An Enhanced CMB Power Spectrum from Quantum Gravity

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The fundamental interaction that has not been quantized as yet is **Gravitation**

A deeper understanding of the quantum version

$\Downarrow$

Find a unified theory

The real structure of nature

Quantum Theory of Gravity

There exist many approaches to a quantum theory of gravity

nowadays no less than 16 !

Either field-theoretical or of sharply different nature.
Characteristic scale of the theory: **Planck scale**

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-33}\mathrm{cm},$$

$$t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.40 \times 10^{-44}\mathrm{s},$$

$$m_P = \frac{\hbar}{l_P c} = \sqrt{\frac{\hbar c}{G}} \approx 1.22 \times 10^{19}\mathrm{GeV}$$

How can we find a way?

Very difficult to test the effects in laboratory

$\Downarrow$

Possible relevant effects at cosmological scale

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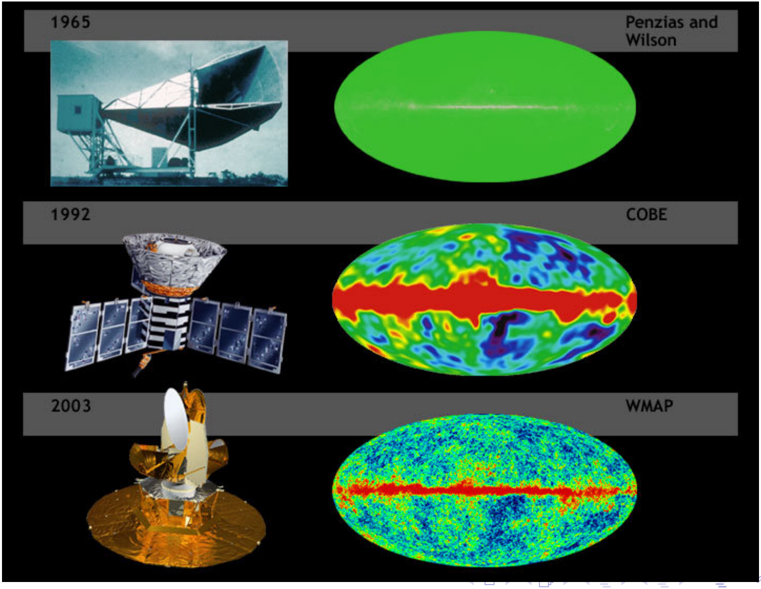
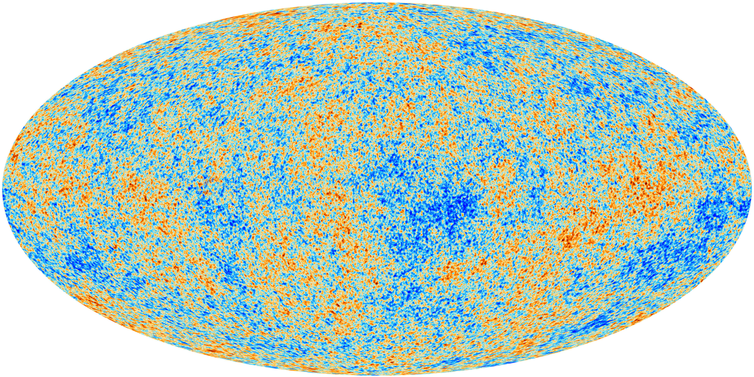
Cosmic Microwave Background Radiation (CMB)

CMB measurement history



CMB measurement history

Planck 2013



Wheeler–DeWitt Equation

Einstein–Hilbert action

$$S = \frac{1}{16\pi} \int_M \mathrm{d}^4x \sqrt{-g}\,{}^{(4)}R$$

$$\Downarrow$$

Arnowitt–Deser–Misner formalism

$$\sqrt{-g}\,{}^{(4)}R = N\sqrt{h}\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right) + 2\partial_t(\sqrt{h}K)$$
$$-2\partial_i\left[\sqrt{h}(N^i K - g^{ij\,(3)}\nabla_j N)\right]$$

$$K_{ij} = \frac{1}{2N}\left({}^{(3)}\nabla_i N_j + {}^{(3)}\nabla_j N_i - \frac{\partial h_{ij}}{\partial t}\right)$$

