Non-perturbative Gauge/Gravity Correspondence

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Gauge Theory from Strings

String theory is a very powerful tool to analyze perturbative field theories, and in particular gauge theories.

Behind this, there is a rather simple and well-known fact: in the field theory limit $\alpha' \to 0$ string theory S-matrix elements reproduce vertices and effective actions in field theory.

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- **Closed Strings amplitudes** $\Rightarrow$ Gravitational amplitudes
- **Open Strings amplitudes** $\Rightarrow$ Gauge amplitudes

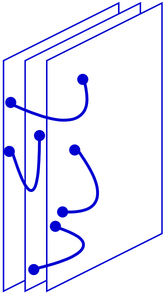
D Branes

The advances of the last 15 years in string theory have opened new perspectives, and new insights on gauge theories can be obtained using D Branes:

- The end points of open strings with Neumann boundary conditions in $(p+1)$ space-time directions and Dirichlet boundary conditions in the other $(9-p)$,

$$\begin{aligned} \partial_\sigma X^\mu|_{\sigma=0,\pi} &= 0 \quad , \qquad \mu = 0,1,\ldots,p \\ X^i|_{\sigma=0,\pi} &= Y^i \quad , \qquad i = p+1,\ldots,9 \end{aligned}$$

  define a $(p+1)$-dimensional hyperplane called Dp-brane.



World-volume Theory

- The presence of a D brane breaks Lorentz and Poincaré invariance:
  - $\mathrm{SO}(1,9) \Rightarrow \mathrm{SO}(1,p) \times \mathrm{SO}(9-p)$
  - translational invariance is broken in the $(9-p)$ directions transverse to the brane

  Open string excitations have no momentum in the transverse directions $\Rightarrow$ massless open strings describe a Field Theory in $(p+1)$ dimensions.

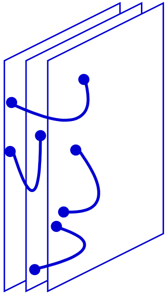
- Also supersymmetry is generically broken, so we have a Supersymmetric Field Theory with at most 16 charges (instead of the original 32 of the bulk theory)!

N D3 branes in Flat Space

Open string spectrum:

$$A_\mu \ , \ \phi^i \qquad \mu = 0, ..., 3 \ ; \ \ i = 1, ..., 6$$
$$\lambda_\alpha{}^A \ , \ \bar{\lambda}^{\dot{\alpha}}{}_A \qquad \alpha, \dot{\alpha} = 1, 2 \ ; \ \ A = 1, 2, 3, 4$$

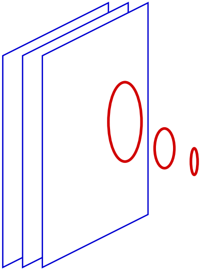
all fields transform in the adjoint representation of SU(N)



Gauge vector multiplet of $\mathcal{N} = 4$ $\mathrm{SU}(N)$ SYM in 4d

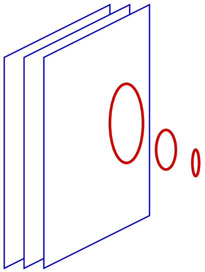
D branes as Gravitational Solitons

- D branes have a dual nature, since they are also sources of closed strings



- A stack of N Dp-branes produces a non-trivial geometry in the 10d-spacetime

D branes as Gravitational Solitons



- For instance D3 Branes are 4-dimensional solitonic solutions of Type IIB Sugra charged under a 4-form RR potential C_4

$$
\begin{aligned}
e^{\phi} &= g_s \\
F_5 &= d(H(r)^{-1} dx^0 \wedge \cdots \wedge dx^3) + *\mathrm{dual} \\
ds^2 &= H(r)^{-\frac{1}{2}} \eta_{\mu\nu} dx^{\mu} dx^{\nu} + H(r)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2)
\end{aligned}
$$

with $r^2 = x^i x_i$ and

$$
H(r) = 1 + \frac{4\pi g_s N \alpha'^2}{r^4} \quad , \quad \int_{S_5} *F_5 = \int_{S_5} F_5 = N
$$

