Quantum geometry, quadratic dynamical algebras and discrete polynomials

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Outline

Standard $SU(2)$ recoupling theory in quantum mechanics

The coupling theory of (eigenstates of) an ordered triple of $SU(2)$ angular momenta operators $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{J}_3$ to states of sharp total angular momentum $\mathbf{J} \equiv \mathbf{J}_4$ is usually based on **binary couplings**.
The admissible schemes are
$\mathbf{J}_1 + \mathbf{J}_2 = \mathbf{J}_{12};\ \mathbf{J}_{12} + \mathbf{J}_3 = \mathbf{J}$ and $\mathbf{J}_2 + \mathbf{J}_3 = \mathbf{J}_{23}; \mathbf{J}_1 + \mathbf{J}_{23} = \mathbf{J}$
with complete sets of eigenvectors given by

$$|j_{12} > := |\,(j_1 j_2) j_{12}\, j_3\, jm >$$
$$|j_{23} > := |\,j_1\,(\,j_2 j_3) j_{23}\, jm >$$

Here (*e.g.*): $\mathbf{J}_i^2\ |j_{12}\ > = j_i(j_i+1)|j_{12}\ >$; labels run over *irreps* $\{0, 1/2, 1, 3/2, \dots\}$ in $\hbar$ units and m is the eigenvalue of J_z with $-j \leq m \leq j$ in integer steps.

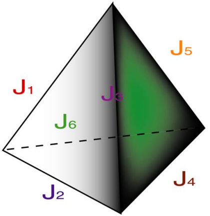
The $6j$ symbol and the tetrahedron

The Racah–Wigner $6j$ symbol is the unitary (orthogonal) transformation or 'recoupling coefficient' relating the two sets of binary coupled states ($\Phi \equiv j_1 + j_2 + j_3$)

$$< j_{23} \,|\, j_{12} \; > = \; (-1)^{\Phi} \left[(2j_{12}+1)\,(2j_{23}+1)\right]^{1/2} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix}$$

According to the geometric, semiclassical view of [Wigner, Ponzano & Regge 1968] the $6j$ is associated with a solid **Euclidean tetrahedron** bounded by triangular faces $\leftrightarrow$ triads of labels

Edge $\leftrightarrow$ j-label ($J = j + 1/2$)



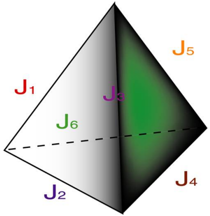
$\leftrightarrow \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$

Labeling as a recoupling coefficient $\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix}$

Labeling as a [4 parameters + 2 variables] array $\begin{Bmatrix} a & b & \ell \\ c & d & \tilde{\ell} \end{Bmatrix}$

$a + b + c + d \equiv j_1 + j_2 + j_3 + j_4$: **perimeter of a quadrilateral**

From 'classical' to Regge symmetries



Each entry of the $6j$ is equivalent to any other $\Rightarrow$
Tetrahedral symmetry: $\mathbf{T} \subset O(3)$ (order 24), isomorphic to the symmetric group S_4

Regge symmetries [1959] are 'functional' relations $[s = (a+b+c+d)/2]$

$$\begin{Bmatrix} a & b & \ell \\ c & d & \tilde{\ell} \end{Bmatrix} = \begin{Bmatrix} s-a & s-b & \ell \\ s-c & s-d & \tilde{\ell} \end{Bmatrix} := \begin{Bmatrix} a' & b' & \ell \\ c' & d' & \tilde{\ell} \end{Bmatrix}$$

The number of classical and Regge symmetries is $144 =$ Order $(S_4 \times S_3)$: the rationale relies on the fact that the $6j$ is a polynomial function (rearranged from the Racah sum rule) $\rightarrow \rightarrow \rightarrow$