Wheeler–DeWitt equation

Hamiltonian formalism

$$H = \int \mathrm{d}^3x \left(N\mathscr{H}_t + N_i \mathscr{H}^i \right)$$

This formalism enables us to use the Dirac quantization method

$$\hat{h}_{ij}\psi = h_{ij}\psi \qquad \hat{\pi}^{jk}\psi = \frac{\hbar}{\mathrm{i}}\frac{\delta\psi}{\delta h_{jk}}$$

$$\Downarrow$$

Wheeler–DeWitt equation (WDW)

$$\hat{\mathscr{H}}_t\psi = \left\{ -\hbar^2 G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} - \sqrt{h}^{(3)}R \right\} \psi[h_{ij}] = 0$$

Friedmann–Lemaitre–Robertson–Walker (FLRW) Universe

Considering a **spatially flat**, **homogeneus** and **isotropic** Universe, one can describe it by a FLRW metric

$$ds^2 = d\tau^2 - a^2(\tau)\delta_{ij}dx^i dx^j .$$

The Wheeler–DeWitt equation, if one assumes an **inflationary field** ϕ, becomes

$$\left[\frac{1}{m_P^2}\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} + e^{6\alpha} m^2\phi^2\right]\psi(\alpha,\phi) = 0 \quad \alpha = \ln a$$

Slow-Roll Condition

We can get a simpler form of the WDW equation if we assume the **slow-roll condition** for the inflationary field ϕ

$$\frac{\partial^2\psi}{\partial\phi^2} \ll e^{6\alpha} m^2 \phi^2 \psi \quad m\phi \to m_P H$$

Thus the equation becomes

$$\left[\frac{1}{m_P^2}\frac{\partial^2}{\partial\alpha^2} + e^{6\alpha} m_P^2 H^2\right] \psi(\alpha, \phi) = 0$$

Born–Oppenheimer Approximation and Inhomogeneous Fluctuations

We now consider the fluctuations of an inhomogeneous inflaton field on top of its homogeneous part

$$\phi \to \phi(t) + (\delta\phi)(\mathbf{x}, t) \quad (\delta\phi)(\mathbf{x}, t) = \sum_{\kappa} f_{\kappa}(t) e^{i\kappa \cdot x}$$

The smallness of the fluctuations' self-interaction and the **Born–Oppenheimer (BO)** approximation enable us to factorize the wave functional

$$\psi(\alpha, \phi, \{f_{\kappa}\}_{\kappa=1}^{\infty}) = \psi_0(\alpha, \phi) \prod_{\kappa=1}^{\infty} \tilde{\psi}_{\kappa}(\alpha, \phi, f_{\kappa})$$

Hamiltonian Factorization

Thus the WDW equation can be rewritten

$$\left[\mathcal{H}_0 + \sum_{\kappa=1}^{\infty} \mathcal{H}_\kappa\right] \psi(\alpha, \phi, \{f_\kappa\}_{\kappa=1}^{\infty}) = 0$$

$$\mathcal{H}_0 = \frac{e^{-3\alpha}}{2}\left[\frac{1}{m_P^2}\frac{\partial^2}{\partial\alpha^2} + e^{6\alpha} m_P^2 H^2\right]$$

$$\mathcal{H}_\kappa = \frac{e^{-3\alpha}}{2}\left[-\frac{\partial^2}{\partial f_\kappa^2} + W_\kappa(\alpha) f_\kappa^2\right]$$

$$W_\kappa(\alpha) = \kappa^2 e^{4\alpha} + m^2 e^{6\alpha}$$

The Jeffreys–Wentzel–Kramers–Brillouin Method

Identify the quantum gravitational contributions to the terms of the expansion of the WDW in powers of m_P^2
(Effective Theory)

On writing every single mode in the form

$$\psi_\kappa(\alpha, f_\kappa) = \mathrm{e}^{\mathrm{i}S(\alpha, f_\kappa)} \quad S(\alpha, f_\kappa) = m_P^2 S_0 + m_P^0 S_1 + m_P^{-2} S_2 + \dots$$

m_P^0 Order

We note that at the zeroth stage of the JWKB approximation one obtains the usual evolution equation for matter **Schwinger–Tomonaga**

$$\mathrm{i}\frac{\partial}{\partial t}\psi_\kappa^{(0)} = \mathcal{H}_\kappa \psi_\kappa^{(0)} \quad \psi_\kappa^{(0)} \equiv \gamma(\alpha)\mathrm{e}^{\mathrm{i}S_1(\alpha,f_\kappa)}$$