AdS/CFT correspondence

- In the near horizon limit $\Rightarrow$ D3 brane geometry reduces to $AdS_5 \times S_5$
- On the field theory side, this limit is a decoupling limit $\Rightarrow$ gauge degrees of freedom decouple from the gravitational ones $\Rightarrow$ $\mathcal{N} = 4$ SU(N) SYM in d=4
- This is at the origin of the AdS/CFT correspondence, that conjectures a duality between Type II B String Theory in $AdS_5 \times S_5$ and $\mathcal{N} = 4$ SU(N) SYM in d=4.

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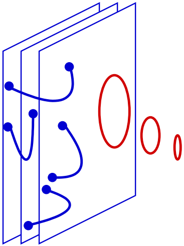
- Strong quantum gauge effects can be computed using classical gravitational calculations!
- This has been tested by
  - comparison of correlation functions in SYM and SUGRA (non-renormalization theorems)
  - comparison of spectra in SYM and SUGRA (matching between conformal dimensions in SYM with Kaluza-Klein masses)

Gauge/Gravity Correspondence

Can this correspondence be established also in case of non conformal theories and/or with reduced supersymmetry?

In this talk I will discuss in a specific example how a quantum field theory with reduced supersymmetry can be described, also non-perturbatively, by string theory and show that the two-fold nature of D branes



allows to establish a quantitative non-perturbative

GAUGE / GRAVITY CORRESPONDENCE

Plan of the talk

$\mathcal{N}$=2 SYM from Fractional D3 branes

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Non-perturbative Effective Action for Gravitational Fields

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Conclusions and Outlooks

Branes on Orbifolds

- To construct gauge theories in 4d with reduced SUSY one must change the 10d geometry:

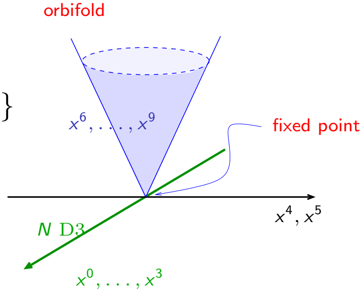
$$\mathbb{R}^{1,3} \times \mathbb{R}^6 \longrightarrow \mathbb{R}^{1,3} \times \mathbb{X}^6$$

  for suitable choices of $\mathbb{X}^6$.

- For example one could consider the $\mathbb{Z}_2$-orbifold:

$$\mathbb{R}^{1,3} \times \mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$$

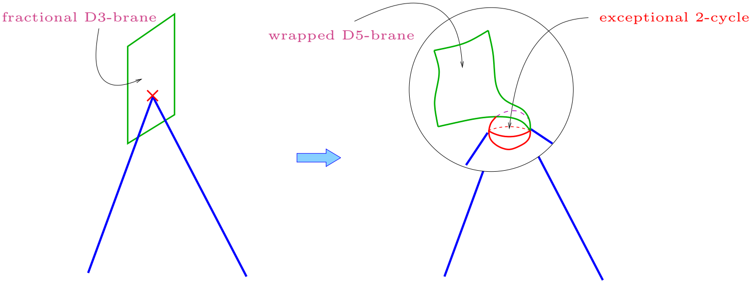
  with $\mathbb{Z}_2 : \{x_6, \ldots, x_9\} \to \{-x_6, \ldots, -x_9\}$



Branes on Orbifolds

- To adjust the matter content one must consider fractional D3 branes.

- They can be interpreted as a D5 branes wrapped around singular exceptional 2-cycles $\mathcal{S}_2$ (with vanishing volume) of the singular space X^6:



For this reason they are stuck at the singular points and cannot move.

