Gauge-Stringy Instantons

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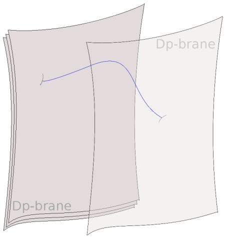
Brane Configuration

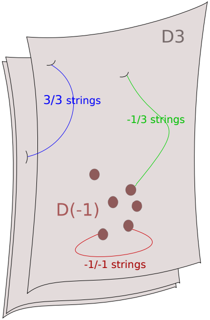
- Supersymmetry gauge theories in string theory can be realized by coinciding D-branes:



- The $\mathcal{N} = 4$ $D = 4$ $U(N)$ gauge theories lives on a stack of N D3-branes .
- Instanton in gauge theories are realized by Dp/D(p-4) brane configurations.

- Gauge instantons with charge k in $\mathcal{N}=4\ U(N)$ in 4D are realized by $N\ D3$-branes and $k\ D(-1)$-branes.

x^μ	1	2	3	4	5	6	7	8	9	10
D3	–	–	–	–	×	×	×	×	×	×
D(-1)	×	×	×	×	×	×	×	×	×	×



– : Neumann boundary condition
× : Dirichlet boundary condition

- Spectrum:

	Neveu-Schwarz	Ramond
3/3	$\hat{A}_\mu, \hat{\Phi}^i$	$\hat{\Lambda}^{\alpha A}, \hat{\Lambda}_{\dot{\alpha} A}$
(-1)/(-1)	$\hat{a}_\mu, \hat{\chi}^i$	$\hat{M}^{\alpha A}, \hat{\lambda}_{\dot{\alpha} A}$
3/(-1) & (-1)/3	$\hat{w}_{\dot{\alpha}}, \hat{\bar{w}}_{\dot{\alpha}}$	$\hat{\mu}^A, \hat{\bar{\mu}}^A$

- Gauge instantons with charge k in $\mathcal{N}=4$ $U(N)$ in 4D are realized by N $D3$-branes and k $D(-1)$-branes.

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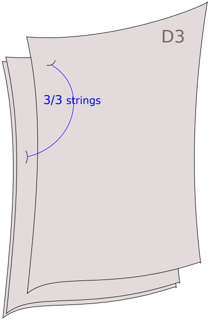

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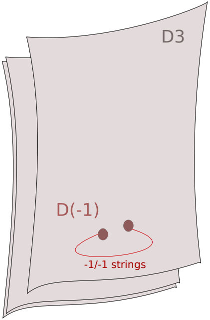
D(-1)	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$



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D(-1)	×	×	×	×	×	×	×	×	×	×

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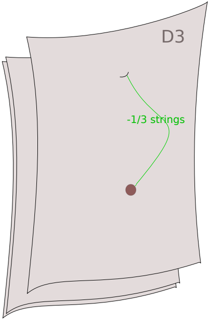

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- We are interested in studying instantons in $\mathcal{N} = 2\ U(N)$ gauge theories in 4D.
- One way to reduce the number of of supersymmetries is to add *orbifolds* in the background.
- An example of orbifold group that we consider in our model:

$$\mathbb{Z}_3 = \{1, \xi, \xi^{-1}\} \qquad \xi = e^{\frac{2\pi i}{3}}$$

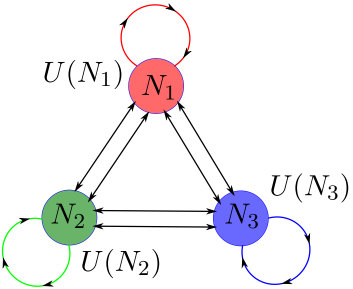
- The orbifold group acts only on the first two complex coordinates in the internal space. The manifold in the internal space:

$$\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$$

- At the singularity of orbifold the SUSY breaks down:

$$\mathcal{N}=4 \quad U(N) \to \mathcal{N}=2 \quad U(N_1)\times U(N_2)\times U(N_3)$$

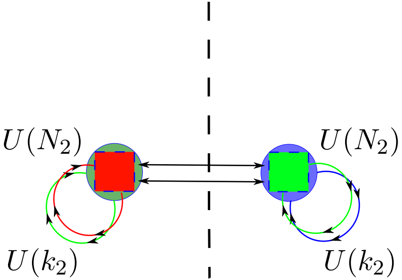
- In the presence of the orbifold N D3-branes split into N_1,N_2 and N_3 *fractional* D3-branes.
- Three gauge theories on fractional branes can be demonstrated by a *quiver diagram*:



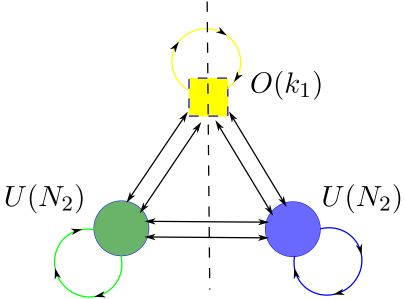
- To our model we include another background: An "O3-plan" along the D3-branes world-volume.
- Orientifold projection imposes extra symmetry on the world-sheet.
- Adding O3-plane do not change the number of supersymmetries; It reduces the number of moduli degrees of freedom.
- The orientifold projection also reduces the unitary groups:

$$U(N), U(k) \rightarrow USp(N), O(k)$$

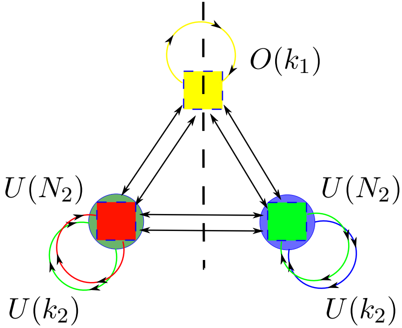
- Gaude instanton configuration:
  The gauge branes and D-instantons are in the same representation of the orbifold group, i.e. they occupy the same node of quiver diagram.