Where we have defined the **JWKB time**

$$\frac{\partial}{\partial t} \equiv -\mathrm{e}^{-3\alpha}\frac{\partial S_0}{\partial \alpha}\frac{\partial}{\partial \alpha}$$

m_P^2 Order

To second order we obtain the first **quantum-gravitational corrections** to the matter wave functional

$$\mathrm{i}\frac{\partial\psi_\kappa^{(1)}}{\partial t} = \mathcal{H}_\kappa\psi_\kappa^{(1)} - \frac{\mathrm{e}^{3\alpha}}{2m_P^2\psi_\kappa^{(0)}}\left[\frac{(\mathcal{H}_\kappa)^2}{V(\alpha)}\psi_\kappa^{(0)} + \mathrm{i}\left(\frac{\psi_\kappa^{(0)}}{V(\alpha)}\frac{\partial\mathcal{H}_\kappa}{\partial t} - \frac{1}{V^2(\alpha)}\frac{\partial V(\alpha)}{\partial t}\mathcal{H}_\kappa\psi_\kappa^{(0)}\right)\right]\psi_\kappa^{(1)}$$

$$\psi_\kappa^{(1)}(\alpha, f_\kappa) \equiv \psi_\kappa^{(0)}(\alpha, f_\kappa)\mathrm{e}^{\mathrm{i}\frac{\eta_2(\alpha, f_\kappa)}{m_P^2}}$$

An Alternative Approach

Consider now a more general form of the WDW equation

$$\left(\frac{1}{2m_P^2} \frac{\partial^2}{\partial a^2} + m_P^2 V(a) + \hat{H}_M \right) \Psi(a, \phi) = 0$$

By inserting into this equation the **BO approximation** in the form

$$\Psi(a, \phi) = \psi(a) \chi(a, \phi)$$

ψ and χ equations

We obtain an equation for $\psi(a)$

$$\left[\frac{1}{2m_P^2}D^2 + V_G + \langle \hat{H}_M \rangle\right]\psi = -\frac{\hbar^2}{2m_P^2}\langle \bar{D}^2 \rangle \psi$$

and an equation for $\chi(a, \phi)$

$$(\hat{H}_M - \langle \hat{H}_M \rangle)\chi + \frac{1}{m_P^2}\frac{(D\psi)}{\psi}\bar{D}\chi = -\frac{1}{2m_P^2}(\bar{D}^2 - \langle \bar{D}^2 \rangle)\chi$$

where we define

$$D \equiv \frac{\partial}{\partial a} + \mathrm{i}A, \qquad \bar{D} \equiv \frac{\partial}{\partial a} - \mathrm{i}A, \qquad A \equiv -\mathrm{i}\left\langle \chi \left| \frac{\partial}{\partial a} \right| \chi \right\rangle$$

ψ and χ equations

If we introduce the gauge invariant wave funcional as

$$\tilde{\psi} \equiv \exp\left[\imath \int^{a} A da'\right] \psi, \qquad \tilde{\chi} \equiv \exp\left[-\imath \int^{a} A da'\right] \chi$$

$$\tilde{\Psi} \equiv \tilde{\psi}\tilde{\chi} = \Psi.$$

thus the D and $\bar{D}$ operators become $\frac{\partial}{\partial a}$

The BO Approximation

Now we can rewrite the BO approximation as

$$\Psi \approx \left(\frac{1}{N_K(a)} \exp\left[\mathrm{i} m_P^2 S_0 + \frac{\mathrm{i}}{m_P^2}\sigma_2\right]\right)\left(N_K(a)\exp\left[\mathrm{i}S_1 + \frac{\mathrm{i}\eta_2}{m_P^2}\right]\right)$$

$$= \tilde{\psi}_K \tilde{\chi}_K$$

Clearly from this we note that

$$\psi_\kappa^{(1)} \equiv \tilde{\chi}_K$$

Substituting these expressions in the equation for ψ, and collecting the terms proportional to m_P^2, m_P^0 and m_P^{-2} respectively

$$-\frac{1}{2}{S_0'}^2 + V = 0$$

$$\langle \hat{H}_M \rangle_0 - \mathrm{i}\frac{N_K' S_0'}{N_K} + \mathrm{i}\frac{S_0''}{2} = 0$$

$$\frac{1}{2}\left(2\frac{{N_K'}^2}{N_K^2} - 2S_0'\sigma_2' - \frac{N_K''}{N_K}\right) + \langle \hat{H}_M \rangle_{-2} = -\frac{1}{2}\langle \bar{D}^2 \rangle_0$$