$\mathcal{N} = 2$ SYM Theory from Fractional Branes

- It is realized by the massless d.o.f. of open strings attached to N fractional D3 branes in the orbifold background

$$\mathbb{R}^{1,3} \times \mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$$

where

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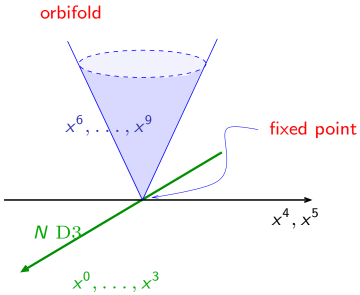

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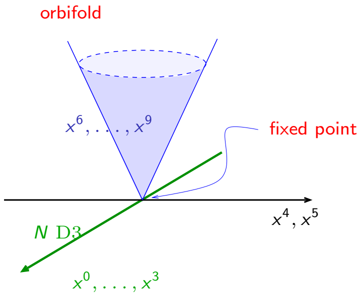

- The orbifold breaks $1/2$ SUSY in the bulk, the D3 branes break a further $1/2$:

$$32 \times \frac{1}{2} \times \frac{1}{2} = 8 \ \text{ real supercharges} \quad \rightarrow \quad \mathcal{N} = 2 \text{ SUSY}$$

	0	1	2	3	4	5	6	7	8	9
D3	–	–	–	–	*	*	*	*	*	*

Since

$$\mathrm{SO}(1,9) \ \rightarrow \ \mathrm{SO}(1,3) \times \mathrm{SO}(2) \times \mathrm{SO}(4)$$

the ten dimensional string coordinates X^M, ψ^M and spin fields $S^{\mathcal{A}}$ split as follows

$$X^M \ \rightarrow \ X^\mu \, , \, X, \bar{X} \, , \, X^i \quad , \quad \psi^M \ \rightarrow \ \psi^\mu \, , \, \Psi, \bar{\Psi} \, , \, \psi^i$$

$$S^{\mathcal{A}} \ \rightarrow \ S_\alpha S_- S_{\dot{a}} \, , \, S^{\dot{\alpha}} S^+ S^{\dot{a}} \, , \, S_\alpha S_+ S_a \, , \, S^{\dot{\alpha}} S^- S^a$$

- The $\mathbb{Z}_2$ orbifold acts as follows

NS sector	$\mathbb{Z}_2$ parity	
$A_\mu \leftrightarrow \psi^\mu_{-\frac{1}{2}}\vert k\rangle$	$+$	
$\phi \leftrightarrow \bar{\Psi}_{-\frac{1}{2}}\vert k\rangle$	$+$	
$\phi^i \leftrightarrow \psi^i_{-\frac{1}{2}}\vert k\rangle$	$-$	

R sector	$\mathbb{Z}_2$ parity	
$\Lambda^{\alpha a} \leftrightarrow S_\alpha S_+ S_a \vert k\rangle$	$+$	
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- Thus the surviving fields are: A_μ, ϕ, $\Lambda^{\alpha a}$, which build a

$$\mathcal{N} = 2 \text{ vector multiplet in } d = 4$$

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- The surviving fields A_μ, ϕ, $\Lambda^{\alpha a}$, can be organized in an $\mathcal{N} = 2$ chiral superfield:

$$\Phi = \phi + \theta\Lambda + \frac{1}{2}\theta\gamma^{\mu\nu}\theta F^{\mu\nu}$$

Classical Geometry of N Fractional D3 branes

There is a non trivial profile for the usual (untwisted) fields associated to D3 Branes:

- Metric
- Dilaton
- R-R self-dual 5-form field strength

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- Dilaton
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There is a non trivial profile also for new (twisted) fields:

- NS-NS twisted scalar:

$$B_2 = b\,\Omega_2 \quad \Rightarrow \qquad b = \int_{\mathcal{S}_2} B_2$$

- R-R twisted scalar:

$$C_2 = c\,\Omega_2 \quad \Rightarrow \qquad c = \int_{\mathcal{S}_2} C_2$$

where Ω_2 is the form dual to the singular cycle $\mathcal{S}_2$ of the orbifold.

