# Noncommutative Geometry in the LHC-Era

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# Overview

# Overview

## The Aim of Noncommutative Geometry:

### Aim (what everybody tries to do):

To unify general relativity (GR) and the standard model of particle physics (SM) on the same geometrical level. This means to describe gravity and the electro-weak and strong forces as gravitational forces of a generalised “space-time”.

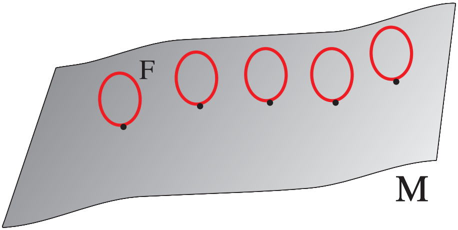
### Idea (what many people have done before):

Find generalised “space-time” with symmetries of General Relativity and the Standard Model (and perhaps more).

### Possible Solution (A. Connes):

Try a special type of noncommutative geometry

**Analogy:** Almost-comm. geometry $\leftrightarrow$ Kaluza-Klein space

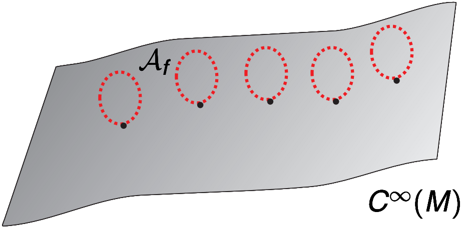



**Idea:**

$M \longrightarrow C^\infty(M)$, $F \longrightarrow$ some "finite space",

differential geometry $\longrightarrow$ spectral triple

## Almost-commutative Geometry



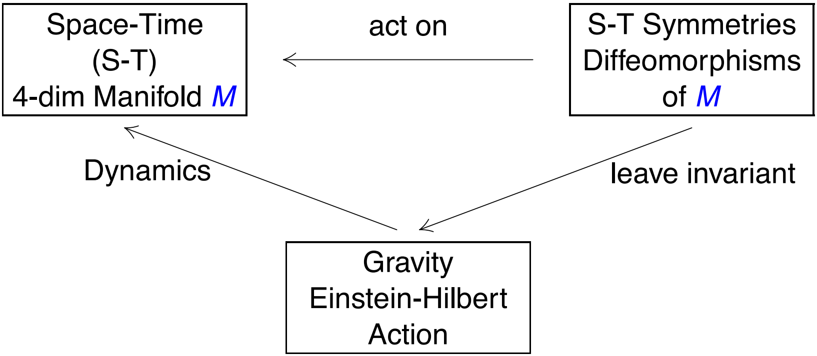

### Replacing manifolds by algebras

extra dimension: $F \to \mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \ldots$

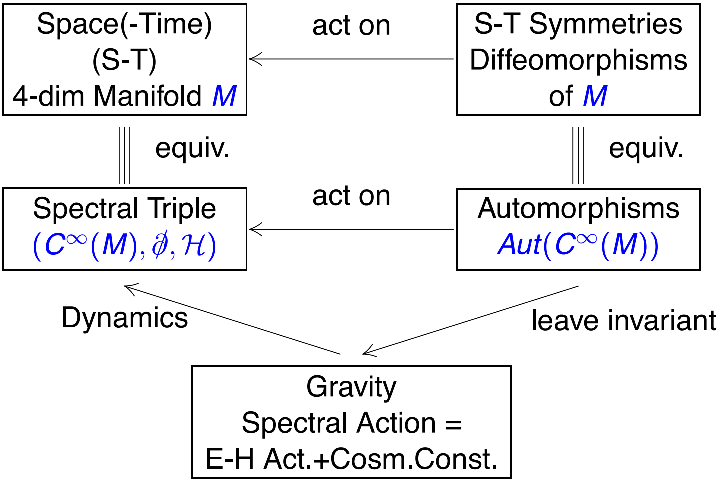
Kaluza-Klein space: $M \times F \to \mathcal{A} = C^\infty(M) \otimes \mathcal{A}_f$

