not evenly–spaced spectrum \{\nu_n, n = 0, 1, 2, \ldots \}. Even in finite–dimensional situations the diagonalization can be carried out analytically only for the lowest eigenvalues. If $\varphi_n$ are the eigenfunctions of $K_3$, the overlap functions

$$< \varphi_n | \psi_p > \quad \text{and} \quad < \psi_p | \varphi_n >$$

are orthogonal, ‘dual’ to each other but not trivially recognized as hypergeometric.
Generalized recoupling theory & the algebra $\mathbf{R}(3)$

The dynamical Racah algebra underlying the evolution of a single 'quantum of space'

- is generated by the operators

$$
K_1 := J^2_{12}; \quad K_2 := J^2_{23};
K_3 := [K_1, K_2] \equiv -4i \, J_1 \cdot (J_2 \times J_3) \equiv -4iK,
$$

with either the volume operator $K$, or $J^2_{12}$, or $J^2_{23}$ playing the role of the Hamiltonian

- has $S_4 \times S_3$ as automorphism group, thus providing a proper group–theoretic interpretation of (classical + Regge) discrete symmetries characterizing both a 'fluttery' quadrilateral configuration and a tetrahedron out of $SU(2)$ symmetric and binary re–coupling theory, respectively
The geometry of \( SU(2) \) recoupling theory

Algebraic background & Askey scheme

Dictionary & eigenfunctions of a quantum of space

The Nauru graph (drawn in the flag of the Nauru Republic)

- \((1, 2, 3, 4: \) vertices of the quadrilateral or of the tetrahedron permuted under \( S_4 \))
- Is a generalized Petersen graph with 24 vertices and 36 edges
- Is vertex and edge–transitive ('symmetric') \( \Rightarrow \) its automorphism group, \( S_4 \times S_3 \), acts transitively
- Can be looked at as a Cayley graph of \( S_4 \) generated by the action of \( S_3 \) which swaps: \((1 \leftrightarrow 2), (1 \leftrightarrow 3), (1 \leftrightarrow 4)\)
Up to relabeling \((j_1, j_2, j_3, j_4) \mapsto (a, b, c, d)\), and by resorting to Regge symmetry for suitably defining the (finite) ranges of such parameters, the binary and symmetric normalized recoupling functionals are denoted

\[
< \tilde{\ell} | \ell > (a, b, c, d) \propto \begin{pmatrix} a & b & \ell \\ c & d & \tilde{\ell} \end{pmatrix}
\]

\[
\Phi^{(k)}_{\ell}(a, b, c, d) = < \ell | k > (a, b, c, d)
\]

\[
\Psi^{(\ell)}_{k}(a, b, c, d) = < k | \ell > (a, b, c, d)
\]

(and similar for \(< \tilde{\ell} | k > (a, b, c, d)\))

**NB.** The eigenvalue problem for the volume operator \(K\) is rewritten

\[
K |k > = \lambda_k |k > (k = 0, 1, 2, \ldots)
\]

to emphasize the fact that the spectrum is not evenly–spaced.
## Families of discrete orthogonal polynomials with \((a, b, c, d)\) fixed\(^5\)

<table>
<thead>
<tr>
<th>family</th>
<th>orthogonality on lattice</th>
<th>eigenvalue ((\rightarrow \text{variable}))</th>
<th>degree ((\text{rel. to}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(&lt; \tilde{\ell}</td>
<td>\ell &gt;)</td>
<td>(\sum_{\ell} &lt; \tilde{\ell}</td>
<td>\ell' &gt; = \delta_{\ell' \ell})</td>
</tr>
<tr>
<td>(&lt; \ell</td>
<td>\tilde{\ell} &gt;)</td>
<td>(\sum_{\ell} &lt; \ell</td>
<td>\tilde{\ell}' &gt; = \delta_{\tilde{\ell}' \tilde{\ell}})</td>
</tr>
<tr>
<td>(&lt; \ell</td>
<td>k &gt;)</td>
<td>(\sum_{\ell} &lt; \ell</td>
<td>\ell' &gt; = \delta_{\ell' \ell})</td>
</tr>
<tr>
<td>(&lt; k</td>
<td>\ell &gt;)</td>
<td>(\sum_k &lt; \ell</td>
<td>\ell' &gt; = \delta_{\ell' \ell})</td>
</tr>
<tr>
<td>(&lt; \tilde{\ell}</td>
<td>k &gt;)</td>
<td>(\sum_{\tilde{\ell}} &lt; \tilde{\ell}</td>
<td>\ell' &gt; = \delta_{\ell' \ell})</td>
</tr>
<tr>
<td>(&lt; k</td>
<td>\tilde{\ell} &gt;)</td>
<td>(\sum_k &lt; \ell</td>
<td>\tilde{\ell}' &gt; = \delta_{\tilde{\ell}' \tilde{\ell}})</td>
</tr>
</tbody>
</table>


Annalisa Marzuoli  
Dipartimento di Matematica 'F. Casorati' & INFN (Pavia)  
Quantum geometry, quadratic dynamical algebras and discrete polynomials
• **Self-duality** of the black family $\rightarrow$ 6j-symbols $\leftrightarrow$ Racah polynomial $\leftrightarrow$ $4F_3$ on the top of the Askey scheme

$$\sum_{\tilde{\ell}} < \ell' | \tilde{\ell} > < \tilde{\ell} | \ell > = \delta_{\ell', \ell}$$

• Duality in the red family

$$\sum_{\ell} < k' | \ell > < \ell | k > = \delta_{k', k} \text{ and } \sum_{k} < \ell' | k > < k | \ell > = \pm \delta_{\ell', \ell}$$

• (Similar for the blue family)

• The reduction of such families to specific hypergeometric functions of type $4F_3$ would require to find out a closed algebraic form for eigenvalues of the volume operator for given parameters

• **Triangular relation(s)**, transversal with respect to the families

$$\sum_{k} < \tilde{\ell} | k > < k | \ell > = \pm < \tilde{\ell} | \ell >$$
Further developments can be addressed in parallel, from algebraic–analytical and geometric viewpoints

- Improve interpretations of the Regge symmetries on the geometric (scissor–congruent tetrahedra$^6$) and algebraic (quaternionic reparametrization) sides
- Explore $q$–deformed extensions and limiting cases of the dual sets of orthogonal polynomials in view of applications to quantum gravity, integrable systems and quantum chemistry

---

$^6$Two polyhedra are said to be scissor–congruent if they can be divided into finitely many pairwise congruent tetrahedra
Find out convolution rules for overlap functions (symmetric–binary recoupling coefficients) of Racah algebra, e.g.

$$\sum_{\ell'} <\ell'|k> \cdots <\ell'|k'> = \sum_{\ell''} <\ell''|k> \cdots <\ell''|k'>$$

Geometrically, this would mean to look for composition rules of collections of quadrilaterals able to provide new classes of (integrable) quantum systems → extended quantum geometries
Collection of tetrahedra ($\rightarrow$ triangulation)

Fluttery quadrilateral by Osvaldo Licini

Annalisa Marzuoli
Dipartimento di Matematica 'F. Casorati' & INFN (Pavia)

Quantum geometry, quadratic dynamical algebras and discrete polynomials