The twisted scalar

It is convenient to combine the twisted fields in a complex scalar

$$t = c + \frac{\mathrm{i}}{g_s} b$$

that is the lowest component of a bulk scalar superfield

$$T = t + \cdots + \theta^4 \left(\partial^2 \bar{t} + \cdots \right)$$

and has a standard quadratic bulk action

$$S_{bulk} = -\frac{(\pi^2 \alpha')^2}{\kappa^2} \int d^6 x \, \left(\partial \bar{t} \cdot \partial t + \cdots \right) \, , \quad \kappa = 8 \pi^{7/2} \alpha'^2$$

World-volume Action

The scalar field t appears also in the world-volume action of the fractional D3 brane:

$$S_{brane} \propto -2N \int d^6x \, \bar{t} \, \delta^2(z) + \int d^4x \, t \, \mathrm{Tr} F^2 \quad + \mathrm{c.c.}$$

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- it generates a source term for the equation of motion of t

$$\Box t = J_{cl}\, \delta^2(z)\ , \qquad J_{cl} = -\frac{\delta}{\delta \bar{t}} S_{brane}$$

Gauge/Gravity Correspondence

- The solution of the field equation for t is

$$\pi\mathrm{i}\, t(z) = \pi\mathrm{i}\, t_0 - 2N \log\frac{z}{\epsilon}\ , \qquad t_0 = \frac{\mathrm{i}}{2g_s}$$

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we have

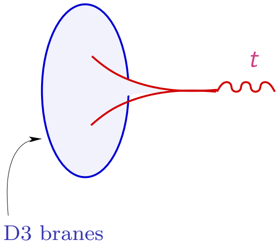
$$\pi\mathrm{i}\, t(a) = \pi\mathrm{i}\, t_0 - 2N \log\frac{a}{\mu}$$

i.e. the running coupling constant of $\mathcal{N} = 2$ SU(N) SYM:

$$\pi\mathrm{i} t \equiv \pi\mathrm{i}\tau_{gauge} = \pi\mathrm{i}\, t_0 - 2N \log\frac{a}{\mu} = -2N \log\frac{a}{\Lambda}$$

where $\Lambda = \mu\,\mathrm{e}^{-\pi/(4Ng_s)}$ is the dynamically generated scale of the theory.

Gauge/Gravity Correspondence



$$t(z) \equiv \tau_{gauge}$$

In this correspondence

- the N source D3 branes are placed at $z = 0$, *i.e.* the expectation value of the adjoint scalar of the world-volume theory is vanishing: $\langle \phi \rangle = 0$.

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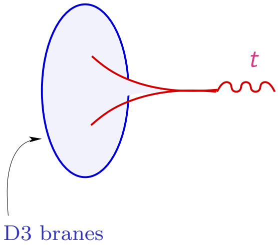

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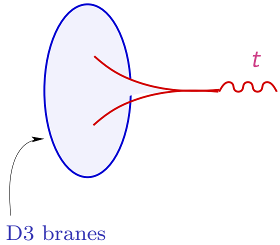
- When the D branes are displaced from the origin, *i.e.* $\langle \phi \rangle = diag(a_1, a_2, \cdot, a_N)$ the solution is

$$\pi i t(z) = -2N \,\mathrm{Tr} \log \frac{z - \langle \phi \rangle}{\Lambda}$$

Gauge/Gravity Correspondence



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- the point z at which we evaluate t(z) is identified with the (complexified) energy scale at which we compute τ_{gauge}!

Is That all?

- The fractional brane solution has problems:
  - it has a short distance (IR) singularity at $|z| = \rho_s$
  - at $|z| = 2\pi\alpha'\Lambda e^{-\pi/4g_s} \geq \rho_s$, the YM coupling diverges : $g_{YM}^2 \to \infty$ and massive probes become tensionless $\Rightarrow$ Enhançon
- This description breaks down for small z , that is for small a (IR region !)

Is That all?

- From the Seiberg & Witten exact solution of $\mathcal{N} = 2$ gauge theories we know that the complete gauge coupling τ has non-perturbative contributions due to instantons !

