FRUSTRATION EFFECTS AND PHASE DIAGRAM OF TOPOLOGICAL SUPERCONDUCTOR - LUTTINGER LIQUID JUNCTIONS

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Motivation

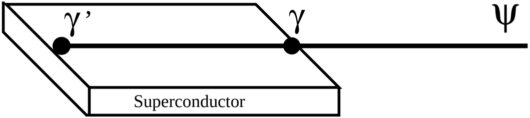
- Frustration effects arise when a system "cannot decide" which phase to flow into. They often lead to the emergence of novel phases, with unconventional properties, such as non-Fermi liquid phases [two-channel Kondo effect], spin liquids, etc.;
- Using the properties of Majorana fermions coupled to a Luttinger liquid to design a junction where the system "switches" between conventional phases by passing through a (partially stable) new, unconventional phase.

Plan of the presentation

- Junction between a single Luttinger liquid and a topological superconductor;
- Junction between two Luttinger liquid and a topological superconductors: description;
- Phase diagram, "trivial" fixed points, nontrivial fixed point;
- Detecting the NTFP in a transport experiment;
- The g-theorem and the junction phase diagram;
- Conclusions and further developments.

[1] Single Luttinger liquid - topological superconductor

- Effective low-energy description of the topological superconductor: two localized Majorana fermions $\gamma^{'}, \gamma$ at the two endpoints [For instance, a spinless quantum wire with p-wave pairing and open boundary conditions [A Yu Kitaev Phys.-Usp. 44, 131 (2001)]. This can be realized, for instance, in a semiconducting quantum wire with strong spin-orbit interaction embedded into a superconducting quantum interference device [R. M. Lutchyn, J. D. Sau, and S. Das Sarma, PRL 105, 077001 (2010)], or simply in proximity to a "standard" *s*-wave superconductor [Y. Oreg, G. Refael, and F. von Oppen, PRL 105 177002 (2010)].



[1] Single Luttinger liquid - topological superconductor

- Sub-gap scattering processes $\Leftarrow$ effective boundary interaction Hamiltonian

$$\begin{aligned} H_B &= V_B \psi^\dagger(0)\psi(0) + \Delta_B\{\psi(0)\partial_x\psi(0) + \mathrm{h.c.}\} \\ &+ t\gamma[\psi(0) - \psi^\dagger(0)] \end{aligned}$$

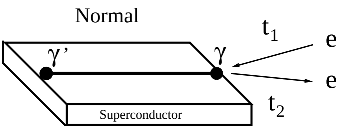
- Scaling dimensions [noninteracting case]
  $d_{V_B} = 1$ [marginal]; $d_{\Delta_B} = 2$ [irrelevant]; $d_t = 1/2$ [relevant].
  All the low-energy physics is determined by the term $\propto t$

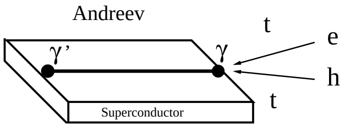
[1] Single Luttinger liquid - topological superconductor

- Relevance of $t \Rightarrow$ resonant (sub-gap) Andreev backscattering at the Fermi level (in the p/h-basis)

$$S(E=0; t \neq 0) = \begin{bmatrix} 0 & e^{i\beta} \\ e^{-i\beta} & 0 \end{bmatrix} \; ; S(E=0; t=0) = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{bmatrix}$$

- Physical interpretation: resonant backscattering in the $p-h$-channel [K. T. Law, P. A. Lee, and T. K. Ng, PRL 103, 237001 (2009)]





[At the Fermi level the resonance in the $e \to h$ Andreev scattering processes with $|A| = 1$ is assured by the "natural" condition $t_1 = t_2 = t$]

[1] Single Luttinger liquid - topological superconductor

- Adding interaction in the normal wire $\Rightarrow$ standard bosonization recipe: "bulk" Hamiltonian, boundary fermionic field

$$H_0 = \frac{u}{2} \int_0^\infty dx \left[K(\partial_x\phi)^2 + K^{-1}(\partial_x\theta)^2\right], \; \psi(0) \sim i\Gamma e^{i\sqrt{\pi}\phi(0)}$$