The *Racah polynomial* can be written in terms of the ${}_4F_3$ hypergeometric function evaluated for the argument $z = 1$

$$\begin{Bmatrix} a & b & d \\ c & f & e \end{Bmatrix} = \Delta(abe)\,\Delta(cde)\,\Delta(acf)\,\Delta(bdf)\,(-)^{\beta_1}(\beta_1+1)!$$

$$\times \frac{{}_4F_3\begin{pmatrix} \alpha_1-\beta_1 & \alpha_2-\beta_1 & \alpha_3-\beta_1 & \alpha_4-\beta_1 & \\ -\beta_1-1 & \beta_2-\beta_1+1 & \beta_3-\beta_1+1 & & ;1 \end{pmatrix}}{(\beta_2-\beta_1)!(\beta_3-\beta_1)!(\beta_1-\alpha_1)!(\beta_1-\alpha_2)!(\beta_1-\beta_3)!(\beta_1-\alpha_4)!}$$

$\beta_1 = \min(a+b+c+d; a+d+e+f; b+c+e+f)$; β_2, β_3 are identified in either way with the pair remaining in the 3–tuple $(a+b+c+d; a+d+e+f; b+c+e+f)$; α's may be identified with any permutation of $(a+b+e; c+d+e;\ a+c+f; b+d+f)$;
$\Delta\,(abc) = [((a+b-c)!(a-b+c)!(-a+b+c)!)/(a+b+c+1)!]^{1/2}$

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Symmetry Properties of Racah's Coefficients.

T. Regge
Istituto Nazionale di Fisica Nucleare, Sezione di Torino - Torino

(ricevuto il 9 Ottobre 1958)

We have shown in a previous letter (1) that the true symmetry of Clebsch-Gordan coefficients is much higher that is was before believed. A similar result has been now obtained for Racah's coefficients. Although no direct connection has been established between these wider symmetries it seems very probable that it will be found in the future. We shall merely state here the results which can be checked very easily with the help of the well known Racah's formula:

The volume operator

Alternatively to the binary coupling of angular momenta, the volume operator

$$K := \mathbf{J}_1 \cdot \mathbf{J}_2 \times \mathbf{J}_3$$

acts 'democratically' on vectors $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ plus a fourth one, $\mathbf{J}_4$,[1] which close a (not necessarily planar) **quadrilateral vector diagram** $\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 + \mathbf{J}_4 = 0$

Generalized recoupling coefficients between a binary–coupled state, say $|j_{12} > \equiv |\ell >$ and a ket $|j_1, j_2, j_3, j_4; k > \equiv |k >$ where $K|k >= k|k >$ (eigenvalues k come in pairs $\pm k$).

Their symmetrized form is denoted in short $\Phi_\ell^{(k)}$ (**eigenfunctions of the volume operator expanded in the ℓ-representation**)

[1] *i.e.* on either a composite system of 4 objects with vanishing total angular momentum, or a system of 3 objects with total angular momentum $\mathbf{J}_4$

The eigenfunctions $\Phi_{\ell}^{(k)}$ satisfy a three–terms recursion relation[2] which can be turned into a a real, finite–difference Schrödinger–like equation [3]

$$\alpha_{\,\ell+1}\;\Phi_{\ell+1}^{(k)}\;+\;\alpha_{\,\ell}\;\Phi_{\ell-1}^{(k)}\;=\;k\,\Phi_{\ell}^{(k)}$$

The matrix elements α_ℓ are given in terms of geometric quantities

$$\alpha_\ell=\frac{F(\ell;j_1+1/2;j_2+1/2)F(\ell;j_3+1/2;j_4+1/2)}{\sqrt{(2\ell+1)(2\ell-1)}}$$

$F(A,B,C)=\frac{1}{4}[(A+B+C)\;(-A+B+C)\;(A-B+C)\;(A+B-C)]^{\frac{1}{2}}$
is the area of a triangle with side lengths A, B and C

[2] Levy-Leblond (1965) up to [Carbone, Carfora, Marzuoli (2002)]
[3] Aquilanti, Marzuoli, Marinelli, *J Phys A: Math Theor* **46** (2013) 175303