**Schematic Structure of GR:**

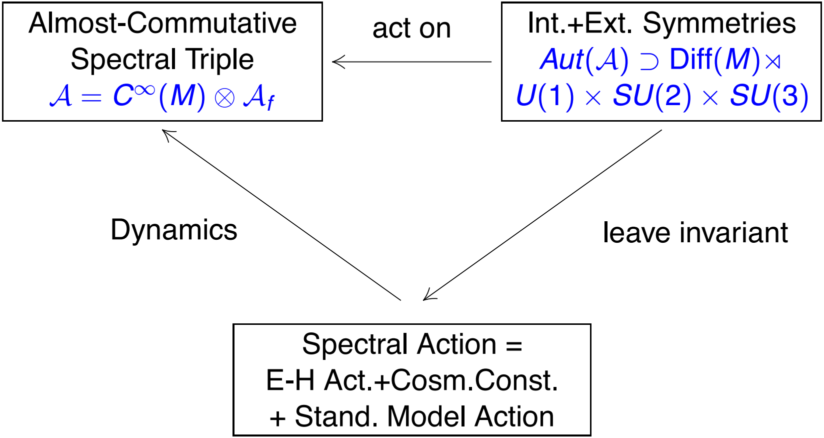
Gravity emerges as a pseudo-force associated to the space-time symmetries, i.e. the diffeomorphisms of the manifold $M$.



# Euclidean space(-time)!



**Almost-Commutative Standard Model** (A.Chamseddine, A.Connes):



The almost-commutative standard model automatically produces:

- The combined Einstein-Hilbert and standard model action
- A cosmological constant
- The Higgs boson with the correct quartic Higgs potential

The Dirac operator plays a multiple role:

$$D = \partial\!\!\!/ \otimes 1_f + \phi\!\!\!/ + \gamma_5 \otimes \Phi$$

| Higgs & Gauge Bosons | Particle Dynamics, Ferm. Mass Matrix | Metric of $M$, Internal Metric |
|---|---|---|

# Overview

**An even, real spectral triple $(\mathcal{A}, \mathcal{H}, D)$**

## The ingredients (A. Connes):

- A real, associative, unital pre-$C^*$-algebra $\mathcal{A}$
- A Hilbert space $\mathcal{H}$ on which the algebra $\mathcal{A}$ is faithfully represented via a representation $\rho$
- A self-adjoint operator $D$ with compact resolvent, the Dirac operator
- An anti-unitary operator $J$ on $\mathcal{H}$, the real structure or charge conjugation
- A unitary operator $\gamma$ on $\mathcal{H}$, the chirality or volume element

**The axioms of noncommutative geometry (A. Connes):**

Axiom 1: Classical Dimension $n$ (we assume $n$ even)

Axiom 2: Regularity

Axiom 3: Finiteness

Axiom 4: First Order of the Dirac Operator

Axiom 5: Reality

Axiom 6: Orientability

Axiom 7: Poincaré Duality

## Connes' Reconstruction Theorem (sloppy version):

Compact Riemannian spin manifolds are equivalent to real spectral triples with $\mathcal{A}$ commutative.

One can therefore replace a compact $4$-dim. Riemannian space-time $\mathcal{M}$ by the spectral triple $(C^\infty(\mathcal{M}), \mathcal{H}, \partial\!\!\!/)$.

## Finite spectral triples:

- $\mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \dots$
  $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$
- $Aut^e(M_n(\mathbb{C})) = U^{nc}(M_n(\mathbb{C})) = U(n)$,
- $\mathcal{H}_f \simeq \mathbb{C}^N$
  $N$ is the total number of particles
  left-/right-handed particles/antiparticles counted separately
- $D_f \in M_N(\mathbb{C})$, $D_f$ is the fermionic mass matrix.

Axioms $\longrightarrow$ Restrictions for $D_f$ and $\mathcal{H}_f$

## Almost-commutative geometry:

An almost-commutative spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is a tensor product of a spectral triple of a manifold $M$ $(\mathcal{A}_M = C^\infty(\mathcal{M}), \mathcal{H}_M, D_M = \partial\!\!\!/)$ with dimensions $n_M > 0$ and a finite spectral triple $(\mathcal{A}_f, \mathcal{H}_f, D_f)$ with metric dimension $n_f = 0$ (i.e. $\mathcal{A}_f$ matrix algebra).

