Non-perturbative Gauge/Gravity Correspondence

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String theory is a very powerful tool to analyze perturbative field theories, and in particular gauge theories.

Behind this, there is a rather simple and well-known fact: in the field theory limit $\alpha' \to 0$ string theory S-matrix elements reproduce vertices and effective actions in field theory.
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- **Closed Strings amplitudes $\Rightarrow$ Gravitational amplitudes**
Gauge Theory from Strings

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Behind this, there is a rather simple and well-known fact: in the field theory limit $\alpha' \rightarrow 0$ string theory S-matrix elements reproduce vertices and effective actions in field theory.

- **Closed Strings amplitudes** $\Rightarrow$ Gravitational amplitudes
- **Open Strings amplitudes** $\Rightarrow$ Gauge amplitudes
D Branes

The advances of the last 15 years in string theory have opened new perspectives, and new insights on gauge theories can be obtained using D Branes:

- The end points of open strings with Neumann boundary conditions in \((p + 1)\) space-time directions and Dirichlet boundary conditions in the other \((9 - p)\),

\[
\partial_\sigma X^\mu|_{\sigma=0,\pi} = 0 , \quad \mu = 0, 1, \ldots, p \\
X^i|_{\sigma=0,\pi} = Y^i , \quad i = p + 1, \ldots, 9
\]

define a \((p + 1)\)-dimensional hyperplane called \(Dp\)-brane.
The presence of a D brane breaks Lorentz and Poincaré invariance:

- \( \text{SO}(1, 9) \Rightarrow \text{SO}(1, p) \times \text{SO}(9 - p) \)
- translational invariance is broken in the \((9 - p)\) directions transverse to the brane

Open string excitations have no momentum in the transverse directions \(\Rightarrow\) massless open strings describe a Field Theory in \((p + 1)\) dimensions.

Also supersymmetry is generically broken, so we have a Supersymmetric Field Theory with at most 16 charges (instead of the original 32 of the bulk theory)!
$N$ D3 branes in Flat Space

Open string spectrum:

\[ A_{\mu} \, , \, \phi^i \mu = 0, \ldots, 3 \; ; \; i = 1, \ldots, 6 \]
\[ \lambda_\alpha^A \, , \, \bar{\lambda}^{\dot{\alpha}}_A \alpha, \dot{\alpha} = 1, 2 \; ; \; A = 1, 2, 3, 4 \]

all fields transform in the adjoint representation of $SU(N)$

Gauge vector multiplet of $\mathcal{N} = 4$ $SU(N)$ SYM in 4d
D branes as Gravitational Solitons

- D branes have a dual nature, since they are also sources of closed strings

- A stack of $N$ D$p$-branes produces a non-trivial geometry in the 10d-spacetime
D branes as Gravitational Solitons

- For instance D3 Branes are 4-dimensional solitonic solutions of Type IIB Sugra charged under a 4-form RR potential $C_4$

$$e^\phi = g_s$$

$$F_5 = d(H(r)^{-1} dx^0 \wedge \cdots \wedge dx^3) + \ast \text{dual}$$

$$ds^2 = H(r)^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + H(r)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2)$$

with $r^2 = x^i x_i$ and

$$H(r) = 1 + \frac{4\pi g_s N \alpha'}{r^4}, \quad \int_{S_5} \ast F_5 = \int_{S_5} F_5 = N$$
AdS/CFT correspondence

- In the near horizon limit \( \Rightarrow \) D3 brane geometry reduces to \( AdS_5 \times S_5 \).
- On the field theory side, this limit is a decoupling limit \( \Rightarrow \) gauge degrees of freedom decouple from the gravitational ones \( \Rightarrow \mathcal{N} = 4 \) SU(\( N \)) SYM in d=4.
- This is at the origin of the AdS/CFT correspondence, that conjectures a duality between Type II B String Theory in \( AdS_5 \times S_5 \) and \( \mathcal{N} = 4 \) SU(\( N \)) SYM in d=4.