The role of Regge symmetry

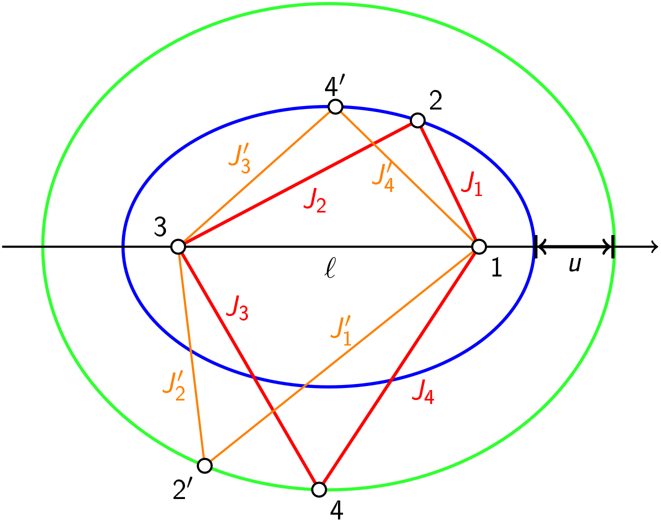
$$J_1 = j_1 + \frac{1}{2},\ J_2 = j_2 + \frac{1}{2},\ J_3 = j_3 + \frac{1}{2}, J_4 = j_4 + \frac{1}{2}$$

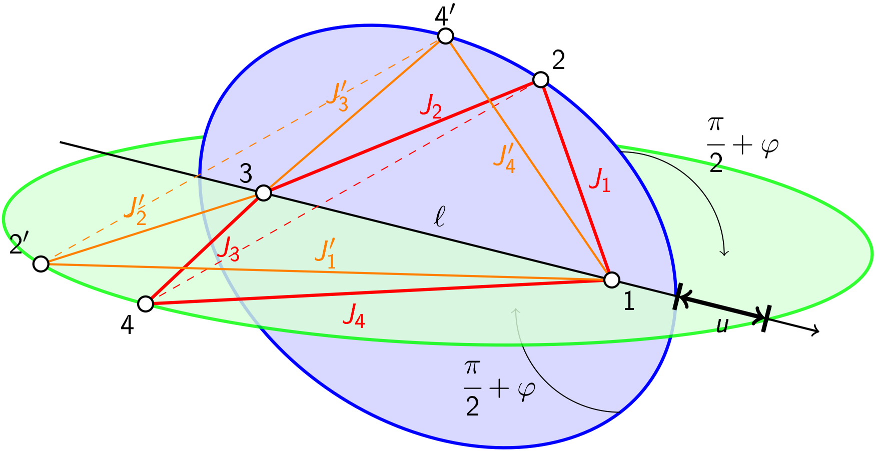
are parameters labeling a quadrilateral with vertices (1,2,3,4), which, together with its **Regge-conjugate** [vertices (1',2',3',4')]

$$J'_1 = j'_1 + \frac{1}{2},\ J'_2 = j'_2 + \frac{1}{2},\ J'_3 = j'_3 + \frac{1}{2}, J'_4 = j'_4 + \frac{1}{2}$$

characterize the analysis of the discrete, Regge–invariant Schrödinger Eq for a **quantum of space** 'bounded' by two confocal ellipses

NB The parameter u in the figures comes out once a reparametrization is performed: then the (1,2,3,4) quadrilateral $\leftrightarrow$ a quaternion Q, while (1',2',3',4') $\leftrightarrow$ $\tilde{Q}$, its conjugate (whose common real part is s, the semi–perimeter in Regge's formula)


4′
2
J'_3
J'_4
J_1
J_2
3
ℓ
1
u
J_3
J'_1
J'_2
J_4
2′
4


4′
2
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1
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J_3
J_4
J'_1
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J'_3
J'_4
ℓ
u
$\frac{\pi}{2} + \varphi$
$\frac{\pi}{2} + \varphi$