In the same way we have for χ two equations proportional to m_P^0 and m_P^{-2}, respectively

$$H_M^0 - \langle \hat{H}_M \rangle_0 + \mathrm{i} S_0' \left(\frac{N_K'}{N_K} + \mathrm{i} S_1' \right) = 0$$

$$H_M^{-2} - \langle \hat{H}_M \rangle_{-2} - \frac{N_K'}{N_K} \left(\frac{N_K'}{N_K} + \mathrm{i} S_1' \right) - S_0' \eta_2'$$

$$= \frac{1}{2} \left(\langle \bar{D}^2 \rangle_0 - \frac{N_K''}{N_K} - \mathrm{i} S_1'' + S_1''^2 - 2\mathrm{i} S_1' \frac{N_K'}{N_K} \right)$$

where we have defined the c-number H_M by

$$\hat{H}_M \tilde{\chi}_K \equiv H_M \tilde{\chi}_K$$

WDW Corrections

Comparing the equations of the same order in m_P

$$H_M^0 + \frac{\mathrm{i}S_0''}{2} - S_0' S_1' = 0$$

$$-S_0'\eta_2' + H_M^{-2} - S_0'\sigma_2' + \mathrm{i}\frac{S_1''}{2} - \frac{{S_1'}^2}{2} = 0$$

They become identical to ours' by choosing

$$\hat{H}_M = \mathcal{H}_k$$

Unitarity Violation

To verify a possible violation of unitarity one has to consider the equation

$$\frac{\partial}{\partial \eta}\langle \tilde{\chi}_{Ks} | \tilde{\chi}_{Ks} \rangle = \frac{\partial}{\partial \eta}\left(\int \mathrm{d}\phi \tilde{\chi}_K^* \tilde{\chi}_K \right)$$

where

$$\frac{\partial}{\partial \eta} \equiv -\frac{1}{m_P^2}\frac{\partial}{\partial a}$$

that is proportional to our **JWKB time** previously defined and

$$\tilde{\chi}_{Ks} \equiv \exp\left[-\mathrm{i} \int^{\eta} \mathrm{d}\eta' \left(\langle \hat{H}_M \rangle_0 + \frac{1}{m_P^2} \langle \hat{H}_M \rangle_{-2} \right) \right] \tilde{\chi}_K$$

this results from the comparison of equations for $\tilde{\chi}_K$ to all orders.

Unitarity Violation

It is important to note that, with this notation, the **BO approximation** requires the condition

$$\langle \chi | \bar{D} | \chi \rangle = \langle \tilde{\chi} | \frac{\partial}{\partial a} | \tilde{\chi} \rangle = 0$$

$$\langle \tilde{\chi}_{Ks} | \frac{\partial}{\partial a} | \tilde{\chi}_{Ks} \rangle = \int \mathrm{d}\phi \left[\tilde{\chi}_K^* \left(\frac{N_K'}{N_K} + \mathrm{i} S_1' + \mathrm{i} \frac{\eta_2'}{m_P^2} \right) \tilde{\chi}_K \right] = 0.$$

So we obtain

$$\mathrm{i} \frac{\partial}{\partial \eta} \langle \tilde{\chi}_{Ks} | \tilde{\chi}_{Ks} \rangle = - \frac{\mathrm{i} S_0'}{m_P^2} \int \mathrm{d}\phi \left[\tilde{\chi}_K \frac{\partial}{\partial a} \tilde{\chi}_K + C.C. \right] = 0.$$

Gaussian Hypothesis

By making a Gaussian ansatz

$$\psi_\kappa^{(0)}(t, f_\kappa) = \mathcal{N}_\kappa^{(0)} \mathrm{e}^{-\frac{1}{2}\Omega_\kappa^{(0)} f_\kappa^2}$$

we obtain a coupled system of non-linear differential equations

$$\dot{\mathcal{N}}_\kappa^{(0)}(t) = -\mathrm{i}\frac{\mathrm{e}^{-3\alpha}}{2}\mathcal{N}_\kappa^{(0)}(t)\Omega_\kappa^{(0)}(t)$$

$$\dot{\Omega}_\kappa^{(0)}(t) = \mathrm{i}\mathrm{e}^{-3\alpha}\left[-(\Omega_\kappa^{(0)}(t))^2 + W_\kappa(t)\right]$$

