Noncommutative Geometry in the LHC-Era

Christoph A. Stephan
Institut für Mathematik
Universität Potsdam

Napoli
2013
Overview

1. Basic Ideas
2. Geometry
3. Physics
4. Beyond the Standard Model
5. Conclusions
Overview

1. Basic Ideas
2. Geometry
3. Physics
4. Beyond the Standard Model
5. Conclusions
The Aim of Noncommutative Geometry:

Aim (what everybody tries to do):
To unify general relativity (GR) and the standard model of particle physics (SM) on the same geometrical level. This means to describe gravity and the electro-weak and strong forces as gravitational forces of a generalised “space-time”.

Idea (what many people have done before):
Find generalised “space-time” with symmetries of General Relativity and the Standard Model (and perhaps more).

Possible Solution (A. Connes):
Try a special type of noncommutative geometry
Analogy: Almost-comm. geometry $\leftrightarrow$ Kaluza-Klein space

M $\rightarrow C^\infty(M)$, $F \rightarrow$ some "finite space",

Differential geometry $\rightarrow$ spectral triple
Almost-commutative Geometry

Replacing manifolds by algebras

extra dimension: $F \rightarrow \mathcal{A}_f = M_1(K) \oplus M_2(K) \oplus \ldots$

Kaluza-Klein space: $M \times F \rightarrow \mathcal{A} = C^\infty(M) \otimes \mathcal{A}_f$
Schematic Structure of GR:
Gravity emerges as a pseudo-force associated to the space-time symmetries, i.e. the diffeomorphisms of the manifold $M$. 

Space-Time (S-T) 4-dim Manifold $M$  \hspace{1cm} act on \hspace{1cm} Dynamics  

Gravitiy Einstein-Hilbert Action  

S-T Symmetries Diffeomorphisms of $M$  \hspace{1cm} leave invariant
Euclidean space(-time)!

Space(-Time) (S-T)
4-dim Manifold $M$

act on

equiv.

S-T Symmetries
Diffeomorphisms
of $M$

equiv.

Spectral Triple
$(C^\infty(M), \phi, \mathcal{H})$

act on

Dynamics

Automorphisms
$Aut(C^\infty(M))$

leave invariant

Gravity
Spectral Action =
Almost-Commutative Standard Model (A. Chamseddine, A. Connes):

Almost-Commutative Spectral Triple
\[ \mathcal{A} = C^\infty(M) \otimes \mathcal{A}_f \]

Int.+Ext. Symmetries
\[ \text{Aut}(\mathcal{A}) \supset \text{Diff}(M) \times U(1) \times SU(2) \times SU(3) \]

Dynamics


leave invariant
The almost-commutative standard model automatically produces:

- The combined Einstein-Hilbert and standard model action
- A cosmological constant
- The Higgs boson with the correct quartic Higgs potential

The Dirac operator plays a multiple role:

\[ D = \bar{\phi} \otimes 1_f + \phi + \gamma_5 \otimes \Phi \]

- Higgs & Gauge Bosons
- Particle Dynamics, Ferm. Mass Matrix
- Metric of $M$, Internal Metric
Overview

1. Basic Ideas
2. Geometry
3. Physics
4. Beyond the Standard Model
5. Conclusions
An even, real spectral triple \((\mathcal{A}, \mathcal{H}, D)\)

The ingredients (A. Connes):

- A real, associative, unital pre-\(C^*\)-algebra \(\mathcal{A}\)
- A Hilbert space \(\mathcal{H}\) on which the algebra \(\mathcal{A}\) is faithfully represented via a representation \(\rho\)
- A self-adjoint operator \(D\) with compact resolvent, the Dirac operator
- An anti-unitary operator \(J\) on \(\mathcal{H}\), the real structure or charge conjugation
- A unitary operator \(\gamma\) on \(\mathcal{H}\), the chirality or volume element
The axioms of noncommutative geometry (A. Connes):

Axiom 1: Classical Dimension $n$ (we assume $n$ even)
Axiom 2: Regularity
Axiom 3: Finiteness
Axiom 4: First Order of the Dirac Operator
Axiom 5: Reality
Axiom 6: Orientability
Axiom 7: Poincaré Duality
Connes’ Reconstruction Theorem (sloppy version):
Compact Riemannian spin manifolds are equivalent to real spectral triples with $\mathcal{A}$ commutative.