  For example, for SU(2) we have

$$\pi \mathrm{i}\,\tau = -4\log\left(\frac{a}{\Lambda}\right) + \frac{3}{2}\frac{\Lambda^4}{a^4} + \frac{105}{64}\frac{\Lambda^8}{a^8} + \cdots$$

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- Is there a way to obtain these instanton effects from the gauge/gravity correspondence?

Instantons in Gauge Theories

- A gauge instanton in four dimensions is a non trivial solution $A^a_\mu(x)$ of the field equations with

$$*F^a_{\mu\nu} = F^a_{\mu\nu} \qquad\qquad S = \frac{8\pi^2}{g^2}\,k$$

- For instance the $k = 1$ SU(2) instanton in the regular gauge is
't Hooft

$$A^a_\mu \;=\; 2\,\frac{\eta^a_{\mu\nu}(x - x_0)^\nu}{(x - x_0)^2 + \rho^2}$$

- The $SU(N)$ instanton depends on $4N$ moduli.

Instantons in Gauge Theories

- The more general instantonic solution of a gauge theory can be constructed using the ADHM construction.

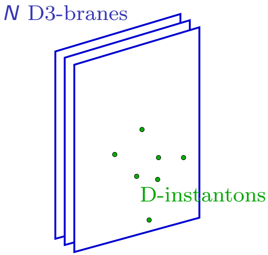
  Atiyah, Drinfeld, Hitchin, Manin

- The gauge connection $A_\mu(x)$ is built up algebrically in terms of a $(k+1)$ dimensional vector of quaternions, which satisfies a certain number of conditions so that the field strenght $F_{\mu\nu}$ is automatically self-dual.
- In this way $A_\mu(x)$ depends on a number of parameters which are more then the real instanton moduli, but are subject to a number of bosonic and fermionic constraints: ADHM constraint.
- To compute the instantonic contribution to the gauge degrees of freedom effective action we have to integrate over the instanton moduli space with an appropriate measure.

String description of Gauge Instantons

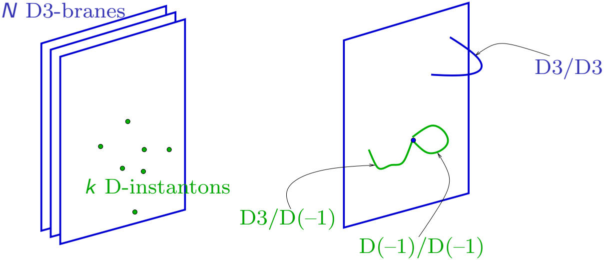
- Instanton-charge k solutions of $SU(N)$ gauge theories correspond to k D-instantons inside N D3 branes.

Witten 1995, Douglas 1995, Dorey 1999, ...



D-Instantons

In a system of k D-instantons inside N D3 branes there are different types of open strings:



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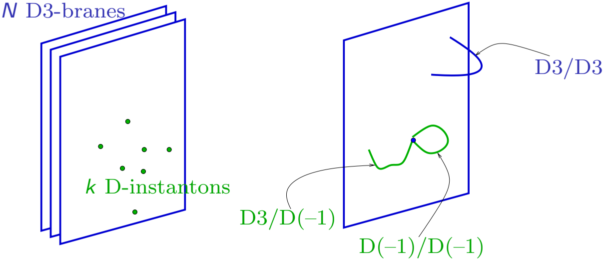

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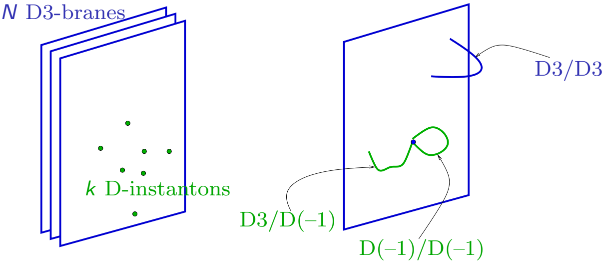

- D3/D3 open strings describe gauge degrees of freedom
- D(–1)/D(–1) and D(–1)/D3 open strings account for the instanton moduli