$K < 1$ **for a repulsive interaction**

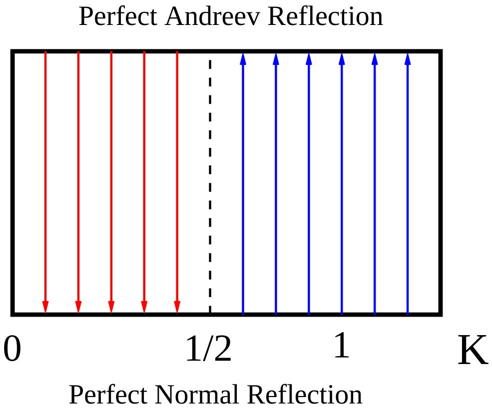
- Boundary interaction Hamiltonian

$$H_B \sim \tilde{V}_N \partial_x\phi(0) + \tilde{\Delta}_B \cos[2\sqrt{\pi}\phi(0)] + 2i\tilde{t}\gamma\Gamma\cos[\sqrt{\pi}\phi(0)]$$

- Scaling dimensions [interacting case]: $d_{\tilde{V}_B} = 1$ [marginal]; $d_{\tilde{\Delta}_B} = 2/K$ [irrelevant]; $d_{\tilde{t}} = 1/(2K)$ [relevant, for $K > 1/2$].

[1] Single Luttinger liquid - topological superconductor

- Phase diagram for a generic Luttinger parameter K [L. Fidkowski, J. Alicea, N. Lindner, R. M. Lutchyn, and M. P. A. Fisher, PRB 85, 245121 (2012)]



[1] Single Luttinger liquid - topological superconductor

- Fixed point dc conductance

$$
\begin{aligned}
G(V) &= 0 \;, (N) \\
G(V) &= \frac{2e^2}{h} \;, (A)
\end{aligned}
$$

- Nonlinear corrections to the dc conductance

$$
\begin{aligned}
G(V) &= \left|\frac{V}{V_*}\right|^{-2+\frac{1}{K}} \;, (V \gg V_*) \\
G(V) &= \frac{2e^2}{h} - \left|\frac{V}{V_*}\right|^{-2+4K} \;, (V \ll V_*)
\end{aligned}
$$

[1] Single Luttinger liquid - topological superconductor

- Analysis of the A-fixed point

$$H_t = 2t\left(d^\dagger d - \frac{1}{2}\right)\cos[\sqrt{\pi}\phi(0)] \ , \ (d = \frac{1}{2}(\gamma + i\Gamma))$$

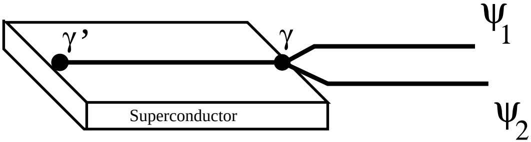
- $\cos[\sqrt{\pi}\phi(0)]$ pinned at ± 1, residual boundary interaction given by $\tilde{V}_N\psi^\dagger(0)\psi(0) \Rightarrow$ leading boundary perturbation

$$\tilde{H}_B \sim 2\bar{t}\cos[2\sqrt{\pi}\theta(0)]$$

The scaling dimension is $d_{\tilde{H}_B} = 2K$, which motivates the exponent in the subleading power-law dependence of $G(V)$ on V [C. L. Kane, M. P. A. Fisher, PRB 46, 15233 (1992)].

[2] Two-Luttinger liquid - topological superconductor

- Contact two one-dimensional normal channels with the superconductor: this may either be realized as a "T"-junction with two actual wires, or by interfacing two channels with the topological superconductor (?) [I. Affleck and D. Giuliano, J. Stat. Mech. (2013) P06011]



- Coupling to the Majorana fermion

$$H_t = \gamma \sum_{j=1}^{2} t_j [\psi_j(0) - \psi_j^\dagger(0)]$$

[2] Two-Luttinger liquid - topological superconductor

- Noninteracting case $\Rightarrow$ one-channel case

$$
\begin{aligned}
\psi_1' &= \frac{t_1}{\sqrt{t_1^2 + t_2^2}}\psi_1 + \frac{t_2}{\sqrt{t_1^2 + t_2^2}}\psi_2 \\
\psi_2' &= -\frac{t_2}{\sqrt{t_1^2 + t_2^2}}\psi_1 + \frac{t_2}{\sqrt{t_1^2 + t_2^2}}\psi_2
\end{aligned} \tag{1}
$$

- Interacting case with "rotational symmetry" between the channles $\Rightarrow$ one-channel case

[2] Two-Luttinger liquid - topological superconductor

- T-junction (no inter-channel interaction)

$$H_0 = \sum_{j=1}^{2} \frac{u_j}{2} \int_0^\infty dx \, [K_j(\partial_x \phi_j)^2 + K_j^{-1}(\partial_X \theta_j)^2]$$

$$H_M = 2i\gamma \sum_{j=1}^{2} t_j \Gamma_j \cos[\sqrt{\pi}\phi_j(0)]$$

- Phase diagram $\Leftarrow$ ϵ-expansion and renormalization group equations [Set $d_j = \frac{1}{2K_j}$ with $d_j = 1 - \epsilon_j$, $(0 < \epsilon_j \ll 1)$, then write RG equations to leading nontrivial order in the ϵ_j's]