Strong quantum gauge effects can be computed using classical gravitational calculations! This has been tested by:
- Comparison of correlation functions in SYM and SUGRA (non-renormalization theorems)
- Comparison of spectra in SYM and SUGRA (matching between conformal dimensions in SYM with Kaluza-Klein masses).
AdS/CFT correspondence

- In the near horizon limit \(\Rightarrow\) D3 brane geometry reduces to \(AdS_5 \times S_5\).
- On the field theory side, this limit is a decoupling limit \(\Rightarrow\) gauge degrees of freedom decouple from the gravitational ones \(\Rightarrow N = 4\ SU(N)\ SYM\ in\ d=4\).
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- Strong quantum gauge effects can be computed using classical gravitational calculations!
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Gauge/Gravity Correspondence

Can this correspondence be established also in case of non conformal theories and/or with reduced supersymmetry?

In this talk I will discuss in a specific example how a quantum field theory with reduced supersymmetry can be described, also non-perturbatively, by string theory and show that the two-fold nature of D branes allows to establish a quantitative non-perturbative GAUGE / GRAVITY CORRESPONDENCE.
Plan of the talk

\( \mathcal{N}=2 \) SYM from Fractional D3 branes
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$\mathcal{N}=2$ SYM from Fractional D3 branes

Gauge Instantons in String Models
Plan of the talk

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Non-perturbative Effective Action for Gravitational Fields
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The Non-perturbative $\tau$ Profile
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The Non-perturbative \( \tau \) Profile

Gauge/Gravity Correspondence

Conclusions and Outlooks
Branes on Orbifolds

- To construct gauge theories in 4d with reduced SUSY one must change the 10d geometry:

\[ \mathbb{R}^{1,3} \times \mathbb{R}^6 \rightarrow \mathbb{R}^{1,3} \times X^6 \]

for suitable choices of \( X^6 \).

- For example one could consider the \( \mathbb{Z}_2 \)-orbifold:

\[ \mathbb{R}^{1,3} \times \mathbb{R}^2 \times \mathbb{R}^4 / \mathbb{Z}_2 \]

with \( \mathbb{Z}_2 : \{x_6, \ldots, x_9\} \rightarrow \{-x_6, \ldots, -x_9\} \)

\[ x^0, \ldots, x^3 \]

\[ x^4, x^5 \]

\[ N \text{ D3} \]
Branes on Orbifolds

- To adjust the matter content one must consider fractional D3 branes.

- They can be interpreted as a D5 branes wrapped around singular exceptional 2-cycles $S_2$ (with vanishing volume) of the singular space $X^6$:

For this reason they are stuck at the singular points and cannot move.
It is realized by the massless d.o.f. of open strings attached to $N$ fractional D3 branes in the orbifold background

$$\mathbb{R}^{1,3} \times \mathbb{R}^2 \times \mathbb{R}^4 / \mathbb{Z}_2$$

where

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\( \mathcal{N} = 2 \) SYM Theory from Fractional Branes

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where

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- The orbifold breaks 1/2 SUSY in the bulk, the D3 branes break a further 1/2:

\[ 32 \times \frac{1}{2} \times \frac{1}{2} = 8 \text{ real supercharges} \rightarrow \mathcal{N} = 2 \text{ SUSY} \]
Since

\[ \text{SO}(1,9) \rightarrow \text{SO}(1,3) \times \text{SO}(2) \times \text{SO}(4) \]

the ten dimensional string coordinates \( X^M, \psi^M \) and spin fields \( S^A \) split as follows

\[ X^M \rightarrow X^\mu, X, \bar{X}, X^i, \quad \psi^M \rightarrow \psi^\mu, \psi, \bar{\psi}, \psi^i \]

\[ S^A \rightarrow S_\alpha S_- S_\dot{\alpha}, S_\dot{\alpha} S^+ S_\dot{\alpha}, S_\alpha S_+ S_a, S_\dot{\alpha} S^- S^a \]
The $\mathbb{Z}_2$ orbifold acts as follows

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Thus the surviving fields are: $A_\mu$, $\phi$, $\Lambda^\alpha_a$, which build a $\mathcal{N} = 2$ vector multiplet in $d = 4$
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The surviving fields $A_\mu$, $\phi$, $\Lambda^{\alpha a}$, can be organized in an $\mathcal{N} = 2$ chiral superfield:

$$\Phi = \phi + \theta \Lambda + \frac{1}{2} \theta \gamma^{\mu\nu} \theta F^{\mu\nu}$$
Classical Geometry of $N$ Fractional D3 branes