In terms of the shift operator $e^{\pm i\varphi}\, \Phi_{\ell}^{(k)} = \Phi_{\ell\pm 1}^{(k)}$, the Hamiltonian operator is

$$\hat{H} = \left(\alpha_\ell e^{-i\,\varphi} + \alpha_{\ell+1} e^{i\,\varphi}\right) \quad \text{with} \quad \varphi = -i\,\frac{\partial}{\partial\,\ell}$$

representing the variable canonically conjugate to ℓ.
The semiclassical behavior of the discrete SE is studied by resorting to a discrete WKB approach [Braun 1993]: the two-dimensional phase space (ℓ, φ) supports the classical Hamiltonian

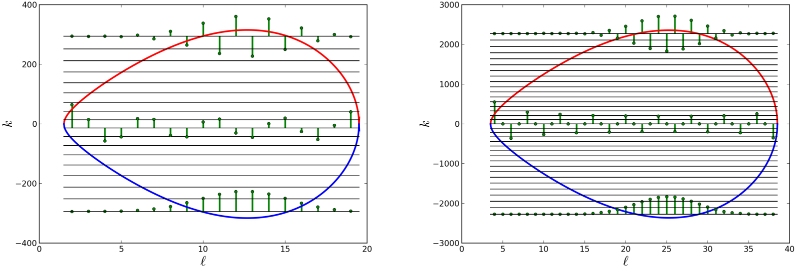
$$H = 2\,\alpha_{\ell+\frac{1}{2}} \cos\varphi$$

Geometrically, the two Regge–conjugate 'fluttery' quadrilaterals fold along the common diagonal ℓ with φ perceived as a torsion angle $\rightarrow$ (figure above)

Analytical and numerical results

- Analytical expressions for (lower) eigenvalues and explicit form of $\Phi_\ell^{(k)}$ [Carbone et al 2002] (complicated)
- Characterization of $\{\Phi_\ell^{(k)}\}$ as a family of orthogonal polynomials of the discrete variable k (not evenly–spaced) with degree given by ℓ (see below)
- Characterization of the 'dual' family $\{\Psi_k^{(\ell)}\}$ (see below)
- Spectrum of the volume operator (numerically):
  the horizontal lines represent the eigenvalues k, the curves are the caustics (the turning points of the semiclassical analysis), which limit the classically allowed region:
  in red $U_\ell^+ = 2\alpha_\ell$, in blue $U_\ell^- = -2\alpha_\ell$

Numerical evaluation



Left: parameters j_1, j_2, j_3, j_4=8.5, 10.5, 13.5, 14.5 or s, u, r, v=23.5, -4.5, 1.5, 0.5.

Right: all four parameters are doubled.

Quantum mechanical dynamical algebras on 3 generators

K_1, K_2, K_3: generators of quadratic operator algebras[4]
(if $K_{1,2}$ are Hermitian then K_3 is anti–Hermitian).
Commutation relations, where $\{\,,\,\}$ is the anticommutator:

$$[K_1, K_2] = K_3$$

$$[K_2, K_3] = 2R\, K_2 K_1 K_2 + A_1\,\{K_1, K_2\} + A_2\, K_2^2 + C_1\, K_1 + D\, k_2 + G_1$$

$$[K_3, K_1] = 2R\, K_1 K_2 K_1 + A_1\, K_1^2 + A_2\,\{K_1, K_2\} + C_2\, K_2 + D\, K_1 + G_2$$

[4] Classically: Poisson algebras on three dynamical variables k_1, k_2, k_3. The further property of 'mutual integrability' of $k_{1,2}$ or K_1, K_2 is required.