[2] Two-Luttinger liquid - topological superconductor

- To construct the β-functions for the t_j's, we use the O.P.E. approach in "deformed" conformal field theories: we expand the partition functions in powers of the t_j's and require that a small change in the short-[imaginary time] cutoff τ_0 is reabsorbed by redefining the t_j's. In general, it is enough knowing two-point O.P.E.'s [J.L. Cardy, Scaling and Renormalization in Statistic Physics, Cambridge University Press, 1996.]. Here, the first nonlinear contribution arises from three-point O.P.E.'s

$$\mathbf{T}_\tau[\prod_{u=1}^{3} O_j(\tau_u)] \sim \frac{O_j(\tau_1)}{4}\left[\left|\frac{\tau_{12}}{\tau_{13}\tau_{23}}\right|^{\frac{1}{K_j}} + \left|\frac{\tau_{13}}{\tau_{13}\tau_{23}}\right|^{\frac{1}{K_j}} + \left|\frac{\tau_{23}}{\tau_{13}\tau_{13}}\right|^{\frac{1}{K_j}}\right]$$

$$\mathbf{T}_\tau[(\prod_{u=1}^{2} O_1(\tau_u))O_2(\tau_3)] \sim \frac{1}{2}\left|\frac{1}{\tau_{12}}\right|^{\frac{1}{K_1}} O_2(\tau_3)$$

[2] Two-Luttinger liquid - topological superconductor

- Renormalization group equations

$$\frac{d\bar{t}_1}{d\ln(D/D_0)} = \epsilon_1\bar{t}_1 - 4\bar{t}_1\bar{t}_2^2 \ , \ \frac{d\bar{t}_2}{d\ln(D/D_0)} = \epsilon_2\bar{t}_2 - 4\bar{t}_2\bar{t}_1^2$$

- Phase diagram stable $[A]\otimes[N], [N]\otimes[A]$ FP's, unstable $[N]\otimes[N], [A]\otimes[A]$ FP's, partially stable NTCP

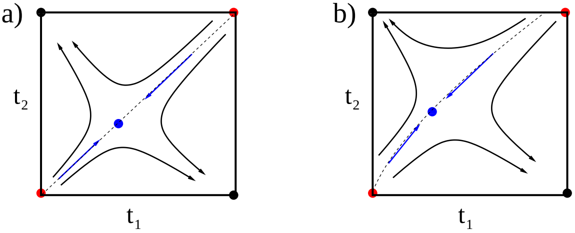


Figure: a) $\epsilon_1 = \epsilon_2$, b) $\epsilon_1 \neq \epsilon_2$.

[3] Phase diagram

- Set $\epsilon_1 = \epsilon_2 = \epsilon$, $d_j = 1 - \epsilon \Rightarrow$ unstable $N \otimes N$-fixed point
  Assume $t_1(D = D_0) > t_2(D = D_0)$ and take $A \otimes N$ as "putative" fixed point
- At strong t_1, $\cos[\sqrt{\pi}\phi_1(0)]$ is "pinned" at $\alpha = \pm 1$ and γ "entangles" with Γ_1, so that the ground state $|G\rangle_\alpha$ is determined by $(\gamma + i\alpha\Gamma_1)|G\rangle_\alpha = 0$
- As $\phi_1(0)$ is pinned, the bosonized operators at $A \otimes N$ are built according to the bosonization rules $\psi_1(0) \propto e^{\pm i\sqrt{\pi}\theta_1(0)}$, $\psi_2(0) \propto e^{i\sqrt{\pi}\phi_2(0)}$
- Construct the various operators allowed by symmetry at the $A \otimes N$-fixed point

[3] Phase diagram

- Intra-wire normal backscattering
  $H_{IN} = V_I \psi_1^\dagger(0)\psi_1(0) \sim \tilde{V}_I \cos[2\sqrt{2}\theta_1(0)]$, $[d_{IN} = 2K_1 > 1]$,
  irrelevant
- Inter-wire normal backscattering
  $H_{IW} = V_W \psi_1^\dagger(0)\psi_2(0) + \mathrm{h.c.} \sim \tilde{V}_W \cos[\sqrt{\pi}\theta_1(0)] e^{i\sqrt{\pi}\phi_2(0)} + \mathrm{h.c.}$,
  $[d_{IN} = (K_1 + 1/K_2)/2 > 1]$, **irrelevant**
- Inter-wire pairing
  $H_{IP} = \Delta_B \psi_1(0)\psi_2(0) + \mathrm{h.c.} \sim \tilde{\Delta}_B \cos[\sqrt{\pi}\theta_1(0)] e^{i\sqrt{\pi}\phi_2(0)} + \mathrm{h.c.}$,
  $[d_{IN} = (K_1 + 1/K_2)/2 > 1]$, **irrelevant**