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_f, \quad \mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_f,$$

$$J = J_M \otimes J_f, \quad \gamma = \gamma_5 \otimes \gamma_f,$$

$$D = \partial\!\!\!/ \otimes 1_f + \gamma_5 \otimes D_f$$

$$\mathrm{Aut}(C^\infty(\mathcal{M}) \otimes \mathcal{A}_f) \simeq \mathrm{Diff}(M) \rtimes U^{nc}(\mathcal{A}_f)$$

**The geometric setup imposes constraints:**

- mathematical axioms
  $\longrightarrow$ Restrictions on particle content
- symmetries of finite space
  $\longrightarrow$ determines gauge group
- representation of matrix algebra
  $\longrightarrow$ representation of gauge group
  (only fundamental and adjoint representations)
- Dirac operator $\longrightarrow$ allowed mass terms / Higgs fields

# Overview

## The fluctuated Dirac operator

The Dirac operator $\partial\!\!\!/ \otimes 1_f + \gamma_5 \otimes D_f$ is fluctuated with inner unitaries $\mathcal{U}^{nc}(\mathcal{A}_f)$ and becomes

$$D = \partial\!\!\!/ \otimes 1_f + \phi\!\!\!/ + \gamma_5 \otimes \Phi$$

## The Spectral Action (A. Connes, A. Chamseddine 1996)

$$(\Psi, D\Psi) + S_D(\Lambda) \quad \text{with } \Psi \in \mathcal{H}$$

- $(\Psi, D\Psi)$ = fermionic action
  includes Yukawa couplings
  & fermion–gauge boson interactions
- $S_D(\Lambda)$ = $\sharp$ eigenvalues of $D$ up to cut-off $\Lambda$
  = Einstein-Hilbert action + Cosm. Const.
  + full bosonic SM action + constraints at $\Lambda$
- constraints => less free parameters than classical SM

## The bosonic spectral action:

Spectral action (bosonic part) of Dirac operator $D$ is the number of eigenvalues of $D$ in the interval $[-\Lambda, \Lambda]$ (with $\Lambda \in \mathbb{R}^+$):

$$S_D(\Lambda) = \mathrm{Tr}\, f\left(\frac{D^2}{\Lambda^2}\right),$$

where $\mathrm{Tr}$ is the $L^2$-trace over the space of spinor fields,
and $f$ is cut-off function with support in $[0, +1]$ which is constant near $0$.

## Asymptotic expansion of the Spectral Action

From the heat trace asymptotics for $\Lambda \to \infty$

$$\mathrm{Tr}\left(e^{-\frac{D^2}{\Lambda^2}}\right) \sim \sum_{n\geq 0} \Lambda^{2-n} a_{2n}(D^2)$$

(with Seeley-deWitt coefficients $a_{2n}(D^2)$)
one gets an asymptotics for the spectral action

$$S_D(\Lambda) = \mathrm{Tr}\, f\left(\frac{D^2}{\Lambda^2}\right) \sim \Lambda^4\, f_4\, a_0(D^2) + \Lambda^2\, f_2\, a_2(D^2) + \Lambda^0\, f_0\, a_4(D^2)$$

as $\Lambda \to \infty$.
Here $f_4, f_2, f_0$ are moments of the cut-off function $f$.

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## Spectral action for Connes-Chamseddine Dirac operator

For the Connes-Chamseddine Dirac operator
(or fluctuated Dirac operator)

$$D := \partial\!\!\!/ + \psi\!\!\!/ + \gamma_5 \otimes \Phi$$

we find the Seeley-deWitt coefficients

$$\begin{aligned}
a_2(D^2) &= -\frac{dim(\mathcal{H}_f)}{96\pi^2} \int_M R\, dvol - \frac{1}{48\pi^2} \int_M \mathrm{tr}(\Phi^2) dvol \\
a_4(D^2) &= \frac{11\, dim(\mathcal{H}_f)}{720} \chi(M) - \frac{dim(\mathcal{H}_f)}{320\pi^2} \int_M \|W\|^2\, dvol \\
&\quad + \frac{1}{8\pi^2} \int_M \mathrm{tr}\left([\nabla^{\mathcal{H}_f}, \Phi]\right) + \mathrm{tr}(\Phi^4) dvol \\
&\quad + \frac{5}{96\pi^2} \int_M \mathrm{tr}\left(\Omega_f^2\right) dvol + \frac{1}{48\pi^2} \int_M R\, \mathrm{tr}(\Phi^2) dvol
\end{aligned}$$

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## The standard model

- $\mathcal{A}_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})(\oplus\mathbb{C})$
- $\mathcal{U}^{nc}(\mathcal{A}_f) = SU(2) \times U(3)$
- $\mathcal{H}_f = \mathcal{H}_{SM}$ Hilbert space of minimal standard model fermion multiplets
- $D_f$: Fermionic mass matrix with CKM matrix and PMNS matrix
  $D_f \to \Phi$ Higgs field(s) by inner fluctuations
- Majorana masses and SeeSaw mechanism for right-handed neutrinos (J. Barrett & A. Connes '06)