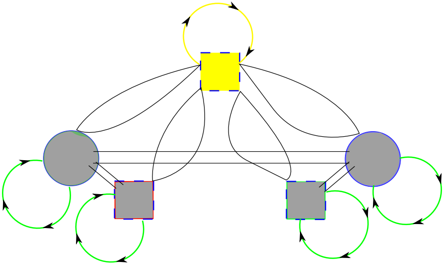
- Stringy instanton configuration:
  The gauge branes and D-instantons are in two different representations of the orbifold group, .i.e they occupy different nodes of quiver diagram.



- Gauge-Stringy Model:



Gauge-Stringy Spectrum



- The NS sector of -1/-1 string states are called the Neutral Bosonic Moduli:

$$\phi^M \to a^\mu \oplus \chi^p \equiv a^\mu \oplus \chi^i \quad M = 0, .., 9; \quad \mu = 0, .., 3; \quad i = 1, 2, 3$$

- Under orbifold transformation:

$$a_\mu = \gamma(g)\, a_\mu\, \gamma(g)^{-1} \qquad \chi_i = \xi^i\, \gamma(g)\, \chi_i\, \gamma(g)^{-1}$$

- $g(\gamma)$ is a representation of the orbifold group acting on Chan-Paton matrices:

$$\gamma(g) = \begin{pmatrix} \mathbb{1}_{k_s} & 0 & 0 \\ 0 & \xi\, \mathbb{1}_{k_g} & 0 \\ 0 & 0 & \xi^{-1}\, \mathbb{1}_{k_g} \end{pmatrix}$$

- Under orientifold transformation:

$$a_\mu = \gamma_+(\Omega)\, a_\mu^T\, \gamma_+(\Omega)^{-1} \quad \chi_i = -\gamma_+(\Omega)\, \chi_i^T\, \gamma_+(\Omega)^{-1}$$

- $\gamma_+(\Omega)$ is a symmetric representation of orientifold group acting on Chan-Paton matrices:

$$\gamma_+(\Omega) = \begin{pmatrix} \mathbb{1}_{k_s} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{k_g} \\ 0 & \mathbb{1}_{k_g} & 0 \end{pmatrix}$$

- $\mathbb{1}_{k_s}$ and $\mathbb{1}_{k_g}$ are respectively $k_s \times k_s$ and $k_g \times k_g$ unit matrices.

- Neutral Bosonic Chan-Patons satisfying orbifold and orienfifold conditions:

$$a^\mu = \begin{pmatrix} a^\mu_{(s)} & 0 & 0 \\ 0 & a^\mu_{(g)} & 0 \\ 0 & 0 & {a^\mu_{(g)}}^T \end{pmatrix} \quad \chi_3 = \begin{pmatrix} \chi_{(s)} & 0 & 0 \\ 0 & \chi_{(g)} & 0 \\ 0 & 0 & -{\chi_{(g)}}^T \end{pmatrix}$$

$$a^\mu_{(s)} = {a^\mu_{(s)}}^T \qquad \chi_{(s)} = -{\chi_{(s)}}^T$$

$$\chi^1 = \begin{pmatrix} 0 & \chi^1_{(gs)} & 0 \\ 0 & 0 & \chi^1_{(g)} \\ -{\chi^1_{(gs)}}^T & 0 & 0 \end{pmatrix} \quad \chi^2 = \begin{pmatrix} 0 & 0 & \chi^2_{(gs)} \\ -{\chi^2_{(gs)}}^T & 0 & 0 \\ 0 & \chi^2_{(g)} & 0 \end{pmatrix}$$

$$\chi^1_{(g)} = -{\chi^1_{(g)}}^T \qquad \chi^2_{(g)} = -{\chi^2_{(g)}}^T$$

- The R sector of –1/–1 string states are the Neutral Fermionic Moduli:

$$\Lambda_{\dot{\mathcal{A}}} \to \lambda_{\dot{\alpha}A} \oplus M^{\alpha A}$$

$$\dot{\mathcal{A}} = 1, .., 16 \qquad \dot{\alpha}, \alpha = 1, 2 \qquad A = 1, .., 4$$

- Through GSO projection we have chosen only the anti-chiral Ramond spinor i.e. $\Lambda_{\dot{\mathcal{A}}}$.
- The index $\dot{\alpha}$ (α) in $\lambda_{\dot{\alpha}A}$ ($M^{\alpha A}$) is anti-chiral (chiral) in the Lorentz space. The lower (upper) index A is chiral (anti-chiral) in 6d internal space.