$\Omega_\kappa^{(0)}$ Solution

On defining

$$\xi = \frac{\kappa}{Ha(t)} \quad \mu = \frac{m}{H} \quad \nu = \frac{1}{2}\sqrt{9 - 4\mu^2} \quad h = \frac{H^2}{\kappa^3}$$

we get the solution

$$\Omega_\kappa^{(0)}(\xi) = \frac{1}{h\xi^2} \frac{1}{(C_1 Y_\nu(\xi) + J_\nu(\xi))}$$

$$\times \left[-\mathrm{i} C_1 Y_{\nu+1}(\xi) + \frac{\mathrm{i}}{2\xi} \left(C_1 Y_\nu(\xi)(3 + 2\nu) - 2\xi J_{\nu+1}(\xi) + J_\nu(\xi)(3 + 2\nu) \right) \right]$$

Massless Case

We find that the solution of the equation considering $\mu \ll 1$ coincides, in the limit $\mu \to 0$, i.e. $\nu \to \frac{3}{2}$, with the solution of the complete equation

$$\Omega_\kappa^{(0)}(\xi) = \frac{\mathrm{i}}{h\xi} \frac{C_1 \cos\xi - \sin\xi}{[(C_1 + \xi)\cos\xi + (C_1\xi - 1)\sin\xi]}$$

$$= \frac{\mathrm{i}}{h\xi} \frac{\left(C_1 J_{-\frac{1}{2}} - J_{\frac{1}{2}}\right)}{\left[(C_1 + \xi)J_{-\frac{1}{2}} + (C_1\xi - 1)J_{\frac{1}{2}}\right]}$$

Massless Case

This solution can be re-expressed substuting $C_1 = \zeta \mathrm{e}^{\mathrm{i}\beta}$ so that

$$\Omega_k^{(0)}(\xi) = \frac{k^3}{H^2}\frac{\mathrm{i}}{\xi}\frac{AB^*}{|B|^2}$$

where

$$A = \rho + \mathrm{i}\sigma \quad B = \gamma + \mathrm{i}\delta$$

$$\rho = 2(\zeta \cos\beta \cos\xi - \sin\xi) \qquad \sigma = 2\zeta \sin\beta \cos\xi$$

$$\gamma = 2\zeta \left[\cos\beta(\cos\xi + \xi \sin\xi) - (\sin\xi - \xi\cos\xi)\right]$$

$$\delta = 2\zeta \sin\beta[\cos\xi + \sin\xi]$$

Power Spectrum

We define the **power spectrum**

$$\mathcal{P}^{(0)}(k) := \frac{k^3}{2\pi^2}\,|\delta_k(t_{\rm enter})|^2$$

where

$$\delta_k(t_{\rm enter}) = \frac{4}{3}\left.\frac{\dot{\sigma}_k(t)}{\dot{\phi}(t)}\right|_{t=t_{\rm exit}}$$

and we have set

$$\sigma_\kappa^2(t) \equiv \langle\psi_\kappa|f_\kappa^2|\psi_\kappa\rangle = \frac{1}{2\Re\mathrm{e}\Omega_\kappa(t)}$$

Power Spectrum

In light of the definition of σ_κ one has

$$|\dot{\sigma}_\kappa(t)| = \left| \frac{H\xi}{\sqrt{2}} \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\left(\Re\mathrm{e}\Omega_\kappa(\xi) \right)^{-\frac{1}{2}} \right] \right|$$

At m_P^0 order we have for the general solution $\Omega_\kappa^{(0)}$

$$\left| \dot{\sigma}_k^{(0)}(t) \right|_{t_{\mathrm{exit}}} = \frac{2\sqrt{2}\pi^2 H^2}{k^{\frac{3}{2}}} \left| \frac{\sqrt{\zeta}(\zeta + 2\pi\cos\beta)}{\sqrt{\sin\beta}\sqrt{\zeta^2 + 4\pi\cos\beta + 4\pi^2}} \right|$$