## Constraints on the SM parameters at the cut-off $\Lambda$:

$$5\, g_1^2 = N_{SM}\, g_2^2 = N_{SM}\, g_3^2 = 3\, \frac{Y_2^2}{H}\, \frac{\lambda}{24} = \frac{3}{4}\, Y_2$$

- $g_1$, $g_2$, $g_3$: $U(1)_Y$, $SU_w(2)$, $SU_c(3)$ gauge couplings
- $\lambda$: quartic Higgs coupling
- $Y_2$: trace of the Yukawa matrix squared
- $H$: trace of the Yukawa matrix to the fourth power
- $N_{SM}$: number of standard model generations

## Consequences from the SM constraints:

Input:

- Big Desert
- $g_1(m_Z) = 0.3575, \; g_2(m_Z) = 0.6514, \; g_3(m_Z) = 1.221$
- renormalisation group equations
- ($m_{top} = 171.2 \pm 2.1$ GeV)

Output:

- $g_2^2(\Lambda) = g_3^2(\Lambda)$ at $\Lambda = 1.1 \times 10^{17}$ GeV
- $m_{top} < 190$ GeV
- no $4^{th}$ SM generation

Excluded by Tevatron & LHC since:

- $m_{SMS} \neq 168.3 \pm 2.5$ GeV
- $\frac{5}{3} g_1(\Lambda)^2 \neq g_2(\Lambda)^2$

## How unique is the Standard Model?

The aim: Classifying the internal spaces

$$\mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \dots$$

- with respect to the number of summands in the algebra
- with respect to physical criteria

## Little Reminder

For the Standard Model we have

$$\mathcal{A}_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})(\oplus\mathbb{C})$$

or alternatively

$$\mathcal{A}_f = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C}$$

### Physicist's ”shopping list” (B. Iochum, T. Schücker, C.S. ’03):

The physical models emerging from the spectral action are required to

- be irreducible i.e. to have the smallest possible internal Hilbert space (minimal approach)
- allow a non-degenerate Fermionic mass spectrum
- be free of harmful anomalies
- have unbroken colour groups
- possess no uncharged massless Fermions

## Classification Results (B.Iochum, J.-H. Jureit, T.Schücker, C.S. 2003-2008):

| # sum. in $\mathcal{A}_f$ | $KO$ 0 | $KO$ 6 |
|---|---|---|
| 1 | no model | no model |
| 2 | no model | no model |
| 3 | SM[2] | no model |
| 4 | SM[2], el.-str.[1] | SM[2], el.-str.[1] |
| 6 | | SM[2] + el.-str.[1], $2 \times$ el.-str.[1] |

[1] Electro-Strong Model: "electron+proton", no Higgs,

$$\mathcal{A}_f = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_n(\mathbb{C}),$$
$$G_{gauge} = U(1) \times SU(n)/SO(n)/Sp(n)$$

[2] first family, colour group = $SU(n)/SO(n)/Sp(n)$

# Overview

## Beyond SM: the general strategy (bottom-up approach)

- find finite geometry that has SM as sub-model (tricky) => particle content, gauge group & representation
- make sure everything is anomaly free
- compute the spectral action => constraints on parameters
- determine the cut-off scale $\Lambda$ with suitable sub-set of the constraints
- use renorm. group equations to obtain low energy values of (hopefully) interesting parameters (Higgs couplings, Yukawa couplings)
- check with experiment! (and here we usually fail)

## Excluded models:

Extensions of the Standard Model:

- $AC$-model (C.S. '05):
  new particles ($A$ and $C$) with opposite hypercharge
  dark matter as bound $AC$-states (Fargion,Khlopov,C.S. '05)
- $\theta$-model (C.S. '07): new particles with $SU_c(D)$-colour
- Vector-Doublet Model (Squellari, C.S. '07):
  new $SU_w(2)$-vector doublets

Problem for models: • $m_{SMS} \geq 170$ GeV
• constraints on $g_1$, $g_2$, $g_3$ at $\Lambda$.