- The Chan-Paton structure of Neutral Fermionic Moduli:

$$M^{\alpha\dot{a}} = \begin{pmatrix} M^{\alpha\dot{a}}_{(s)} & 0 & 0 \\ 0 & M^{\alpha\dot{a}}_{(g)} & 0 \\ 0 & 0 & M^{\alpha\dot{a}\,T}_{(g)} \end{pmatrix} \qquad M^{\alpha\dot{a}}_{(s)} = M^{\alpha\dot{a}\,T}_{(s)}$$

$$\lambda_{\dot{\alpha}\dot{a}} = \begin{pmatrix} \lambda_{(s)\dot{\alpha}\dot{a}} & 0 & 0 \\ 0 & \lambda_{(g)\dot{\alpha}\dot{a}} & 0 \\ 0 & 0 & -\lambda_{(g)\dot{\alpha}\dot{a}}{}^{T} \end{pmatrix} \qquad \lambda_{(s)\dot{\alpha}\dot{a}} = -\lambda_{(s)\dot{\alpha}\dot{a}}{}^{T}$$

- The entries in the Chan-Paton matrices are of either stringy or gauge type.

- The off-diagonal Chan-Patons of Neutral Fermionic Moduli:

$$M^{\alpha 3} = \begin{pmatrix} 0 & M^{\alpha}_{(gs)} & 0 \\ 0 & 0 & M^{\alpha}_{(g)} \\ {M^{\alpha}_{(gs)}}^T & 0 & 0 \end{pmatrix} \qquad M^{\alpha}_{(g)} = {M^{\alpha}_{(g)}}^T$$

$$\lambda_{\dot\alpha 3} = \begin{pmatrix} 0 & 0 & \lambda_{(gs)\dot\alpha} \\ -{\lambda_{(gs)\dot\alpha}}^T & 0 & 0 \\ 0 & \lambda_{(g)\dot\alpha} & 0 \end{pmatrix} \qquad \lambda_{(g)\dot\alpha} = -{\lambda_{(g)\dot\alpha}}^T$$

$$M^{\alpha 4} = \begin{pmatrix} 0 & 0 & M'^{\alpha}_{(gs)} \\ {M'^{\alpha}_{(gs)}}^T & 0 & 0 \\ 0 & M'^{\alpha}_{(g)} & 0 \end{pmatrix} \qquad M'^{\alpha}_{(g)} = {M'^{\alpha}_{(g)}}^T$$

$$\lambda_{\dot\alpha 4} = \begin{pmatrix} 0 & \lambda'_{(gs)\dot\alpha} & 0 \\ 0 & 0 & \lambda'_{(g)\dot\alpha} \\ -{\lambda'_{(gs)\dot\alpha}}^T & 0 & 0 \end{pmatrix} \qquad \lambda'_{(g)\dot\alpha} = -{\lambda'_{(g)\dot\alpha}}^T$$

- The charged moduli stem from –1/3 string states which enjoy the mixed boundary conditions.
- The string endpoint on D3-branes transform under anti-symmetric representation $\gamma_-(\Omega)$ of orienfifold group and string endpoint on D-instanton transform under the symmetric representation $\gamma_+(\Omega)$:

$$\gamma_-(\Omega) = \begin{pmatrix} \epsilon_{N_1\times N_1} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{N\times N} \\ 0 & -\mathbb{1}_{N\times N} & 0 \end{pmatrix}$$

$$\gamma_+(\Omega) = \begin{pmatrix} \mathbb{1}_{k_s} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{k_g} \\ 0 & \mathbb{1}_{k_g} & 0 \end{pmatrix}$$

- Cherged bosonic moduli $w_{\dot{\alpha}}$ and $\bar{w}_{\dot{\alpha}}$ are Weyl spinors in Lorentz space and scalars in internal space.
- Orbifold and orienfifold conditions on $w_{\dot{\alpha}}$ and $\bar{w}_{\dot{\alpha}}$ are:

$$\bar{w}_{\dot{\alpha}} = \gamma\left(g\right) w_{\dot{\alpha}} \gamma\left(g\right)^{-1} \qquad \bar{w}_{\dot{\alpha}} = \gamma_{+}\left(\Omega\right) w_{\dot{\alpha}}^{T} \gamma_{-}\left(\Omega\right)^{-1}$$

- Chan-Paton matrices takes the form:

$$w_{\dot{\alpha}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & w_{(g)\dot{\alpha}} & 0 \\ 0 & 0 & w'_{(g)\dot{\alpha}} \end{pmatrix} \quad \bar{w}_{\dot{\alpha}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & {w'_{(g)\dot{\alpha}}}^{T} & 0 \\ 0 & 0 & -{w_{(g)\dot{\alpha}}}^{T} \end{pmatrix}$$