Classification of quadratic algebras [1986 $\rightarrow$ Zhedanov *et al*]

	R	A_1	A_2	C & D
AW(3) (*Askey–Wilson*)	*	*	*	*
R(3) (*Racah*)	0	*	*	*
H(3) (*Hahn*)	0	0	*	*
J(3) (*Jacobi*)	0	0	*	0
Lie algebras: $su(2)$, $su(1,1)$, $h(1)$	0	0	0	*

The case $R = 0$ corresponds to the **Racah algebra** and the explicit construction of the relevant Hilbert (representation) spaces can be done in particular when K_1, K_2 have discrete spectra

Ladder representations and eigenvalue problems

Starting from the eigenvalue problem for K_1

$$K_1\,\psi_p \;=\; \lambda_p\,\psi_p\,, \quad p = 0, 1, 2, \ldots;\ \{\lambda_p\}\ \text{evenly–spaced}$$

the commutation relations imply that the operator K_2 is **tridiagonal** in this basis

$$K_2\,\psi_p \;=\; \alpha_{p+1}\,\psi_{p+1} \;+\; \alpha_p\,\psi_{p-1} \;+\; \beta_p\,\psi_p$$

and similarly

$$K_2\,\phi_s \;=\; \mu_s\,\phi_s \;\;(s = 0, 1, 2, \ldots) \;\;\Rightarrow$$

$$K_1\,\phi_s \;=\; \gamma_{s+1}\,\phi_{s+1} \;+\; \gamma_s\,\phi_{s-1} \;+\; \delta_s\,\phi_s$$

(The matrix coefficients α, β, γ are evaluated from the specific commutation relations)

The *overlap functions* of the two (normalized) bases ψ_p, ϕ_s

$$<\phi_s|\psi_p> \equiv <s|p> \quad \text{and} \quad <\psi_p|\phi_s> \equiv <p|s>$$

are hypergeometric orthogonal polynomials of one discrete variable, here ($R = 0$ and eigenvalues of both K_1 and K_2 varying uniformly) Racah polynomials → on the top of the Askey scheme →

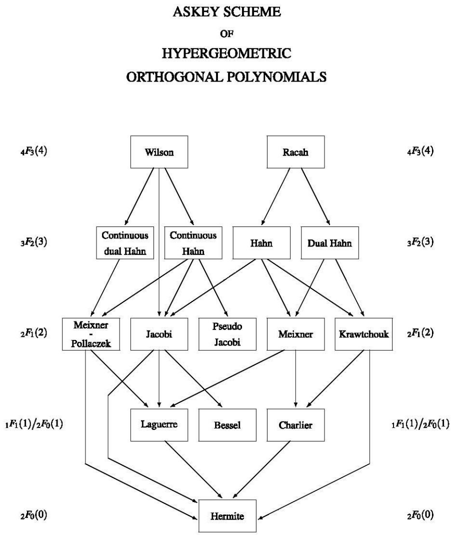
Duality property of the algebra **R(3)**

• The exchange of generators $K_1 \leftrightarrows K_2$, $K_3 \to -K_3$ represents an **automorphism** of the Racah algebra **R(3)**

• Under this automorphism: the (discrete) variables of the two families of overlap functions and their degrees as polynomials are $\leftrightarrows$

NB All other functions in the lower levels of the Askey (–Wilson) scheme are obtained by suitable limiting procedures on parameters (dropped in the present notation) and/or on spectral parameters.

Relation with special function theory



The operator K_3 and associated overlap functions

In the K_1–eigenbasis K_3 satisfies

$$K_3\,\psi_p \;=\; (\lambda_{p+1}-\lambda_p)\,\alpha_{p+1}\,\psi_{p+1}\;-\;(\lambda_p-\lambda_{p-1})\,\alpha_p\,\psi_{p-1}$$

$(\alpha, \lambda$ as before)
K_3 has a discrete, but not evenly–spaced spectrum $\{\nu_n\,, n = 0, 1, 2, \dots\}$. Even in finite–dimensional situations the diagonalization can be carried out analytically only for the lowest eigenvalues. If φ_n are the eigenfunctions of K_3, the **overlap functions**

$$<\varphi_n\,|\psi_p\;> \quad \text{and} \quad <\psi_p\,|\varphi_n\;>$$

are orthogonal, 'dual' to each other but **not trivially recognized as hypergeometric**.