One can therefore replace a compact 4-dim. Riemannian space-time $\mathcal{M}$ by the spectral triple $(\mathcal{C}^\infty(\mathcal{M}), \mathcal{H}, \partial)$. 
Finite spectral triples:

- \( \mathcal{A}_f = M_1(K) \oplus M_2(K) \oplus \ldots \)
  \( K = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \)
- \( \text{Aut}^e(M_n(\mathbb{C})) = U^{nc}(M_n(\mathbb{C})) = U(n) \),
- \( \mathcal{H}_f \simeq \mathbb{C}^N \)
  \( N \) is the total number of particles
  left-/right-handed particles/antiparticles counted separately
- \( D_f \in M_N(\mathbb{C}), D_f \) is the fermionic mass matrix.

Axioms \( \rightarrow \) Restrictions for \( D_f \) and \( \mathcal{H}_f \)
Almost-commutative geometry:

An almost-commutative spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is a tensor product of a spectral triple of a manifold \(M\) 
\((\mathcal{A}_M = C^\infty(M), \mathcal{H}_M, D_M)\) with dimensions \(n_M > 0\)
and a finite spectral triple 
\((\mathcal{A}_f, \mathcal{H}_f, D_f)\) with metric dimension \(n_f = 0\)
(i.e. \(\mathcal{A}_f\) matrix algebra).

\[
\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_f, \quad \mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_f,
\]

\[
J = J_M \otimes J_f, \quad \gamma = \gamma_5 \otimes \gamma_f,
\]

\[
D = \emptyset \otimes 1_f + \gamma_5 \otimes D_f
\]

\[
\text{Aut}(C^\infty(M) \otimes \mathcal{A}_f) \cong \text{Diff}(M) \rtimes \text{U}^{\text{nc}}(\mathcal{A}_f)
\]
The geometric setup imposes constraints:

- mathematical axioms → Restrictions on particle content
- symmetries of finite space → determines gauge group
- representation of matrix algebra → representation of gauge group (only fundamental and adjoint representations)
- Dirac operator → allowed mass terms / Higgs fields
Overview

1. Basic Ideas
2. Geometry
3. Physics
4. Beyond the Standard Model
5. Conclusions
The fluctuated Dirac operator

The Dirac operator $\not{\partial} \otimes 1_f + \gamma_5 \otimes D_f$ is fluctuated with inner unitaries $U^{nc}(A_f)$ and becomes

$$D = \not{\partial} \otimes 1_f + \phi + \gamma_5 \otimes \Phi$$

The Spectral Action (A. Connes, A. Chamseddine 1996)

$$(\psi, D\psi) + S_D(\Lambda) \quad \text{with} \quad \psi \in \mathcal{H}$$

- $(\psi, D\psi) = \text{fermionic action}$
  - includes Yukawa couplings
  - & fermion–gauge boson interactions

- $S_D(\Lambda) = \# \text{eigenvalues of} \ D \text{ up to cut-off} \ \Lambda$
  - + full bosonic SM action + constraints at $\Lambda$

- constraints $\Rightarrow$ less free parameters than classical SM
The bosonic spectral action:

Spectral action (bosonic part) of Dirac operator $D$ is the number of eigenvalues of $D$ in the interval $[-\Lambda, \Lambda]$ (with $\Lambda \in \mathbb{R}^+$):

$$S_D(\Lambda) = \text{Tr} f \left( \frac{D^2}{\Lambda^2} \right),$$

where $\text{Tr}$ is the $L^2$-trace over the space of spinor fields, and $f$ is cut-off function with support in $[0, +1]$ which is constant near 0.
Asymptotic expansion of the Spectral Action

From the heat trace asymptotics for $\Lambda \to \infty$

$$\text{Tr} \left( e^{-\frac{D^2}{\Lambda^2}} \right) \sim \sum_{n \geq 0} \Lambda^{2-n} a_{2n}(D^2)$$

(with Seeley-deWitt coefficients $a_{2n}(D^2)$)

one gets an asymptotics for the spectral action

$$S_D(\Lambda) = \text{Tr} f \left( \frac{D^2}{\Lambda^2} \right) \sim \Lambda^4 f_4 a_0(D^2) + \Lambda^2 f_2 a_2(D^2) + \Lambda^0 f_0 a_4(D^2)$$

as $\Lambda \to \infty$.