- The R sector of 3/–1 string states μ^A and $\bar{\mu}^A$ are charged fermionic moduli:

$$\bar{\mu}^A = R(g)^A{}_B \gamma(g) \mu^B \gamma(g)^{-1} \qquad \bar{\mu}^A = R(\Omega)^A{}_B \gamma_+(\Omega) (\mu^B)^T \gamma_-(\Omega)^{-1}$$

$$\mu^a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu^a_{(g)} & 0 \\ 0 & 0 & \mu'^a_{(g)} \end{pmatrix} \qquad \bar{\mu}^a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & {\mu'^a_{(g)}}^T & 0 \\ 0 & 0 & -{\mu^a_{(g)}}^T \end{pmatrix}$$

$$\mu^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu_{(g)} \\ \mu_{(s)} & 0 & 0 \end{pmatrix} \qquad \bar{\mu}^3 = \begin{pmatrix} 0 & {\mu_{(s)}}^T & 0 \\ 0 & 0 & -{\mu_{(g)}}^T \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mu^4 = \begin{pmatrix} 0 & 0 & 0 \\ \mu'_{(s)} & 0 & 0 \\ 0 & \mu'_{(g)} & 0 \end{pmatrix} \qquad \bar{\mu}^4 = \begin{pmatrix} 0 & 0 & -{\mu'_{(s)}}^T \\ 0 & 0 & 0 \\ 0 & {\mu'_{(g)}}^T & 0 \end{pmatrix}$$

- The quartic terms in the moduli action can be rewritten in quadratic form by introducing "auxiliary moduli":

$$D^c \equiv \bar{\eta}^c_{\mu\nu}\,[a^\mu, a^\nu] + \bar{\zeta}^c_{mn}\,[\chi^m, \chi^n]\,.$$

where

$$\bar{\eta}^c_{\mu\nu}\;\bar{\zeta}^c_{mn} = 0$$

$$D^c = \begin{pmatrix} D^c_{(s)} & 0 & 0 \\ 0 & D^c_{(g)} & 0 \\ 0 & 0 & -{D^c_{(g)}}^T \end{pmatrix}$$

with $D^c_{(s)} = -{D^c_{(s)}}^T$.

$$C^{\alpha\dot{\alpha}} \equiv (\bar{\sigma}_\mu)_{\dot{\alpha}}^{\ \alpha} \left[a^\mu, \chi^{\dot{\alpha}a}\right]$$

where

$$\chi^{\dot{a}b} \equiv (\bar{\sigma}^m)^{\dot{a}b} \chi_m \qquad \chi_{a\dot{b}} \equiv (\sigma^m)_{a\dot{b}} \chi_m$$

therefore:

$$C^{\alpha 3} = \begin{pmatrix} 0 & C^\alpha_{(gs)} & 0 \\ 0 & 0 & C^\alpha_{(g)} \\ {C^\alpha_{(gs)}}^T & 0 & 0 \end{pmatrix} \qquad C^{\alpha 4} = \begin{pmatrix} 0 & 0 & C'^\alpha_{(gs)} \\ {C'^\alpha_{(gs)}}^T & 0 & 0 \\ 0 & C'^\alpha_{(g)} & 0 \end{pmatrix}$$

$$C^\alpha_{(g)} = {C^\alpha_{(g)}}^T \qquad C'^\alpha_{(g)} = {C'^\alpha_{(g)}}^T$$

$$h_a \equiv \bar{w}^{\dot{\alpha}} \chi_{\dot{\alpha} a}$$

$$h^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h_{(g)} \\ h_{(s)} & 0 & 0 \end{pmatrix} \quad \bar{h}^1 = \begin{pmatrix} 0 & h^T_{(s)} & 0 \\ 0 & 0 & -{h_{(g)}}^T \\ 0 & 0 & 0 \end{pmatrix}$$

$$h^2 = \begin{pmatrix} 0 & 0 & 0 \\ h'_{(s)} & 0 & 0 \\ 0 & h'_{(g)} & 0 \end{pmatrix} \quad \bar{h}^2 = \begin{pmatrix} 0 & 0 & -{h'_{(s)}}^T \\ 0 & 0 & 0 \\ 0 & {h'_{(g)}}^T & 0 \end{pmatrix}$$

Gauge-Stringy Q-Exact Action

- The moduli action is known from ADHM construction. It can also be rederived from disk amplitudes couplings in type IIB:

$$S = S_{\text{cubic}} + S_{\text{quartic}} + S_{\text{charged}}$$

$$g_0^2\, S_{\text{quartic}} = \frac{1}{2} D_c^2 + \frac{1}{2} D_c \left(\bar{\eta}^c_{\mu\nu} \left[a^\mu , a^\nu \right] + \bar{\zeta}^c_{mn} \left[\chi^m , \chi^n \right] \right) - \frac{1}{4} \left[a_\mu , \chi \right] \left[a_\mu , \bar{\chi} \right]$$
$$- \frac{1}{2} \left[a^\mu , \chi^{\dot{\alpha} a} \right] \left[a_\mu , \chi_{a \dot{\alpha}} \right] - \frac{1}{4} \left[\chi , \bar{\chi} \right] \left[\chi , \bar{\chi} \right] - \frac{1}{4} \left[\chi , \chi^{\dot{\alpha} a} \right] \left[\bar{\chi} , \chi_{a \dot{\alpha}} \right] ,$$