There is a non trivial profile for the usual (untwisted) fields associated to D3 Branes:

- Metric
- Dilaton
- R-R self-dual 5-form field strength
Classical Geometry of $\mathcal{N}$ Fractional D3 branes

There is a non trivial profile for the usual (untwisted) fields associated to D3 Branes:

- Metric
- Dilaton
- R-R self-dual 5-form field strength

There is a non trivial profile also for new (twisted) fields:

- NS-NS twisted scalar:
  \[ B_2 = b \Omega_2 \quad \Rightarrow \quad b = \int_{S_2} B_2 \]

- R-R twisted scalar:
  \[ C_2 = c \Omega_2 \quad \Rightarrow \quad c = \int_{S_2} C_2 \]

where $\Omega_2$ is the form dual to the singular cycle $S_2$ of the orbifold.
The twisted scalar

It is convenient to combine the twisted fields in a complex scalar

\[ t = c + \frac{i}{g_s}b \]

that is the lowest component of a bulk scalar superfield

\[ T = t + \cdots + \theta^4(\partial^2 \bar{t} + \cdots) \]

and has a standard quadratic bulk action

\[ S_{bulk} = -\frac{(\pi^2 \alpha')^2}{\kappa^2} \int d^6x \left( \partial \bar{t} \cdot \partial t + \cdots \right), \quad \kappa = 8\pi^{7/2} \alpha'^2 \]
The scalar field $t$ appears also in the world-volume action of the fractional D3 brane:

$$S_{brane} \propto -2N \int d^6 \bar{t} \delta^2(z) + \int d^4 x \ t \ TrF^2 + \text{c.c.}$$
World-volume Action

The scalar field $t$ plays a role also in the world-volume action of the fractional D3 brane:

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- It plays the role of the gauge coupling constant

$$t \equiv \tau_{gauge} = \frac{\theta}{\pi} + i \frac{8\pi}{g^2}$$
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$$t \equiv \tau_{gauge} = \frac{\theta}{\pi} + i \frac{8\pi}{g^2}$$

- it generates a source term for the equation of motion of $t$

$$\Box t = J_{cl} \delta^2(z) \ , \quad J_{cl} = -\frac{\delta}{\delta t} S_{brane}$$
Gauge/Gravity Correspondence

The solution of the field equation for $t$ is

$$\pi i \, t(z) = \pi i \, t_0 - 2N \, \log \frac{z}{\epsilon}, \quad t_0 = \frac{i}{2g_s}$$

Bertolini, Di Vecchia, M.F., Lerda, Marotta, Pesando; Polchinski; M.F., Liccardo, Musto.
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- Identifying $z$ with quantities with mass dimension 1

\[
z \equiv 2\pi \alpha' a \quad \epsilon \equiv 2\pi \alpha' \mu
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we have

$$\pi i t(a) = \pi i t_0 - 2N \log \frac{a}{\mu}$$

i.e. the running coupling constant of $\mathcal{N} = 2$ SU($N$) SYM:

$$\pi i t \equiv \pi i \tau_{gauge} = \pi i t_0 - 2N \log \frac{a}{\mu} = -2N \log \frac{a}{\Lambda}$$

where $\Lambda = \mu e^{-\pi/(4Ng_s)}$ is the dynamically generated scale of the theory.
In this correspondence

- the N source D3 branes are placed at $z = 0$, i.e. the expectation value of the adjoint scalar of the world-volume theory is vanishing: $\langle \phi \rangle = 0$. 

$t(z) \equiv \tau_{\text{gauge}}$
In this correspondence

- the N source D3 branes are placed at \( z = 0 \), i.e. the expectation value of the adjoint scalar of the world-volume theory is vanishing: \( \langle \phi \rangle = 0 \).