[3] Phase diagram

- Coupling to channel-2 is $H_2 \propto \gamma \Gamma_2 \cos[\sqrt{\pi}\phi_2(0)]$. This looks like $d_2 = 1/(2K_2) < 1 \Rightarrow$ relevant ??
- Correct approach to the problem $\Rightarrow$ Schrieffer-Wolff transformation using $\gamma \Gamma_2 |G\rangle_{\pm 1} \propto |G\rangle_{\mp 1}$
- Second-order SW transformation $\Rightarrow$ $\tilde{H}_2 \propto \frac{t_2^2}{t_1} \cos[2\sqrt{\pi}\phi_2(0)]$, $[d_2 = 2/K_2 > 1]$, **irrelevant**
- Conclusion: $A \otimes N$ and $N \otimes A$ are both stable. The transition between the two of them cannot pass through $N \otimes N$ (which is unstable in any direction). Thus, it may either pass through a $A \otimes A$ fixed point, or through a novel (partially stable) NTCP

[3] Phase diagram

- Ruling out the $A \otimes A$ option $\Rightarrow$ $A \otimes A$ fully unstable (analysis of the operators allowed by symmetry)
- Assuming $t_1(D_0) = t_2(D_0)$ might lead to a "putative" fixed point, with $\cos[\sqrt{\pi}\phi_j(0)]$ pinned at α_j and with groundstate $|G\rangle_{A\otimes A}$ such that $\left[\gamma + \frac{i}{\sqrt{2}}(\alpha_1\Gamma_1 + \alpha_2\Gamma_2)\right] |G\rangle_{A\otimes A} = 0$
- Intra-channel backscattering
  $H_{NI} \propto \sum_{j=1}^{2} V_{NI,j}\psi_j^\dagger(0)\psi_j(0) \propto \sum_{j=1}^{2} \tilde{V}_{NI,j} \cos[2\sqrt{\pi}\theta_j(0)]$,
  $[d_{NI} = 2K > 1]$, **irrelevant**
- Inter-channel backscattering
  $H_{WI} \propto V_W \psi_1^\dagger(0)\psi_2(0) + \mathrm{h.c.} \propto \tilde{V}_W \cos[\sqrt{\pi}(\theta_1(0) - \theta_2(0))]$,
  $[d_w = K]$, **relevant** if $K < 1$
- Inter-channel pairing
  $H_{PI} \propto \Delta_B \psi_1(0)\psi_2(0) + \mathrm{h.c.} \propto \tilde{\Delta}_B \cos[\sqrt{\pi}(\theta_1(0) + \theta_2(0))]$,
  $[d_B = K]$, **relevant** if $K < 1$

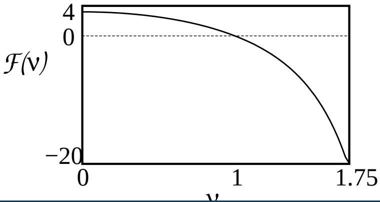
[3] Phase diagram

- Our results may be generalized to the case of nonzero inter-channel **bulk** interaction $[\phi(\theta)_{\rho(\sigma)} = (\phi(\theta)_1 \pm \phi(\theta)_2)/\sqrt{2}]$

$$H_0 = \sum_{\lambda=\rho,\sigma} \frac{u_\lambda}{2} \int_0^\infty dx \, [K_\lambda(\partial_x \phi_\lambda)^2 + K_\lambda^{-1}(\partial_x \theta_\lambda)^2]$$

- Renormalization Group equations $[\nu = (2K_\rho)^{-1} - (2K_\sigma)^{-1}]$

$$\frac{d\bar{t}_1}{d \ln(D/D_0)} = \epsilon_1 \bar{t}_1 - \mathcal{F}(\nu) \bar{t}_1 \bar{t}_2^2 \; , \; \frac{d\bar{t}_2}{d \ln(D/D_0)} = \epsilon_2 \bar{t}_2 - \mathcal{F}(\nu) \bar{t}_2 \bar{t}_1^2$$



[3] Phase diagram

- **Conclusion**: the $A \otimes A$-fixed point comes out to be unstable, just as the $N \otimes N$-fixed point. A "minimal" hypothesis leads to the necessary existence of a NTCP at intermediate couplings $t_{1,*} = t_{2,*} \propto \sqrt{\epsilon}$.