Here $f_4, f_2, f_0$ are moments of the cut-off function $f$. 

Noncommutative Geometry in the LHC-Era

Physics

The Action

Asymptotic expansion of the Spectral Action

From the heat trace asymptotics for \( \Lambda \to \infty \)

\[
\text{Tr} \left( e^{-\frac{D^2}{\Lambda^2}} \right) \sim \sum_{n \geq 0} \Lambda^{2-n} a_{2n}(D^2)
\]

(with Seeley-deWitt coefficients \( a_{2n}(D^2) \))

one gets an asymptotics for the spectral action

\[
S_D(\Lambda) = \text{Tr} f \left( \frac{D^2}{\Lambda^2} \right) \sim \Lambda^4 f_4 a_0(D^2) + \Lambda^2 f_2 a_2(D^2) + \Lambda^0 f_0 a_4(D^2)
\]

as \( \Lambda \to \infty \).

Here \( f_4, f_2, f_0 \) are moments of the cut-off function \( f \).
For the Connes-Chamseddine Dirac operator (or fluctuated Dirac operator)

\[ D := \partial + \phi + \gamma_5 \otimes \Phi \]

we find the Seeley-deWitt coefficients

\[ a_2(D^2) = -\frac{\dim(H_f)}{96\pi^2} \int_M R \, d\text{vol} - \frac{1}{48\pi^2} \int_M \text{tr}(\Phi^2) \, d\text{vol} \]
\[ a_4(D^2) = \frac{11 \dim(H_f)}{720} \chi(M) - \frac{\dim(H_f)}{320\pi^2} \int_M \|W\|^2 \, d\text{vol} \]
\[ + \frac{1}{8\pi^2} \int_M \text{tr}([\nabla H_f, \Phi]) + \text{tr}(\Phi^4) \, d\text{vol} \]
\[ + \frac{5}{96\pi^2} \int_M \text{tr}(\Omega_f^2) \, d\text{vol} + \frac{1}{48\pi^2} \int_M R \text{tr}(\Phi^2) \, d\text{vol} \]
Spectral action for Connes-Chamseddine Dirac operator

For the Connes-Chamseddine Dirac operator (or fluctuated Dirac operator)

$$D := \slashed{D} + \phi + \gamma_5 \otimes \Phi$$

we find the Seeley-deWitt coefficients

$$a_2(D^2) = - \frac{\text{dim}(\mathcal{H}_f)}{96\pi^2} \int_M R \, d\text{vol} - \frac{1}{48\pi^2} \int_M \text{tr}(\Phi^2) \, d\text{vol}$$

$$a_4(D^2) = \frac{11 \text{dim}(\mathcal{H}_f)}{720} \chi(M) - \frac{\text{dim}(\mathcal{H}_f)}{320\pi^2} \int_M \|W\|^2 \, d\text{vol}$$

$$+ \frac{1}{8\pi^2} \int_M \text{tr}(\nabla^\mathcal{H}_f, \Phi) + \text{tr}(\Phi^4) \, d\text{vol}$$

$$+ \frac{5}{96\pi^2} \int_M \text{tr}(\Omega^2_f) \, d\text{vol} + \frac{1}{48\pi^2} \int_M R \text{tr}(\Phi^2) \, d\text{vol}$$
## The standard model