- When the D branes are displaced from the origin, i.e. \( \langle \phi \rangle = \text{diag}(a_1, a_2, \cdots, a_N) \) the solution is

\[
\pi i t(z) = -2N \text{ Tr} \log \frac{z - \langle \phi \rangle}{\Lambda}
\]
In this correspondence

- the $N$ source D3 branes are placed at $z = 0$, i.e. the expectation value of the adjoint scalar of the world-volume theory is vanishing: $\langle \phi \rangle = 0$.
- the point $z$ at which we evaluate $t(z)$ is identified with the (complexified) energy scale at which we compute $\tau_{\text{gauge}}$!
Is That all?

- The fractional brane solution has problems:
  - it has a short distance (IR) singularity at $|z| = \rho_s$
  - at $|z| = 2\pi\alpha'\Lambda e^{-\pi/4g_s} \geq \rho_s$, the YM coupling diverges: $g_{YM}^2 \to \infty$ and massive probes become tensionless $\Rightarrow$ Enhancion

- This description breaks down for small $z$, that is for small $a$ (IR region !)
Is That all?

- From the Seiberg & Witten exact solution of $\mathcal{N} = 2$ gauge theories we know that the complete gauge coupling $\tau$ has non-perturbative contributions due to instantons!

For example, for $\text{SU}(2)$ we have

$$\pi i \tau = -4 \log \left( \frac{a}{\Lambda} \right) + \frac{3}{2} \frac{\Lambda^4}{a^4} + \frac{105}{64} \frac{\Lambda^8}{a^8} + \cdots$$
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Is there a way to obtain these instanton effects from the gauge/gravity correspondence?
A gauge instanton in four dimensions is a non trivial solution $A^a_\mu(x)$ of the field equations with

$$\star F^a_{\mu\nu} = F^a_{\mu\nu} \quad S = \frac{8\pi^2}{g^2} k$$

For instance the $k = 1$ SU(2) instanton in the regular gauge is

$$A^a_\mu = 2 \frac{\eta^a_{\mu\nu}(x - x_0)\nu}{(x - x_0)^2 + \rho^2}$$

The SU($N$) instanton depends on $4N$ moduli.
The more general instantonic solution of a gauge theory can be constructed using the ADHM construction.

The gauge connection $A_{\mu}(x)$ is built up algebraically in terms of a $(k + 1)$ dimensional vector of quaternions, which satisfies a certain number of conditions so that the field strength $F_{\mu\nu}$ is automatically self-dual.

In this way $A_{\mu}(x)$ depends on a number of parameters which are more than the real instanton moduli, but are subject to a number of bosonic and fermionic constraints: ADHM constraint.

To compute the instantonic contribution to the gauge degrees of freedom effective action we have to integrate over the instanton moduli space with an appropriate measure.
String description of Gauge Instantons

• **Instanton-charge** \( k \) solutions of SU\((N)\) gauge theories correspond to \( k \) D-instantons inside \( N \) D3 branes.

Witten 1995, Douglas 1995, Dorey 1999, ...
D-Instantons

In a system of $k$ D-instantons inside $N$ D3 branes there are different types of open strings:
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- **D(–1)/D(–1) and D(–1)/D3** open strings account for the instanton moduli

Witten 1995; Douglas 1995
D-Instantons

In a system of \( k \) D-instantons inside \( N \) D3 branes there are different types of open strings:

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  Witten 1995; Douglas 1995

- the action of the \( \text{D(–1)/D(–1) and D(–1)/D3} \) strings is the ADHM measure of the instanton moduli space

Green-Gutperle 2000; Billó et al. 2002
Non-perturbative Gravitational Solution?

We could now ask two questions:

▶ Is it possible to compute a Gravitational Solution corresponding to bound states of D branes and D-instantons?
▶ Is this classical profile related to the exact gauge coupling of a dual theory?