$$g_0^2\, S_{\text{cubic}} = 4 (\bar{\sigma}^\mu)_{\dot{\alpha}\beta} \left[M^{\beta a} , a_\mu \right] \lambda^{\dot{\alpha}}{}_a + 4 (\bar{\sigma}^\mu)_{\dot{\alpha}\beta} \left[M^{\beta \dot{a}} , a_\mu \right] \lambda^{\dot{\alpha}}{}_{\dot{a}}$$
$$- \frac{i}{2} \lambda_{\dot{\alpha} a} \left[\chi , \lambda^{\dot{\alpha} a} \right] - \frac{i}{2} \lambda_{\dot{\alpha} \dot{a}} \left[\bar{\chi} , \lambda^{\dot{\alpha} \dot{a}} \right] - i \lambda_{\dot{\alpha} \dot{a}} \left[\chi^{\dot{a} b} , \lambda^{\dot{\alpha}}{}_b \right]$$
$$- \frac{i}{2} M^{\alpha a} \left[\chi , M_{\alpha a} \right] - \frac{i}{2} M^{\alpha \dot{a}} \left[\bar{\chi} , M_{\alpha \dot{a}} \right] - i M^{\alpha a} \left[\chi_{a \dot{b}} , M_\alpha{}^{\dot{b}} \right]$$

$$g_0^2\, S_{\text{charged}} = 2i \left(\bar{\mu}^a w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}} \mu^a \right) \lambda^{\dot{\alpha}}{}_a + 2i \left(\bar{\mu}^{\dot{a}} w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}} \mu^{\dot{a}} \right) \lambda^{\dot{\alpha}}{}_{\dot{a}}$$
$$- i D^c \bar{w}^{\dot{\alpha}} \left(\tau^c \right)^{\dot{\beta}}{}_{\dot{\alpha}} w_{\dot{\beta}} - \chi^{\dot{a} b} \bar{w}_{\dot{\alpha}} w^{\dot{\alpha}} \chi_{b \dot{a}} + 2 \chi \bar{w}_{\dot{\alpha}} w^{\dot{\alpha}} \bar{\chi}$$
$$+ i \bar{\mu}^a \mu_a \chi + i \bar{\mu}^{\dot{a}} \mu_{\dot{a}} \bar{\chi} + i \left(\bar{\mu}^a \mu^{\dot{b}} - \bar{\mu}^{\dot{b}} \mu^a \right) \chi_{a \dot{b}}$$

- All moduli in the above action are 3×3 block Chan-Paton matrices.

- The prepotential of $\mathcal{N}=2$ SYM is obtained through the logarithm of the total partition function $\mathcal{Z}$:

$$\mathcal{Z}=\sum_{k=1}^{\infty} q^k Z_k \qquad\qquad q=\mu^{\gamma(k_s,k_g)}e^{2\pi i\tau}$$

- k-instanton partition function is given by integrating out over all neutral and charged moduli.

$$Z_k=\mathcal{N}_k\int dx^4 d\theta^4\int d\hat{\mathcal{M}}_k e^{-S(\mathcal{M}_k,\phi)} \qquad [d\mathcal{M}_k]=\mu^{-\gamma}$$

- Superspace coordinates $\theta^{\alpha a}\equiv \text{tr}M^{\alpha a}$ and $x^{\mu}\equiv \text{tr}a_{\mu}$ are the center of the instanton. The moduli action does not depend on the center of instanton.
- The moduli integration except for leading instanton numbers is too difficult to perform.

- The action of the instanton moduli space enjoys an important holomorphicity property.
- The holomorphicity becomes evident by a topological twist:

$$SU(2)_R \times SU(2)_I \to SU(2)' = \text{diag}(SU(2)_R \times SU(2)_I)$$

- This identification reorganize the 4 supercharges $Q^{\dot{\alpha}a}$ into a singlet and a triplet:

$$Q = \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}Q^{\dot{\alpha}\dot{\beta}} \qquad Q_c = \frac{i}{2}(\tau_c)_{\dot{\alpha}\dot{\beta}}Q^{\dot{\alpha}\dot{\beta}}$$

- The moduli having an index of right-handed Lorentz subgroup or the internal subgroup are decomposed.

$$\lambda_{\dot{\alpha}a} \;\to\; \lambda_{\dot{\alpha}\dot{\beta}} \equiv \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\eta + \frac{i}{2}(\tau^c)_{\dot{\alpha}\dot{\beta}}\lambda_c$$

$$M^{\alpha a} \to M^{\alpha\dot{\beta}} \equiv \frac{1}{2}M_\mu(\sigma^\mu)^{\alpha\dot{\beta}}$$

The action turns out to be Q-exact; $S = Q\,\Xi$:

$$\Xi = \frac{i}{4} M^{\mu} [\bar{\chi}, a_{\mu}] + \frac{1}{2} A \bar{\eta}^{c}_{\mu\nu} \lambda^{c} [a_{\mu}, a_{\nu}] - \bar{w}^{\dot{\alpha}} (\tau^{c})_{\dot{\alpha}\dot{\beta}} w^{\dot{\beta}} \lambda^{c} + \left(\bar{\mu}^{\dot{\alpha}} w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}} \mu^{\dot{\alpha}}\right) \bar{\chi}$$
$$+ \frac{1}{2} \lambda^{c} D^{c} + \frac{i}{4} [\chi, \bar{\chi}] \eta - \frac{1}{2} \left(\bar{\mu}^{a} h_{a} + \bar{h}^{a} \mu_{a}\right) - \left(\bar{w}^{\dot{\alpha}} \mu^{a} + \bar{\mu}^{a} w^{\dot{\alpha}}\right) \chi_{a\dot{\alpha}}$$
$$+ 4 \left(\bar{\sigma}^{\mu}\right)^{\dot{\alpha}}{}_{\alpha} [\chi_{a\dot{\alpha}}, a_{\mu}] M^{\alpha a} + \frac{1}{2} M^{\alpha a} C_{\alpha a} - \frac{i}{2} \lambda_{\dot{\alpha} a} \left[\chi^{\dot{\alpha} a}, \bar{\chi}\right]$$
$$+ \frac{1}{2} \bar{\zeta}^{c}_{mn} \left(\bar{\sigma}^{m} \sigma^{n}\right)^{\dot{\beta}}_{\dot{\alpha}} \lambda_{c} \left[\chi^{a\dot{\alpha}}, \chi_{\dot{\beta} a}\right]$$