[4] Detecting the NTFP: transport measurement

- Apply a voltage bias V_j to channel-j and measure the (linear) conductance tensor $G_{i,j}$, defined by $I_j = \sum_{i=1}^{2} G_{j,i} V_i$
- For small ("bare") t_j's, $G_{j,i}$ can be computed perturbatively in the t_j's. If we neglect the inter-wire interactions, then $G_{j,i}$ is purely diagonal, and

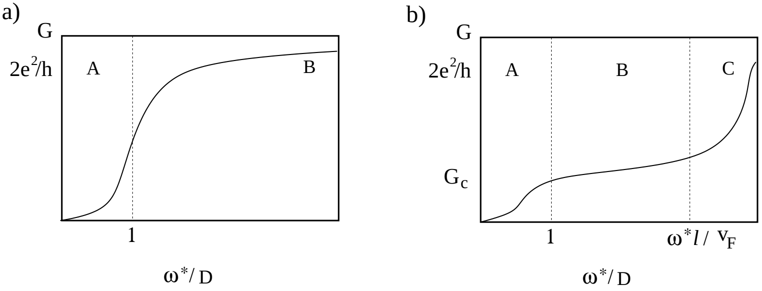
$$G_{j,j} = \frac{e^2}{h}(2\pi \bar{t}_j)^2 \ , \ (\bar{t}_j = t_j (D/D_0)^{-1+\frac{1}{2K_j}})$$

- As the scale $D/D_0 \to 0$, we obtain

$$G_{j,j} \to \frac{e^2}{h}(2\pi \bar{t}_{j,*})^2 = \frac{4\pi^2 e^2}{h} \frac{\epsilon_j}{\mathcal{F}(\nu)}$$

[4] Detecting the NTFP: transport measurement

- **a)** Expected behavior of $G_{j,j}$ if the system flows towards the $A \otimes N$ fixed point
- **b)** Expected behavior of $G_{j,j}$ if the system flows towards the NTFP



[5] g-theorem and phase diagram

- "In a given boundary interaction problem, it is possible to define a function g which always decreases under renormalization between two different boundary fixed points (associated with the same boundary critical point)" [I. Affleck, A. W. W. Ludwig, PRL 67, 161 (1991); PRB 48, 7297 (1992)]
- g is the "boundary analog" of Zamolodchikov's c-function in conformal field theories, which is proportional to the coefficient in the linear specific heat, and decreases under renormalization between two different bulk critical points.
- g is identified with the "groundstate degeneracy" associated with the impurity. More precisely, g is related to the scale-independent contribution to the entanglement entropy associated with the boundary interaction [P. Calabrese, J. Cardy, J.Stat.Mech.0406: P06002 (2004).

[5] g-theorem and phase diagram

- The recipe for computing the g-function at the "conformal" boundary fixed points
- System on a segment of length ℓ with conformally invariant boundary conditions A and B at its endpoints: partition function

$$\mathcal{Z}_{AB}[u/(\ell T)] = \sum_n \exp[-x^n_{AB} u/(\ell T)]$$

- Take the limit $u/(\ell T) \to 0$

$$\mathcal{Z}_{AB}[u/(\ell T)] \to_{u/(\ell T)\to 0} g_A g_B \; e^{\pi \ell T c/6u}$$

[5] g-theorem and phase diagram

- Result for $g_{N\otimes N}$

$$g_{N\otimes N} = 2(K_1 K_2)^{\frac{1}{4}}$$

- Result for $g_{A\otimes N}$

$$g_{A\otimes N} = \sqrt{2}(K_2/K_1)^{\frac{1}{4}}$$

- Result for $g_{A\otimes A}$

$$g_{A\otimes A} = 4/(K_2 K_1)^{\frac{1}{4}}$$

- Result for g_{NTFP} [ϵ-expansion + dimensional regularization]

$$g_{NTFP} = g_{N\otimes N}\left[1 - \frac{2\pi^2(\epsilon_1^2 + \epsilon_2^2)}{\mathcal{F}(\nu)}\right]$$

[6] Conclusions and further developments

- Having more than one normal channel connected to a topological superconductor opens the way to novel physics
- Frustration mechanism: similar to two-channel Kondo effect, but the NTCP has a different nature, which still need to be explored
- Extension to multi-(normal) channel junctions
- Experimental realization of our system
- Fingerprints of the frustration physics in dc Josephson current measurements
- Nonlinear conductance: "Kane-Fisher"-like physics