Let's try to answer these questions!
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Let’s try to answer these questions!
Field Equations for $t$

We have derived the classical $t$ profile solving the field equation

\[ \Box t = J_{cl} \delta^2(z) \]

derived from the bulk supergravity action:

\[ S_{bulk} \propto \int d^6x \ (\partial \bar{t} \cdot \partial t + \cdots) \]

and from the boundary action:

\[ S_{brane} \propto \int d^6x \ J_{cl} \bar{t} \delta^2(z) \]
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and from the boundary action:

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\]

All the relevant information is encoded in \( J_{cl} \Rightarrow \) terms linear in \( \bar{t} \) of the boundary action!
The Current $J_{cl}$ from the Prepotential

- As customary for $\mathcal{N} = 2$ theories, the brane action can be written in terms of a prepotential

$$S_{brane} = \int d^4 x \, d^4 \theta \, F_{pert}(\Phi, T)$$

encoding all classical interactions among (massless) open and closed strings.
The Current $J_{cl}$ from the Prepotential

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$$S_{brane} = \int d^4x d^4\theta F_{pert}(\Phi, T)$$

encoding all classical interactions among (massless) open and closed strings.

- If we recall that

$$T = t + \cdots + \theta^4(\partial^2 t + \cdots)$$

we see that the source terms can be derived from $F_{pert}(\Phi, T)$ as

$$J_{cl} = \frac{\bar{p}^2}{\pi} \left. \frac{\delta F_{pert}}{\delta T} \right|_{T=0, \Phi=\langle \phi \rangle}$$
At this point it is clear how to obtain the source $J$ describing also the non-perturbative contributions to the D3 branes theory!
The Complete Current $J$ from the Prepotential

- At this point it is clear how to obtain the source $J$ describing also the non-perturbative contributions to the D3 branes theory!

- To find $J$ we have to include in $F(\Phi, T)$ the non perturbative contributions induced by D-instantons:

\[
F(\Phi, T) = F_{pert}(\Phi, T) + F_{non-pert}(\Phi, T)
\]
The non-perturbative prepotential

The non-perturbative prepotential is an integral over the centered D-instanton moduli space $\mathcal{M}_k$: 

$$F_{np}(\Phi, T) = \sum_k \int d\mathcal{M}_k e^{-S_{\text{inst}}(\mathcal{M}_k, \Phi, T)}$$
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$$F_{np}(\Phi, T) = \sum_k \int d\widehat{\mathcal{M}}_k e^{-S_{\text{inst}}(\mathcal{M}_k, \Phi, T)}$$

$$S_{\text{inst}}(\mathcal{M}_k, M, T) = S(\mathcal{M}_k) + S(\mathcal{M}_k, \Phi) + S(\mathcal{M}_k, T)$$

- $S(\mathcal{M}_k)$: pure moduli action $\Rightarrow$ ADHM measure on $\mathcal{M}_k$
- $S(\mathcal{M}_k, \Phi)$: mixed moduli-gauge fields action
- $S(\mathcal{M}_k, T)$: mixed moduli-gravity fields action
$S_{\text{inst}}(\mathcal{M}_k, \Phi, T)$

All terms in $S_{\text{inst}}(\mathcal{M}_k, \Phi, T)$ can be computed from strings diagrams:

$S(\mathcal{M}_k)$:

$S(\mathcal{M}_k, \Phi)$:

$S(\mathcal{M}_k, \bar{T})$:
The relevant term in our discussion is $S(M(k), T)$:

$$S(M(k), T) = -i\pi \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \text{Tr}(\chi^\ell) (i\bar{p})^\ell T$$
The Non-perturbative Prepotential

The non-perturbative prepotential is:

\[
F_{np}(\Phi, T) = \sum_k \int d\hat{M}(k)e^{-S_{\text{inst}}(\mathcal{M}(k), \Phi, T)}
\]

\[
= \cdots + i\pi T \sum_k \int d\hat{M}(k) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (i\bar{\rho})^\ell \text{Tr} \chi^\ell e^{-S_{\text{inst}}(\mathcal{M}(k), \Phi)}
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The integrals over the moduli space can be explicitly computed using the localization technique

Nekrasov, 2002; Flume+Poghossian, 2002; Bruzzo et al, 2003; ...
The Non-perturbative Prepotential

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\[ F_{np}(\Phi, T) = \cdots + i\pi T \sum_k \int d\tilde{M}(k) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (i\bar{p})^\ell \text{Tr} \chi^\ell e^{-S_{\text{inst}}(M(k), \Phi)} \]