$$\begin{array}{ll}
Q\chi = 0, \;\; Q\bar{\chi} = \eta & Q\eta = i[\chi, \bar{\chi}] \\
Q a^{\mu} = M^{\mu} & Q M^{\mu} = i[\chi, a^{\mu}] \\
Q\lambda^{c} = D^{c} & Q D^{c} = i[\chi, \lambda^{c}] \\
Q w^{\dot{\alpha}} = \mu^{\dot{\alpha}} & Q\mu^{\dot{\alpha}} = -i w^{\dot{\alpha}} \chi \\
Q\bar{w}_{\dot{\alpha}} = \bar{\mu}_{\dot{\alpha}} & Q\bar{\mu}_{\dot{\alpha}} = i\chi\bar{w}_{\dot{\alpha}} \\
Q\chi^{\dot{\alpha} a} = \lambda^{\dot{\alpha} a} & Q\lambda^{\dot{\alpha} a} = i\left[\chi, \chi^{\dot{\alpha} a}\right] \\
Q\chi_{a\dot{\alpha}} = \lambda_{a\dot{\alpha}} & Q\lambda_{a\dot{\alpha}} = i[\chi, \chi_{a\dot{\alpha}}] \\
Q M^{\alpha a} = C^{\alpha a} & Q C^{\alpha a} = i[\chi, M^{\alpha a}] \\
Q\mu^{a} = h^{a} & Q h^{a} = i\mu^{a}\chi \\
Q\bar{\mu}^{a} = \bar{h}^{a} & Q\bar{h}^{a} = -i\chi\bar{\mu}^{a}
\end{array}$$

- The multi-instanton calculus becomes possible by localization of integral on the instanton moduli space through the introduction of an Ω-background.
- In IIB the Ω-background is provided by a R-R 3-form flux F_{LMN}.
- The flux invariant under orbifold and orientifold projection: $\mathcal{F}_{\mu\nu} \equiv F_{\mu\nu z}$ and $\bar{\mathcal{F}}_{\mu\nu} \equiv F_{\mu\nu\bar{z}}$, z and $\bar{z}$ along the third complex coordinate.
- Interaction with bosonic moduli:

$$\frac{1}{g_0^2}\text{tr}\Big\{\mathcal{F}^{\mu\nu}a_\nu[\bar{\chi},a_\mu] + i\bar{\mathcal{F}}a_\mu[\chi,a_\nu] - i\bar{\mathcal{F}}^{\mu\nu}a_\mu\mathcal{F}_{\nu\rho}a^\rho\Big\}$$

- Interaction with fermionic moduli:

$$\frac{1}{g_0^2}\text{tr}\Big\{-\frac{1}{2}\epsilon_{cde}\lambda^c\lambda^d f^e - f_c\lambda^c\eta + if_cD^c\bar{\chi} + \bar{\mathcal{F}}_{\mu\nu}M^\mu M^\nu\Big\}$$

- The BRST transformation of moduli involve only holomorphic graviphoton field strength.