## Saving these models?

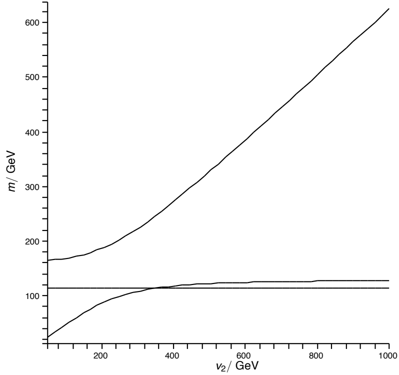
Some of these models, e.g. the $AC$-model may perhaps be extended to comply with experimental data!

## SM + $U(1)_X$ scaler field + $U(1)_X$ fermion singlet (C.S. 2009):

- Internal space: $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$
- Gauge group: $U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X$
- New fermions: $U(1)_X$-vector singlets ($X$-particles)
  neutral w.r.t SM gauge group , $M_X \sim \Lambda$
- New scalar: $U(1)_X$ singlet $\sigma$, neutral w.r.t SM gauge group
- $\mathcal{L}_{scalar} = -\mu_1^2|H|^2 + \frac{\lambda_1}{6}|H|^4 - \mu_2^2|\sigma|^2 + \frac{\lambda_2}{6}|\sigma|^4 + \frac{\lambda_3}{3}|H|^2|\sigma|^2$
- $U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X \to U(1)_{el.} \times SU(3)_c$
- $\mathcal{L}_{ferm+gauge} = \bar{X}_L M_X X_R + g_{\nu,X} \bar{\nu}_R \sigma X_L + \; h.c. \; + 1/g_4^2 F_X^{\mu\nu} F_{X,\mu\nu}$

## The constraints at $\Lambda$:

- only top-quark & $\nu_\tau$
- valid at $g_2 = g_3$
  => $\Lambda = 1.1 \times 10^{17}$ GeV
- $g_2^2 = \frac{\lambda_1}{24} \frac{(3g_t^2+g_\nu^2)^2}{3g_t^4+g_\nu^4}$
- $g_2^2 = \frac{\lambda_2}{24}$
- $g_2^2 = \frac{\lambda_3}{24} \frac{3g_t^2+g_\nu^2}{g_\nu^2}$
- $g_2^2 = \frac{1}{4}(3g_t^2 + g_\nu^2)$
- free parameters: $|\langle\sigma\rangle|$, $g_4$
- $m_{SMS} \sim 120 - 130$ GeV
- Problem: $\sqrt{5/3}g_1 \neq g_2 = g_3$




Mass EVs of scalar fields for $v_2 = \sqrt{2}\,|\langle\sigma\rangle|$, $\sqrt{2}|\langle H\rangle| = 246$ GeV, $g_4(m_Z) = 0.3$

## SM + $U(1)_X$ scalar field + new fermions (C.S. '13):

- SM as a sub-model: comme il faut!
- Internal space: $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \oplus_{i=1}^{6} \mathbb{C}_i$
- gauge group: $U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X$
- new fermions in each SM-generation:

$$X_l^1 \oplus X_l^2 \oplus X_l^3 : (0,1,1,+1) \oplus (0,1,1,+1) \oplus (0,1,1,0)$$
$$X_r^1 \oplus X_r^2 \oplus X_r^3 : (0,1,1,+1) \oplus (0,1,1,0) \oplus (0,1,1,+1)$$
$$V_\ell^w,\ V_r^w :\ (0,\bar{2},1,0)$$
$$V_\ell^c,\ V_r^c :\ (-1/6,1,\bar{3},0)$$

- new scalar: $\sigma : (0,1,1,+1)$

## The Lagrangian (scalar potential & new terms):

- $\mathcal{L}_{scalar} = -\mu_1^2|H|^2 - \mu_2^2|\sigma|^2 + \frac{\lambda_1}{6}|H|^4 + \frac{\lambda_2}{6}|\sigma|^4 + \frac{\lambda_3}{3}|H|^2|\sigma|^2$

- $$\mathcal{L}_{ferm} = g_{\nu,X^1}\bar{\nu}_r\sigma X_\ell^1 + \bar{X}_\ell^1 m_X X_r^1 + g_{X^2}\bar{X}_\ell^2\sigma X_r^2 + g_{X^3}\bar{X}_\ell^3\sigma X_r^3 + \bar{V}_\ell^c m_c V_r^c + \bar{V}_\ell^w m_w V_r^w + h.c.$$

- $\mathcal{L}_{gauge} = \frac{1}{g_4^2}F_X^{\mu\nu}F_{X,\mu\nu}$

- Symmetry breaking:
  $U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X \to U(1)_{el.} \times SU(3)_c \times \mathbb{Z}_2$