Generalized recoupling theory & the algebra $\mathbf{R(3)}$

The dynamical Racah algebra underlying the evolution of a single 'quantum of space'

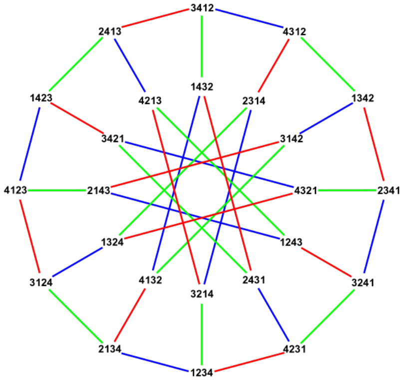
- is generated by the operators

$$K_1 := \mathbf{J}_{12}^2; \; K_2 := \mathbf{J}_{23}^2;$$
$$K_3 := [K_1, K_2] \equiv -4i \, \mathbf{J}_1 \cdot (\mathbf{J}_2 \times \mathbf{J}_3) \equiv -4iK,$$

  with either the volume operator K, or $\mathbf{J}_{12}^2$, or $\mathbf{J}_{23}^2$ playing the role of the Hamiltonian

- has $S_4 \times S_3$ as automorphism group, thus providing a proper group–theoretic interpretation of (classical + Regge) discrete symmetries characterizing both a 'fluttery' quadrilateral configuration and a tetrahedron out of $SU(2)$ symmetric and binary re–coupling theory, respectively

The Nauru graph (drawn in the flag of the Nauru Republic)



- (1, 2, 3, 4: vertices of the quadrilateral or of the tetrahedron permuted under S_4)
- Is a generalized Petersen graph with 24 vertices and 36 edges
- Is vertex and edge–transitive ('symmetric') $\Rightarrow$ its automorphism group, $S_4 \times S_3$, acts transitively
- Can be looked at as a Cayley graph of S_4 generated by the action of S_3 which swaps: $(1 \leftrightarrow 2)$, $(1 \leftrightarrow 3)$, $(1 \leftrightarrow 4)$

Up to relabeling $(j_1, j_2, j_3, j_4) \mapsto (a, b, c, d)$, and by resorting to Regge symmetry for suitably defining the (finite) ranges of such parameters, the binary and symmetric normalized recoupling functionals are denoted

$$< \tilde{\ell} \,|\, \ell > (a,b,c,d) \propto \begin{Bmatrix} a & b & \ell \\ c & d & \tilde{\ell} \end{Bmatrix}$$

$$\Phi_{\ell}^{(k)}(a,b,c,d) = < \ell \,|\, k > (a,b,c,d)$$

$$\Psi_{k}^{(\ell)}(a,b,c,d) = < k \,|\ell > (a,b,c,d)$$

(and similar for $< \tilde{\ell} \,|\, k > (a,b,c,d)$)

NB. The eigenvalue problem for the volume operator K is rewritten $K\,|k> = \lambda_k\,|k> \, (k = 0, 1, 2, \ldots)$
to emphasize the fact that the spectrum is not evenly–spaced

Families of discrete orthogonal polynomials with (a, b, c, d) fixed[5]