$$
\begin{array}{ll}
Q_\Omega \chi = 0,\; Q_\Omega \bar{\chi} = \eta & Q_\Omega \eta = i\,[\chi,\bar{\chi}] \\
Q_\Omega a^\mu = M^\mu & Q_\Omega M^\mu = i\,[\chi, a^\mu] - i\bar{\mathcal{F}}^{\mu\nu} a_\nu \\
Q_\Omega \lambda^c = D^c & Q_\Omega D^c = i\,[\chi,\lambda^c] + \epsilon^{cde}\lambda_d f_e \\
Q_\Omega w^{\dot\alpha} = \mu^{\dot\alpha} & Q_\Omega \mu^{\dot\alpha} = -iw^{\dot\alpha}\chi + i\phi w^{\dot\alpha} - \frac{1}{2}\bar{\mathcal{F}}_{\mu\nu}\left(\bar\sigma^{\mu\nu}\right)^{\dot\alpha}_{\ \dot\beta} w^{\dot\beta} \\
Q_\Omega \bar{w}_{\dot\alpha} = \bar\mu_{\dot\alpha} & Q_\Omega \bar\mu_{\dot\alpha} = i\chi\bar{w}_{\dot\alpha} - i\bar{w}^{\dot\alpha}\phi - \frac{1}{2}\bar{\mathcal{F}}_{\mu\nu}\left(\bar\sigma^{\mu\nu}\right)^{\dot\alpha}_{\ \dot\beta}\bar{w}^{\dot\beta} \\
Q_\Omega \chi^{\dot\alpha a} = \lambda^{\dot\alpha a} & Q_\Omega \lambda^{\dot\alpha a} = i\left[\chi,\chi^{\dot\alpha a}\right] - \frac{1}{2}\bar{\mathcal{F}}_{\mu\nu}\left(\bar\sigma^{\mu\nu}\right)^{\dot\alpha}_{\ \dot\beta}\chi^{\dot\beta a} \\
Q_\Omega \chi_{a\dot\alpha} = \lambda_{a\dot\alpha} & Q_\Omega \lambda_{a\dot\alpha} = i\,[\chi,\chi_{a\dot\alpha}] - \frac{1}{2}\bar{\mathcal{F}}_{\mu\nu}\left(\bar\sigma^{\mu\nu}\right)_{\dot\alpha}^{\ \dot\beta}\chi_{a\dot\beta} \\
Q_\Omega M^{\alpha a} = C^{\alpha a} & Q_\Omega C^{\alpha a} = i\,[\chi, M^{\alpha a}] - \frac{1}{2}\bar{\mathcal{F}}^{\mu\nu}\left(\sigma_{\mu\nu}\right)^{\alpha}_{\ \beta} M^{\beta a} \\
Q_\Omega \mu^a = h^a & Q_\Omega h^a = i\mu^a\chi - i\phi\mu^a \\
Q_\Omega \bar\mu^a = \bar{h}^a & Q_\Omega \bar{h}^a = -i\chi\bar\mu^a + i\bar\mu^a\phi
\end{array}
$$

- The gauge fermion Ξ depends only on anti-holomorphic graviphoton field strength: $\Xi = \Xi + \Xi_{\bar{\mathcal{F}}}$

$$
\Xi_{\bar{\mathcal{F}}} = if_c\lambda^c\bar\chi + \bar{f}_{\mu\nu}a^\mu M^\nu + \bar{f}_{\mu\nu}\left(\bar\sigma^{\mu\nu}\right)_{\dot\alpha\dot\beta}\bar{w}^{\dot\alpha}\mu^{\dot\beta} + \bar{f}^{\mu\nu}\left(\bar\sigma_{\mu\nu}\right)^{\dot\alpha}_{\ \dot\beta}\chi_{\dot\alpha a}\lambda^{a\dot\beta}
$$

- The dimension of the moduli space is defined as the sum over all canonical dimensions of the moduli.
- In number of degrees of freedom of the various Chan-Paton matrices and the moduli dimensions are given by the table:

↓ BRST pairs | moduli →	gauge	stringy	gauge-stringy	$[L]$
(a^μ, M^μ)	k_g^2	$\frac{1}{2}k_s(k_s+1)$	$\times$	$(L, L^{\frac{1}{2}})$
$(\bar{\chi}, \eta)$	k_g^2	$\frac{1}{2}k_s(k_s-1)$	$\times$	$(L^{-1}, L^{-\frac{3}{2}})$
(λ^c, D^c)	k_g^2	$\frac{1}{2}k_s(k_s-1)$	$\times$	$(L^{-\frac{3}{2}}, L^{-2})$
$(\mu^a, h^a), (\bar{\mu}^a, \bar{h}^a)$	$k_g N$	$k_s N$	$\times$	$(L^{\frac{1}{2}}, L^0)$
$\left(w^{\dot{\alpha}}, \mu^{\dot{\alpha}}\right), (\bar{w}_{\dot{\alpha}}, \bar{\mu}_{\dot{\alpha}})$	$k_g N$	$\times$	$\times$	$(L, L^{\frac{1}{2}})$
$(M^{\alpha a}, C^{\alpha a})$	$\frac{1}{2}k_g(k_g+1)$	$\times$	$k_g k_s$	$(L^{\frac{1}{2}}, L^0)$
$(\chi_{\dot{\alpha}a}, \lambda_{\dot{\alpha}a})$	$\frac{1}{2}k_g(k_g-1)$	$\times$	$k_g k_s$	$(L^{-1}, L^{-\frac{3}{2}})$

- The dimension of the moduli space becomes then:

$$[d\mathcal{M}] = \mu^{-b_1(k_g-k_s)}$$

Gauge-Stringy Partition Function

- In a certain localization limit the functional integral becomes Gaussian and easy to perform.

$$Z_k^{(gs)} = \int d\chi d\tilde{\chi}\, \mathcal{I}_{(g)}\, \mathcal{I}_{(s)}\, \mathcal{I}_{(gs)}$$

where

$$\mathcal{I}_{(g)} = \frac{\mathcal{P}_{(g)}(\tilde{\chi})\mathcal{R}_{(g)}(\tilde{\chi})\mathcal{C}_{(g)}(\tilde{\chi})}{\mathcal{Q}_{(g)}(\tilde{\chi})\mathcal{L}_{(g)}(\tilde{\chi})\mathcal{W}_{(g)}(\tilde{\chi})}$$

$$\mathcal{I}_{(s)} = \frac{\mathcal{P}_{(s)}(\chi)\mathcal{R}_{(s)}(\chi)}{\mathcal{Q}_{(s)}(\chi)}$$

$$\mathcal{I}_{(gs)} = \frac{\mathcal{C}_{(gs)}(\tilde{\chi}, \chi)}{\mathcal{L}_{(gs)}(\tilde{\chi}, \chi)}$$

- The functions above are the determinants of the BRST charge in different representations.
- $\chi \equiv \chi_s$ and $\tilde{\chi} \equiv \chi_g$ are unpaired Chan-Paton matrices.