The integrals over the moduli space can be explicitly computed using the localization technique

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but in fact their result is already known, since they are related to the quantum expectation values of chiral correlators:

\[ \sum_k \int d\tilde{M}(k) e^{-S_{\text{inst}}(M(k), \Phi)} \text{Tr} \chi^\ell \propto \langle \text{Tr} \phi^{\ell+2} \rangle_{\text{inst}} \]

For instance:

\[ \langle \text{Tr} \phi^4 \rangle = 2a^4 + 6\Lambda^4 - \frac{9}{8} \frac{\Lambda^8}{a^4} + \frac{7}{8} \frac{\Lambda^{12}}{a^8} + \cdots \]
The Non-perturbative Prepotential

\[ F_{np}(\Phi, T) = \cdots + i\pi T \sum_k \int d\hat{M}(k) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (i\bar{p})^\ell \text{Tr} \chi^\ell e^{-S_{\text{inst}}(\mathcal{M}(k), \Phi)} \]

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so we get

\[ F_{np}(\Phi, T) = -i\pi T \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (i\bar{p})^{\ell-2} \langle \text{Tr} \phi^\ell \rangle_{\text{inst}} \]
We can now compute the complete superpotential:

\[ F(\Phi, T) = F_{pert}(\Phi, T) + F_{np}(\Phi, T) \]
The Exact Current \( J \)

We can now compute the complete superpotential

\[
F(\Phi, T) = -i\pi T \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (i\bar{p})^{\ell-2} \langle \text{Tr} \phi^\ell \rangle,
\]

that depends on the complete quantum correlators \( \langle \text{Tr} \phi^\ell \rangle \) and obtain the exact current \( J \):

\[
J = \frac{\bar{p}^2}{\pi} \frac{\delta F(\Phi, T)}{\delta T} \bigg|_{T=0, \Phi=\langle \phi \rangle} = i \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (i\bar{p})^{\ell} \langle \text{Tr} \phi^\ell \rangle
\]
The Exact $t$ Profile

Solving the Field Equation for $t$ with the exact current:

$$\Box t = J \delta^2(z)$$
The Exact $t$ Profile

Solving the Field Equation for $t$ with the exact current:

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\Box t = J \delta^2(z)
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we find the new profile for $t$:

$$
i\pi t = i\pi t_0 - 2\left\langle \text{Tr} \log \frac{Z - \phi}{\mu} \right\rangle = -2\left\langle \text{Tr} \log \frac{Z - \phi}{\Lambda} \right\rangle
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The Exact $t$ Profile

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The exact solution has the same form of the classical one

$$i\pi t = -2 \text{Tr} \log \frac{z - \langle \phi \rangle}{\Lambda}$$

with the classical expectation values replaced by the quantum ones!
Closed Expression for the $t$ Profile

The complete solution for the $t$ field can be expressed in a closed form exploiting the information about the gauge theory of the source branes contained in the Seiberg-Witten curve.
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The complete solution for the $t$ field can be expressed in a closed form exploiting the information about the gauge theory of the source branes contained in the Seiberg-Witten curve.

In the simple SU(2) case, when the curve is a torus

$$y^2 = (z^2 - u)^2 - 4\Lambda^4$$

where $u = \frac{1}{2} \langle \text{Tr} \phi^2 \rangle$ and $\Lambda$ is the dynamically generated scale of the effective theory, it can be shown that

$$i\pi t = \log \frac{(z^2 - u) - \sqrt{(z^2 - u)^2 - 4\Lambda^4}}{(z^2 - u) + \sqrt{(z^2 - u)^2 - 4\Lambda^4}}.$$
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Similar expressions hold for the SU($N$) cases.
We have computed the exact $t$ profile taking into account the full dynamics of the D3 brane theory, including the non-perturbative effects induced by the D-instantons:
The $t$ Profile

We have computed the exact $t$ profile taking into account the full dynamics of the D3 brane theory, including the non-perturbative effects induced by the D-instantons:

\[ t_{\text{cl}} \rightarrow t \]