- $\mathcal{A}_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})(\oplus \mathbb{C})$
- $\mathcal{U}^{nc}(\mathcal{A}_f) = SU(2) \times U(3)$
- $\mathcal{H}_f = \mathcal{H}_{SM}$ Hilbert space of minimal standard model fermion multiplets
- $D_f$: Fermionic mass matrix with CKM matrix and PMNS matrix
  - $D_f \rightarrow \Phi$ Higgs field(s) by *inner fluctuations*
- Majorana masses and SeeSaw mechanism for right-handed neutrinos (J. Barrett & A. Connes ’06)
Constraints on the SM parameters at the cut-off $\Lambda$:

\[ 5 g_1^2 = N_{SM} g_2^2 = N_{SM} g_3^2 = 3 \frac{Y_2^2}{H} \frac{\lambda}{24} = \frac{3}{4} Y_2 \]

- $g_1, g_2, g_3$: $U(1)_Y$, $SU_w(2)$, $SU_c(3)$ gauge couplings
- $\lambda$: quartic Higgs coupling
- $Y_2$: trace of the Yukawa matrix squared
- $H$: trace of the Yukawa matrix to the fourth power
- $N_{SM}$: number of standard model generations
**Consequences from the SM constraints:**

**Input:**
- Big Desert
- $g_1(m_Z) = 0.3575$, $g_2(m_Z) = 0.6514$, $g_3(m_Z) = 1.221$
- renormalisation group equations
- $(m_{top} = 171.2 \pm 2.1 \text{ GeV})$

**Output:**
- $g_2^2(\Lambda) = g_3^2(\Lambda)$ at $\Lambda = 1.1 \times 10^{17} \text{ GeV}$
- $m_{top} < 190 \text{ GeV}$
- no 4$^{th}$ SM generation

**Excluded by Tevatron & LHC since:**
- $m_{SMS} \neq 168.3 \pm 2.5 \text{ GeV}$
- $\frac{5}{3} g_1(\Lambda)^2 \neq g_2(\Lambda)^2$
How unique is the Standard Model?

The aim: Classifying the internal spaces

\[ \mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \ldots \]

- with respect to the number of summands in the algebra
- with respect to physical criteria

Little Reminder

For the Standard Model we have

\[ \mathcal{A}_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})(\oplus \mathbb{C}) \]

or alternatively

\[ \mathcal{A}_f = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \]
The physical models emerging from the spectral action are required to

- be irreducible i.e. to have the smallest possible internal Hilbert space (minimal approach)
- allow a non-degenerate Fermionic mass spectrum
- be free of harmful anomalies
- have unbroken colour groups
- possess no uncharged massless Fermions
Classification Results

<table>
<thead>
<tr>
<th># sum. in $A_f$</th>
<th>KO 0</th>
<th>KO 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>no model</td>
<td>no model</td>
</tr>
<tr>
<td>2</td>
<td>no model</td>
<td>no model</td>
</tr>
<tr>
<td>3</td>
<td>SM$^2$</td>
<td>no model</td>
</tr>
<tr>
<td>4</td>
<td>SM$^2$, el.-str.$^1$</td>
<td>SM$^2$, el.-str.$^1$</td>
</tr>
<tr>
<td>6</td>
<td>SM$^2$ + el.-str.$^1$, $2 \times$ el.-str.$^1$</td>
<td></td>
</tr>
</tbody>
</table>

$^1$ Electro-Strong Model: "electron+proton", no Higgs,

$$A_f = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_n(\mathbb{C}),$$

$$G_{gauge} = U(1) \times SU(n)/SO(n)/Sp(n)$$

$^2$ first family, colour group = $SU(n)/SO(n)/Sp(n)$
Overview

1. Basic Ideas
2. Geometry
3. Physics
4. Beyond the Standard Model
5. Conclusions
Beyond SM: the general strategy (bottom-up approach)

- find finite geometry that has SM as sub-model (tricky) => particle content, gauge group & representation
- make sure everything is anomaly free
- compute the spectral action => constraints on parameters
- determine the cut-off scale $\Lambda$ with suitable sub-set of the constraints
- use renorm. group equations to obtain low energy values of (hopefully) interesting parameters (Higgs couplings, Yukawa couplings)
- **check with experiment!** (and here we usually fail)
Excluded models:

Extensions of the Standard Model:

- **AC-model (C.S. ’05):**
  new particles (A and C) with opposite hypercharge
dark matter as bound AC-states (Fargion, Khlopov, C.S. ’05)

- **θ-model (C.S. ’07):** new particles with $SU_c(D)$-colour

- **Vector-Doublet Model (Squellari, C.S. ’07):**
  new $SU_w(2)$-vector doublets

Problem for models: • $m_{SMS} \geq 170$ GeV
• constraints on $g_1$, $g_2$, $g_3$ at $\Lambda$.