- The integration still get simplified more if one goes to the Cartan basis. In Cartan basis the partition function integration takes the form:

$$Z_k^{(gs)} = \int d\chi_i d\tilde{\chi}_I \prod_{I=1}^{r[U(k)]} (\frac{d\tilde{\chi}_I}{2\pi i}) \prod_{i=1}^{r[SO(k)]} (\frac{d\chi_i}{2\pi i}) \Delta(\tilde{\chi}_I)\Delta(\chi_i)$$
$$\times \mathcal{I}_{(g)}(\chi_I)\, \mathcal{I}_{(s)}(\chi_i)\, \mathcal{I}_{(gs)}(\chi_i, \chi_I)$$

where

$$\mathcal{I}_{(g)}(\chi_I) = \frac{\mathcal{P}_{(g)}(\tilde{\chi}_I)\mathcal{R}_{(g)}(\tilde{\chi}_I)\mathcal{C}_{(g)}(\tilde{\chi}_I)}{\mathcal{Q}_{(g)}(\tilde{\chi}_I)\mathcal{L}_{(g)}(\tilde{\chi}_I)\mathcal{W}_{(g)}(\tilde{\chi}_I)}$$

$$\mathcal{I}_{(s)}(\chi_i) = \frac{\mathcal{P}_{(s)}(\chi_i)\mathcal{R}_{(s)}(\chi_i)}{\mathcal{Q}_{(s)}(\chi_i)}$$

$$\mathcal{I}_{(gs)}(\chi_i, \tilde{\chi}_I) = \frac{\mathcal{C}_{(gs)}(\tilde{\chi}_I, \chi_i)}{\mathcal{L}_{(gs)}(\tilde{\chi}_I, \chi_i)}$$

- Switching off any of the gauge or stringy moduli we reach respectively to stringy or gauge partition function.
- The partition function we obtain in our string theory calculations is exactly the same as the one extracted from ADHM calculations:

$$Z_k^{(g)} = \frac{\epsilon^k}{(E_1E_2)^k} \int \prod_{I=1}^{k} d\tilde{\chi}_I \prod_{l=1}^{N} \prod_{A=1}^{2} \frac{(2\tilde{\chi}_I + E_A)(\tilde{\chi}_I + \phi_l)}{(\tilde{\chi}_I + \phi_l - \epsilon)(\tilde{\chi}_I + \phi_l + \epsilon)} \times \frac{(\tilde{\chi}_I - \tilde{\chi}_J)^2[(\tilde{\chi}_I - \tilde{\chi}_J)^2 - \epsilon^2](\tilde{\chi}_I + \tilde{\chi}_J + E_A)}{[(\tilde{\chi}_I - \tilde{\chi}_J)^2 - E_A^2][(\tilde{\chi}_I + \tilde{\chi}_J)^2 - \epsilon^2]}$$

- The stringy partition function is also coincides with the one we studied in a separate paper.

- The prepotential is the logarithm of the total partition function:

$$F^{(n.p.)}(\Phi) = \epsilon \log \mathcal{Z}_{tot} \mid_{\phi \to \Phi, E_A \to 0}$$

- The prepotential itself come from the contribution of all instanton numbers:

$$F^{(n.p.)} = \sum_{k=1}^{\infty} F_k q^k \mid_{\phi \to \Phi, E_A \to 0}$$

- Expanding the logarithm of the total partition function one arrives at

$$F_1^{(gs)} = \epsilon Z_1^{(gs)}$$
$$F_2^{(gs)} = \epsilon Z_2^{(gs)} - {F_1^{(gs)}}^2 / 2\epsilon$$

- Performing the integration even for small instanton numbers is very difficult.
- We have done the integration of the partition function for only $k=1$ and $k=2$:

$$Z_1^{(gs)} = \frac{(-1)^N}{2}\mathcal{N}_1 E_1^2$$

$$Z_2^{(gs)} = \frac{(-1)^N}{3}\mathcal{N}_2 E_1^4 \left(8\text{tr}\Phi^4 - 4E_1^2\text{tr}\Phi^2 + 5/16E_1^4\right)$$

- The prepotential corrections due to 1- and 2-instanton in U(N) gauge theory becomes:

$$F_1^{(gs)} = \frac{(-1)^N}{2}\mathcal{N}_1 E_1^4$$

$$F_2^{(gs)} = \frac{(-1)^N}{3}\mathcal{N}_2 E_1^6 \left(8\text{tr}\Phi^4 - 4E_1^2\text{tr}\Phi^2 + 5/16E_1^4\right) - \frac{\mathcal{N}_1^2}{8}E_1^6$$

Conclusion

- One may be interested in studying the gauge-instanton functions in other classical gauge theories.
- It is nice to see if the dimensionless moduli measure can exist in other gauge-stringy configurations.
- Obtaining higher instanton number calculation is still an open problem in gauge-stringy calculation.

Thank You!