Power Spectrum

At m_P^0 order if we consider the **(Bunch–Davies Vacuum)** boundary condition

$$\Omega_\kappa^{(0)}(\infty) = \frac{1}{h\xi^2}$$

we obtain at the $\xi(t_{exit}) = 2\pi$ time

$$\left|\dot{\sigma}_\kappa^{(0)}\right| = \frac{H^2}{\kappa^{\frac{3}{2}}} \frac{2\sqrt{2}\pi^2}{\sqrt{4\pi^2+1}}$$

Power Spectrum: Quantum Corrections

At m_P^2 order, making the same Gaussian ansatz, we can write the wave functional in the form

$$\psi_\kappa^{(1)}(t, f_\kappa) = \left(\mathcal{N}_\kappa^{(0)}(t) + \frac{1}{m_P^2}\mathcal{N}_\kappa^{(1)}(t)\right) \exp\left[-\frac{1}{2}\left(\Omega_\kappa^{(0)}(t) + \frac{1}{m_P^2}\Omega_\kappa^{(1)}(t)\right) f_\kappa^2\right]$$

and inserting it into the m_P^2 order equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\log\left(N_k^{(0)} + \frac{N_1^{(1)}}{m_P^2}\right) - \frac{\mathrm{i}}{2}\left(\dot{\Omega}_k^{(0)} + \frac{\dot{\Omega}_k^{(1)}}{m_P^2}\right) f_k^2 =$$

$$\frac{1}{2}\mathrm{e}^{-3\alpha}\left\{\Omega_k^{(0)} + \frac{1}{m_P^2}\left[\Omega_k^{(1)} - \frac{3}{4V}\left(\left(\Omega_k^{(0)}\right)^2 - \frac{2}{3}W_k\right)\right] + \right.$$

$$\left.\left[W_k - \left(\Omega_k^{(0)} + \frac{\Omega_k^{(1)}}{m_P^2}\right)^2 - \frac{3\Omega_k^{(0)}(W_k - (\Omega_k^{(0)})^2)}{2Vm_P^2}\right] f_k^2 + O(f_k^4)\right\}$$

Power Spectrum: Quantum Corrections

The equation for $\Omega_k^{(1)}$ is

$$\dot{\Omega}_\kappa^{(1)}(t) = -2\mathrm{i}e^{-3\alpha}\Omega_\kappa^{(0)}(t)\left[\Omega_\kappa^{(1)}(t) - \frac{3}{4V(t)}\left((\Omega_\kappa^{(0)}(t))^2 - W_\kappa\right)\right]$$

if we substitute the massless expression for $\Omega_k^{(0)}$ with the Bunch-Davies boundary condition

$$\frac{d\Omega_k^{(1)}}{d\xi} = \frac{2\mathrm{i}\xi}{(\xi - \mathrm{i})}\Omega_k^{(1)} + \frac{3}{2}\xi^3\frac{(2\xi - \mathrm{i})}{(\xi - \mathrm{i})^3}$$

Power Spectrum: Quantum Corrections

One can factorize the correction contributions

$$\left|\dot{\sigma}_\kappa^{(1)}(t)\right| = \left|\sigma_\kappa^{(0)}\right| |C_\kappa|$$

$$C_k(\xi) \equiv \left(1 + \frac{\xi^2+1}{\kappa^3}\frac{H^2}{m_P^2}\Re\mathrm{e}\Omega_\kappa^{(1)}(\xi)\right)^{-\frac{3}{2}} \left(1 - \frac{(\xi^2+1)^2}{2\xi\kappa^3}\Re\mathrm{e}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\Omega_\kappa^{(1)}(\xi)\right]\frac{H^2}{m_P^2}\right)$$

In order to evalute this quantity we have to find the function $\Omega_\kappa^{(1)}$, that is, for the **massless form** of $\Omega_\kappa^{(0)}$, and considering the boundary condition $\Omega_\kappa^{(1)}(0) = 0$

$$\Omega_\kappa^{(1)}(\xi) = \frac{-3\mathrm{e}^{2\mathrm{i}\xi}}{8}\frac{1+\mathrm{Ei}(1,2)\mathrm{e}^2}{(1+\mathrm{i}\xi)^2} + \frac{3}{8}\frac{1+6\mathrm{i}\xi+4\mathrm{Ei}(1,2\mathrm{i}\xi+2)\mathrm{e}^{2\mathrm{i}\xi+2}-4\xi^2-4\mathrm{i}\xi^3}{(1+\mathrm{i}\xi)^2}$$

$$\mathrm{Ei}(a,z) \equiv \int_1^\infty \frac{\mathrm{e}^{-tz}}{t^a}\mathrm{d}t \quad a \in \mathbb{R} \text{ and } \Re\mathrm{e}(z) > 0$$