Saving these models?

Some of these models, e.g. the AC-model may perhaps be extended to comply with experimental data!
Noncommutative Geometry in the LHC-Era
Beyond the Standard Model
New Scalars

SM + $U(1)_X$ scalar field + $U(1)_X$ fermion singlet (C.S. 2009):
- Internal space: $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$
- Gauge group: $U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X$
- New fermions: $U(1)_X$-vector singlets ($X$-particles) neutral w.r.t SM gauge group, $M_X \sim \Lambda$
- New scalar: $U(1)_X$ singlet $\sigma$, neutral w.r.t SM gauge group

\[ \mathcal{L}_{scalar} = -\mu_1^2 |H|^2 + \frac{\lambda_1}{6} |H|^4 - \mu_2^2 |\sigma|^2 + \frac{\lambda_2}{6} |\sigma|^4 + \frac{\lambda_3}{3} |H|^2 |\sigma|^2 \]

\[ U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X \rightarrow U(1)_{el.} \times SU(3)_c \]

\[ \mathcal{L}_{ferm+gauge} = \bar{X}_L M_X X_R + g_{\nu, X} \bar{\nu}_R \sigma X_L + h.c. + \frac{1}{g_4^2} F_X^{\mu \nu} F_{X, \mu \nu} \]
The constraints at $\Lambda$:

- only top-quark & $\nu_T$
- valid at $g_2 = g_3$
  \[ \Rightarrow \Lambda = 1.1 \times 10^{17} \text{ GeV} \]
- $g_2^2 = \frac{\lambda_1}{24} \frac{(3g_t^2 + g_\nu^2)^2}{3g_t^4 + g_\nu^4}$
- $g_2^2 = \frac{\lambda_2}{24}$
- $g_2^2 = \frac{\lambda_3}{24} \frac{3g_t^2 + g_\nu^2}{g_\nu^2}$
- $g_2^2 = \frac{1}{4} (3g_t^2 + g_\nu^2)$
- free parameters: $|\langle \sigma \rangle|$, $g_4$
- $m_{SMS} \sim 120 - 130 \text{ GeV}$
- Problem: $\sqrt{5/3}g_1 \neq g_2 = g_3$

Mass EVs of scalar fields for

$\nu_2 = \sqrt{2} |\langle \sigma \rangle|$, $\sqrt{2} |\langle H \rangle| = 246 \text{ GeV}$, $g_4(m_Z) = 0.3$
**SM + $U(1)_X$ scalar field + new fermions (C.S. ’13):**

- **SM** as a sub-model: comme il faut!
- Internal space: $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \bigoplus_{i=1}^{6} \mathbb{C}_i$
- Gauge group: $U(1)_Y \times SU(2)_W \times SU(3)_C \times U(1)_X$

New fermions in each SM-generation:

- $X^1_\ell \oplus X^2_\ell \oplus X^3_\ell : (0, 1, 1, +1) \oplus (0, 1, 1, +1) \oplus (0, 1, 1, 0)$
- $X^1_r \oplus X^2_r \oplus X^3_r : (0, 1, 1, +1) \oplus (0, 1, 1, 0) \oplus (0, 1, 1, +1)$
- $V^w_\ell, V^w_r : (0, \bar{2}, 1, 0)$
- $V^c_\ell, V^c_r : (-1/6, 1, 3, 0)$

- New scalar: $\sigma : (0, 1, 1, +1)$
The Lagrangian (scalar potential & new terms):

- \( \mathcal{L}_{\text{scalar}} = -\mu_1^2 |H|^2 - \mu_2^2 |\sigma|^2 + \frac{\lambda_1}{6} |H|^4 + \frac{\lambda_2}{6} |\sigma|^4 + \frac{\lambda_3}{3} |H|^2 |\sigma|^2 \)