## The constraints at $\Lambda$:

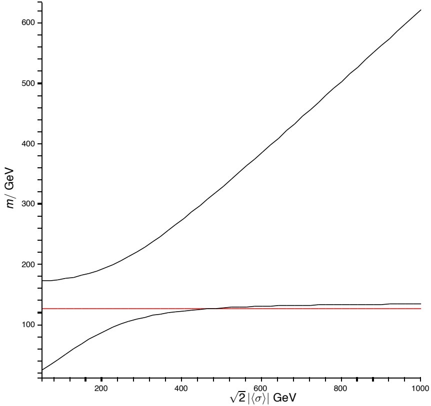
- $g_2(\Lambda) = g_3(\Lambda) = \sqrt{\frac{7}{6}}\, g_1(\Lambda) = \sqrt{\frac{4}{3}}\, g_4(\Lambda)$
- $\lambda_1(\Lambda) = 36\, \frac{H}{Y_2}\, g_2(\Lambda)^2, \;\; \lambda_2(\Lambda) = 36\, \frac{tr(g^4_{\nu,X^1})}{tr(g^2_{\nu,X^1})^2}\, g_2(\Lambda)^2$
- $\lambda_3(\Lambda) = 36\, \frac{tr(g^2_\nu)}{Y_2}\, g_2(\Lambda)^2$
- $Y_2(\Lambda) = tr(g^2_{\nu,X^1})(\Lambda) + tr(g^2_{X^1})(\Lambda) + tr(g^2_{X^2})(\Lambda) = 6\, g_2(\Lambda)^2$

## Some simplifications:

- $Y_2 \approx 3g_{top} + g_{\nu_\tau}$
- $tr(g^2_{X^1})(\Lambda) \approx tr(g^2_{X^2})(\Lambda) \approx 0$
- $tr(g^2_{\nu,X^1})(\Lambda) \approx g_{\nu,X}(\Lambda)^2 = 6\, g_2(\Lambda)^2$
- $(m_w)_{ij} \approx \Lambda$, $(m_c)_{ij} \approx 10^{15}$ GeV

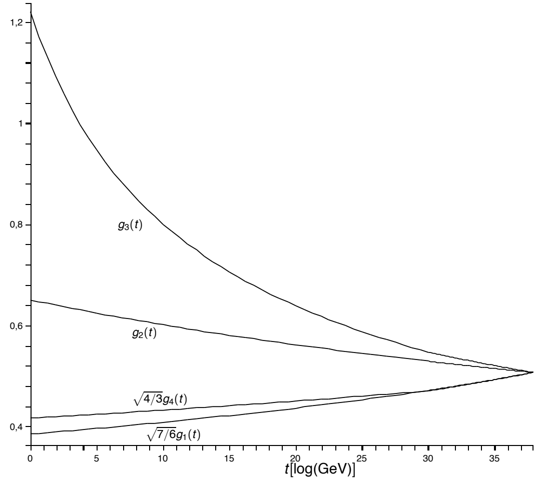
## Results for 1-loop renormalisation groups:

- Constraints
  => $\Lambda \approx 2 \times 10^{18}$ GeV
- $m_{top} \approx 172.9 \pm 1.5$ GeV
- $m_{\sigma_{1,SMS}} \approx 125 \pm 1.1$ GeV
- $m_{\sigma_2} \approx 445 \pm 139$ GeV
- $m_{Z_X} \approx 254 \pm 87$ GeV
- $g_4(m_Z) \approx 0.36$
- $m_{X_2,X_3} \precsim 50$ GeV
- free parameter: $|\langle\sigma\rangle|$




Mass EVs of scalar fields

# Running of the gauge couplings with normalisation factors



# Overview

## Questions & to-do-list

- Is the SM + scalar model compatible with LHC-data?
- Does the SM + scalar model contain viable dark matter candidates?
- Explore parameter space ($g_{\nu,X^1}$, $g^2_{X^2}$, $g^2_{X^3}$, $m_{X^1}$, $m_{V^w}$, $m_{V^c}$)
- Extend renormalisation group analysis to n-loop, $n \geq 2$
- Is the geometry a "sub-geometry" of a Connes-Chamseddine-type geometry?
- Classify Models beyond the Standard Model
- Spectral triples with Lorentzian signature (A. Rennie, M. Paschke, R. Verch, K. van den Dungen...)