Going from $t_{\text{cl}}(z)$ to $t(z)$ the result is naturally expressed as function of $u = \frac{1}{2} \langle \text{Tr} \phi^2 \rangle$ instead of $a^2 = \frac{1}{2} \text{Tr} \langle \phi^2 \rangle$. 
Can this profile be read as the exact gauge coupling of a dual SU(2) gauge theory?

\[ i\pi \, t = \log \frac{(z^2 - u) - \sqrt{(z^2 - u)^2 - 4\Lambda^4}}{(z^2 - u) + \sqrt{(z^2 - u)^2 - 4\Lambda^4}} \]
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- In the classical case, the gauge coupling \( \tau \) was identified with the classical solution of 2 D3 branes sitting at the origin: \( a = 0 \)

- In the quantum case, the equivalent choice is

\[ u = a^2 + \frac{1\Lambda^4}{2a^2} + \frac{5\Lambda^8}{32a^6} + \frac{9\Lambda^{12}}{64a^{10}} + \cdots = 0 \]

\[ \Rightarrow \text{Enhancion} !!! \]
Can this profile be read as the exact gauge coupling of a dual SU(2) gauge theory?

\[ i\pi t = \log \frac{z^2 - \sqrt{z^4 - 4\Lambda^4}}{z^2 + \sqrt{z^4 - 4\Lambda^4}} \]

- In the classical case, the complex variable \( z \) was identified with the classical parameter of the Coulomb branch of the dual theory \( a \) while \( \Lambda \equiv \Lambda \)

- In the quantum case we have to identify \( z \) with the quantum parameter of the Coulomb branch of the dual theory \( u^{1/2} \) and \( \Lambda \equiv \Lambda \)
Can this profile be read as the exact gauge coupling of a dual SU(2) gauge theory?

\[ i\pi t = \log \frac{u - \sqrt{u^2 - 4\Lambda^4}}{u + \sqrt{u^2 - 4\Lambda^4}} \]

Using the Seiberg-Witten curve of an SU(2) theory one can verify that this expression does not coincide with the torus modular parameter, which is in fact the gauge coupling \( \tau \)!
$\tau$ as Function of $t$

There is however a relation between $\tau$ and $t$!
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There is however a relation between $\tau$ and $t$!

- Using again the Seiberg-Witten curve one can verify that $t$ is the anharmonic ratio of the roots of the curve, which is known to be a particular modular function of $\tau$:

$$i \pi t = \log \frac{u - \sqrt{u^2 - 4\Lambda^4}}{u + \sqrt{u^2 - 4\Lambda^4}} = -16 \frac{\eta(4\tau)^8}{\eta(\tau)^8}$$

where $\eta(\tau) = e^{i\pi \tau/24} \prod_n (1 - e^{i\pi n\tau})$ is the Dedekind theta-function.
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- This relation holds for all the SU(2) theories with an arbitrary matter content.

- Similar relations hold for SU(3) and SU(4) (and presumably for all the SU($N$))!
There is however a relation between $\tau$ and $t$!

- Using again the Seiberg-Witten curve one can verify that $t$ is the anharmonic ratio of the roots of the curve, which is known to be a particular modular function of $\tau$:

\[ i\pi t = \log \frac{u - \sqrt{u^2 - 4\Lambda^4}}{u + \sqrt{u^2 - 4\Lambda^4}} = -16 \frac{\eta(4\tau)^8}{\eta(\tau)^8} \]

where $\eta(\tau) = e^{i\pi\tau/24} \prod_n (1 - e^{i\pi n\tau})$ is the Dedekind theta-function.

- $t(z)$ still contains all the information about the gauge theory coupling, even if it does not directly coincide with it!
Conclusions and Outlooks

- **D brane systems** provide a very efficient set-up to describe gauge theories in a “stringy way”.

- Also **non-perturbative effects** can be explicitly described in this set-up introducing **instantonic branes**.

- **Gauge/Gravity** correspondence can, at least in some cases, incorporate also the non perturbative features of the field theory.

- **Possible developments:**
  - **Gravity** $\Rightarrow$ compute the exact solution for the other gravitational fields.
  - **Gauge** $\Rightarrow$ study strong coupling regime of the gauge theories.