- \( \mathcal{L}_{\text{ferm}} = g_{\nu,X1} \bar{\nu}_r \sigma X_\ell^1 + \bar{X}_\ell^1 m_X X_r^1 + g_{X2} \bar{X}_\ell^2 \sigma X_r^2 \\
  + g_{X3} \bar{X}_\ell^3 \sigma X_r^3 + \bar{V}_\ell^c m_c V_r^c + \bar{V}_\ell^w m_w V_r^w + h.c. \)

- \( \mathcal{L}_{\text{gauge}} = \frac{1}{g_4^2} F^\mu \nu_X F_{X,\mu \nu} \)

- Symmetry breaking:
  \( U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X \rightarrow U(1)_{el.} \times SU(3)_c \times \mathbb{Z}_2 \)
The constraints at $\Lambda$:

- $g_2(\Lambda) = g_3(\Lambda) = \sqrt{\frac{7}{6}} g_1(\Lambda) = \sqrt{\frac{4}{3}} g_4(\Lambda)$
- $\lambda_1(\Lambda) = 36 \frac{H}{Y_2} g_2(\Lambda)^2$, $\lambda_2(\Lambda) = 36 \frac{\text{tr}(g_{\nu,X_1}^4)}{\text{tr}(g_{\nu,X_1}^2)^2} g_2(\Lambda)^2$
- $\lambda_3(\Lambda) = 36 \frac{\text{tr}(g_{\nu}^2)}{Y_2} g_2(\Lambda)^2$
- $Y_2(\Lambda) = \text{tr}(g_{\nu,X_1}^2)(\Lambda) + \text{tr}(g_{X_1}^2)(\Lambda) + \text{tr}(g_{X_2}^2)(\Lambda) = 6 g_2(\Lambda)^2$

Some simplifications:

- $Y_2 \approx 3g_{top} + g_{\nu\tau}$
- $\text{tr}(g_{X_1}^2)(\Lambda) \approx \text{tr}(g_{X_2}^2)(\Lambda) \approx 0$
- $\text{tr}(g_{\nu,X_1}^2)(\Lambda) \approx g_{\nu,X}(\Lambda)^2 = 6 g_2(\Lambda)^2$
- $(m_w)_{ij} \approx \Lambda$, $(m_c)_{ij} \approx 10^{15}$ GeV
Results for $1$-loop renormalisation groups:

- Constraints
  $\Lambda \approx 2 \times 10^{18}$ GeV
- $m_{top} \approx 172.9 \pm 1.5$ GeV
- $m_{\sigma_1, SMS} \approx 125 \pm 1.1$ GeV
- $m_{\sigma_2} \approx 445 \pm 139$ GeV
- $m_{Z_X} \approx 254 \pm 87$ GeV
- $g_4(m_Z) \approx 0.36$
- $m_{X_2, X_3} \lesssim 50$ GeV
- Free parameter: $|\langle \sigma \rangle|$

Mass EVs of scalar fields
Running of the gauge couplings with normalisation factors

\[
g_0(t) = \frac{\sqrt{4/3}g_4(t)}{\sqrt{7/6}g_1(t)}
\]
Noncommutative Geometry in the LHC-Era

Conclusions

Overview

1. Basic Ideas
2. Geometry
3. Physics
4. Beyond the Standard Model
5. Conclusions
### Questions & to-do-list

- Is the SM + scalar model compatible with LHC-data?
- Does the SM + scalar model contain viable dark matter candidates?
- Explore parameter space \((g_\nu, \chi_1, g_\chi^2, m_\chi, m_{V^w}, m_{V^c})\)
- Extend renormalisation group analysis to \(n\)-loop, \(n \geq 2\)
- Is the geometry a “sub-geometry” of a Connes-Chamseddine-type geometry?
- Classify Models beyond the Standard Model
- Spectral triples with Lorentzian signature
  (A. Rennie, M. Paschke, R. Verch, K. van den Dungen...)