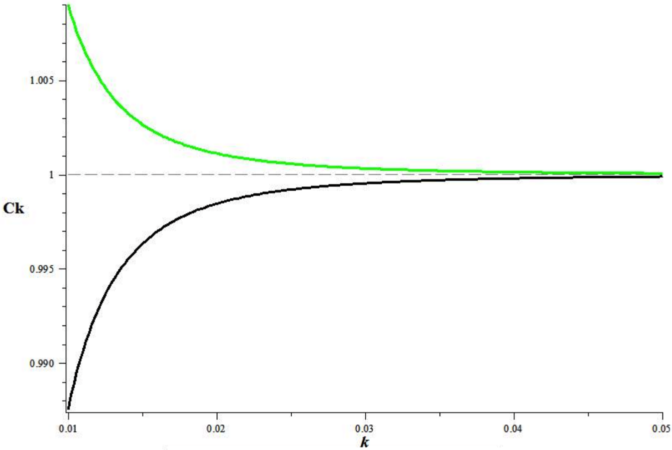
At $t_{exit} \to \xi = 2\pi$ time

$$C_k \equiv \left(1 - \frac{54.37}{\kappa^3}\frac{H^2}{m_P^2}\right)^{-\frac{3}{2}} \left(1 + \frac{7.98}{\kappa^3}\frac{H^2}{m_P^2}\right)$$

and for what concerns the power spectrum

$$\mathcal{P}^{(1)}(k) = \mathcal{P}^{(0)}(k)\, C_k^2 \sim \mathcal{P}^{(0)}(k)\left[1 + \frac{89.54}{k^3}\frac{H^2}{m_P^2} + \frac{1}{k^6}\mathrm{O}\left(\frac{H^4}{m_P^4}\right)\right]^2$$

The C_κ Behavior at Various Scales



Equation in the z Variable

Remarkably, by passing to the new variable

$$z = 1 + \mathrm{i}\xi$$

the m_P^2 order equation can be written

$$\frac{\mathrm{d}\Omega_k^{(1)}}{\mathrm{d}z} = 2\left(1 - \frac{1}{z}\right)\Omega_k^{(1)} + \frac{3}{2}\left(7 - 2z - \frac{9}{z} + \frac{5}{z^2} - \frac{1}{z^3}\right)$$

that leads to the solution

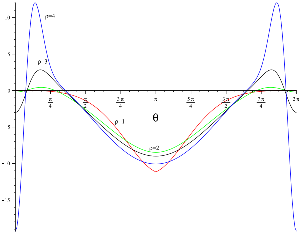
$$\Omega_k^{(1)}(z) = P_1 \frac{\mathrm{e}^{2z}}{z^2} + \frac{3}{8z^2}\left[4z^3 - 8z^2 + 10z - 5 + 4\mathrm{e}^{2z}\,\mathrm{Ei}(1, 2z)\right]$$

Graphical studies

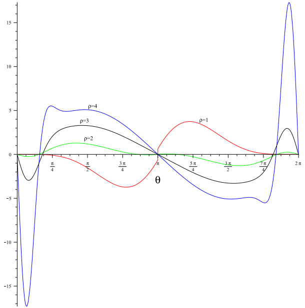
Such a solution can be studied graphically by introducing the complex polar representation for $z = \rho e^{i\theta}$ and defining the functions

$$f_\rho(\theta) = \mathrm{Re}\left[\Omega_k^{(1)}(\rho e^{i\theta})\right] \quad g_\rho(\theta) = \mathrm{Im}\left[\Omega_k^{(1)}(\rho e^{i\theta})\right]$$