family	orthogonality on lattice	eigenvalue ($\to$ variable)	degree (rel. to)
$<\tilde{\ell}\,\vert\,\ell>$	$\sum_{\tilde{\ell}} \overline{<\tilde{\ell}\,\vert\,\ell'>} <\tilde{\ell}\,\vert\,\ell> = \delta_{\ell'\ell}$	$\ell(\ell+1)$	$\tilde{\ell}$
$<\ell\,\vert\,\tilde{\ell}>$	$\sum_{\ell} \overline{<\ell\,\vert\,\tilde{\ell}'>} <\ell\,\vert\,\tilde{\ell}> = \delta_{\tilde{\ell}'\tilde{\ell}}$	$\tilde{\ell}(\tilde{\ell}+1)$	ℓ
$<\ell\,\vert\,k>$	$\sum_{\ell} \overline{<\ell\,\vert\,k'>} <\ell\,\vert\,k> = \delta_{k'k}$	λ_k	ℓ
$<k\,\vert\,\ell>$	$\sum_{k} \overline{<k\,\vert\,\ell'>} <k\,\vert\,\ell> = \delta_{\ell'\ell}$	$\ell(\ell+1)$	k
$<\tilde{\ell}\,\vert\,k>$	$\sum_{\tilde{\ell}} \overline{<\tilde{\ell}\,\vert\,k'>} <\tilde{\ell}\,\vert\,k> = \delta_{k'k}$	λ_k	$\tilde{\ell}$
$<k\,\vert\,\tilde{\ell}>$	$\sum_{k} \overline{<k\,\vert\,\tilde{\ell}'>} <k\,\vert\,\tilde{\ell}> = \delta_{\tilde{\ell}'\tilde{\ell}}$	$\tilde{\ell}(\tilde{\ell}+1)$	k

[5] Aquilanti, Marzuoli, Marinelli, *J Phys: Conf Series* (2013)

- **Self–duality** of the black family $\rightarrow$ $6j$-symbols $\leftrightarrow$ Racah polynomial $\leftrightarrow$ ${}_4F_3$ on the top of the Askey scheme
$\sum_{\tilde{\ell}} < \ell' | \tilde{\ell} >< \tilde{\ell} | \ell > = \delta_{\ell' \ell}$

- Duality in the red family
$\sum_{\ell} < k' | \ell >< \ell | k > = \delta_{k' k}$ and $\sum_{k} < \ell' | k >< k | \ell > = \pm \delta_{\ell' \ell}$

- (Similar for the blue family)

- • The reduction of such families to specific hypergeometric functions of type ${}_4F_3$ would require to find out a closed algebraic form for eigenvalues of the volume operator for given parameters

- **Triangular relation(s)**, transversal with respect to the families

$$\sum_{k} < \tilde{\ell} | k >< k | \ell > == \pm < \tilde{\ell} | \ell >$$

Further developments can be addressed in parallel, from algebraic–analytical and geometric viewpoints

- Improve interpretations of the Regge symmetries on the geometric (*scissor–congruent tetrahedra*)[6] and algebraic (*quaternionic reparametrization*) sides
- Explore q–deformed extensions and limiting cases of the dual sets of orthogonal polynomials in view of applications to quantum gravity, integrable systems and quantum chemistry

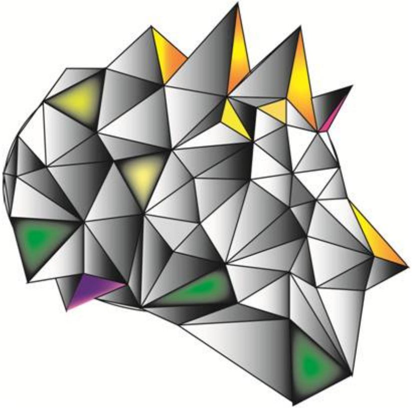
[6] Two polyhedra are said to be scissor–congruent if they can be divided into finitely many pairwise congruent tetrahedra

- Find out convolution rules for overlap functions (symmetric–binary recoupling coefficients) of Racah algebra, *e.g.*

$$\sum_{\ell'} < \ell' | k > \cdots < \ell' | k' > = \sum_{\ell''} < \ell'' | k > \cdots < \ell'' | k' >$$

- Geometrically, this would mean to look for composition rules of collections of quadrilaterals able to provide new classes of (integrable) quantum systems
$\rightarrow$ extended quantum geometries

Collection of tetrahedra ($\rightarrow$ triangulation)



Fluttery quadrilateral by *Osvaldo Licini*