$f_\rho(\theta)$



$g_\rho(\theta)$



Observability of the Corrections

We note that the uncorrected power spetrum is proportional to

$$\mathcal{P}^{(0)}(k) \propto \frac{H^4}{\left|\dot{\phi}(t)\right|^2_{t_{\mathrm{exit}}}}$$

this corresponds, apart from a dimensionless constant, to the standard power spectrum of scalar cosmological perturbations

$$\mathcal{P}_s^{(0)}(k) = \frac{G}{\epsilon\pi} H^2$$

where we have introduced the first slow-roll parameter

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{4\pi G \left|\dot{\phi}\right|^2_{t_{\mathrm{exit}}}}{H^2}$$

Quantum Corrections

The quantum correction takes the approximate form

$$C_k^2 = 1 + \delta^{\pm}_{\mathrm{WDW}}(k) + \frac{1}{k^6}\,\mathcal{O}\left(\left(\frac{H}{m_{\mathrm{P}}}\right)^4\right)$$

where $\delta^{\pm}_{\mathrm{WDW}}(k)$ could take the values

$$\delta^{+}_{\mathrm{WDW}}(k) = \frac{179.09}{k^3}\left(\frac{H}{m_{\mathrm{P}}}\right)^2 \qquad \delta^{-}_{\mathrm{WDW}}(k) = -\frac{247.68}{k^3}\left(\frac{H}{m_{\mathrm{P}}}\right)^2$$

Spectral Index

The basic equations in the theory of the spectral index n_s and its running α_s are

$$n_s - 1 := \frac{\mathrm{d}\log \mathcal{P}_s}{\mathrm{d}\log k} \approx \frac{1}{H}\frac{\mathrm{d}\log \mathcal{P}_s}{\mathrm{d}t} \approx 2\eta - 4\epsilon - 3\delta^{\pm}_{\mathrm{WDW}}$$

$$\alpha_s := \frac{\mathrm{d}n_s}{\mathrm{d}\log k} \approx 2(5\epsilon\eta - 4\epsilon^2 - \Xi^2) + 9\delta^{\pm}_{\mathrm{WDW}}$$

where we have defined the slow-roll parameters

$$\eta := -\frac{\ddot{\phi}}{H\dot{\phi}} \quad \Xi^2 := \frac{1}{H^2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\ddot{\phi}}{\dot{\phi}}.$$

Observability bounds

Reinserting a reference wave number wich can either correspond

to $k_{min} \approx 1.4 \times 10^{-4}\ \mathrm{Mpc}^{-1}$ largest observable scale or to $k_0 = 0.002\ \mathrm{Mpc}^{-1}$ pivot scale

we find the corrections for $k \to \frac{k}{k_0}$

$$\left|\delta^{+}_{\mathrm{WDW}}(k_0)\right| \lesssim 2.9 \times 10^{-9}, \quad \left|\delta^{-}_{\mathrm{WDW}}(k_0)\right| \lesssim 4.0 \times 10^{-9}$$

and for $k \to \frac{k}{k_{min}}$

$$\left|\delta^{+}_{\mathrm{WDW}}(k_0)\right| \lesssim 9.8 \times 10^{-13}, \quad \left|\delta^{-}_{\mathrm{WDW}}(k_0)\right| \lesssim 1.4 \times 10^{-12}$$

the resulting upper bounds for H are

$$H \lesssim 1.67 \times 10^{-2}\, m_{\mathrm{P}} \approx 4.43 \times 10^{17}\ \mathrm{GeV}$$

$$H \lesssim 1.42 \times 10^{-2}\, m_{\mathrm{P}} \approx 3.76 \times 10^{17}\ \mathrm{GeV}$$

Conclusions

- ⋆ Exact form of the functions $\Omega_\kappa^{(0)}$, $\Omega_\kappa^{(1)}$ and $\dot{\sigma}_\kappa^{(0)}$. Enhancement/Suppression of quantum gravitational corrections, hard to discriminate on observational ground.
- ⋆ Unobservable corrections to CMB anisotropy spectrum; nevertheless, their size is bigger than QG corrections in laboratory situations.
- ⋆ Other choices of vacuum besides Bunch–Davies allowed by the general integral of our non linear equation?

Conclusions

- ⋆ Calculation of an upper limit for H in an inflationary model.
- ⋆ Gauge-invariant Mukhanov variables instead of a scalar field.
- ⋆ More complicated quantum state (instead of ground state) to see how the results depend on this choice.
- ⋆ We have found a way of dealing with unitarity violating terms.

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