Witten 1995; Douglas 1995

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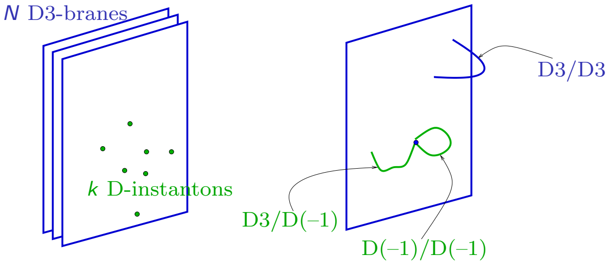

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- the action of the D(–1)/D(–1) and D(–1)/D3 strings is the ADHM measure of the instanton moduli space

Green-Gutperle 2000; Billó et al. 2002

Non-perturbative Gravitational Solution?

We could now ask two questions:

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Non-perturbative Gravitational Solution?

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- Is it possible to compute a Gravitational Solution corresponding to bound states of D branes and D-instantons?
- Is this classical profile related to the exact gauge coupling of a dual theory?

Non-perturbative Gravitational Solution?

We could now ask two questions:

- Is it possible to compute a Gravitational Solution corresponding to bound states of D branes and D-instantons?
- Is this classical profile related to the exact gauge coupling of a dual theory?

Let's try to answer these questions!

Field Equations for t

We have derived the classical t profile solving the field equation

$$\Box\, t = J_{cl}\, \delta^2(z)$$

derived from the bulk supergravity action:

$$S_{bulk} \propto \int d^6x\, \left(\partial\bar{t} \cdot \partial t + \cdots\right)$$

and from the boundary action:

$$S_{brane} \propto \int d^6x\, J_{cl}\, \bar{t}\, \delta^2(z)$$

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All the relevant information is encoded in J_{cl} $\Rightarrow$ terms linear in $\bar{t}$ of the boundary action!

The Current J_{cl} from the Prepotential

- As customary for $\mathcal{N} = 2$ theories, the brane action can be written in terms of a prepotential

$$S_{brane} = \int d^4x \, d^4\theta \, F_{pert}(\Phi, T)$$

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- If we recall that

$$T = t + \cdots + \theta^4 \left(\partial^2 \bar{t} + \cdots\right)$$

we see that the source terms can be derived from $F_{pert}(\Phi, T)$ as

$$J_{cl} = \frac{\bar{p}^2}{\pi} \frac{\delta F_{pert}}{\delta T}\bigg|_{T=0, \Phi=\langle\phi\rangle}$$

The Complete Current J from the Prepotential

- At this point it is clear how to obtain the source J describing also the non-perturbative contributions to the D3 branes theory !

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- At this point it is clear how to obtain the source J describing also the non-perturbative contributions to the D3 branes theory !

- To find J we have to include in $F(\Phi, T)$ the non perturbative contributions induced by D-instantons:

$$F(\Phi, T) = F_{pert}(\Phi, T) + F_{non-pert}(\Phi, T)$$

The non-perturbative prepotential

The non-perturbative prepotential is an integral over the centered D-instanton moduli space $\mathcal{M}_{(k)}$:

$$F_{np}(\Phi, T) = \sum_k \int d\widehat{\mathcal{M}}_{(k)} \mathrm{e}^{-S_{inst}(\mathcal{M}_{(k)}, \Phi, T)}$$

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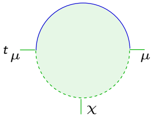
$$S_{inst}(\mathcal{M}_{(k)}, M, T) = S(\mathcal{M}_{(k)}) + S(\mathcal{M}_{(k)}, \Phi) + S(\mathcal{M}_{(k)}, T)$$

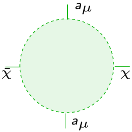
- $S(\mathcal{M}_{(k)})$: pure moduli action $\Rightarrow$ ADHM measure on $\mathcal{M}_{(k)}$
- $S(\mathcal{M}_{(k)}, \Phi)$: mixed moduli-gauge fields action
- $S(\mathcal{M}_{(k)}, T)$: mixed moduli-gravity fields action

$S_{inst}(\mathcal{M}_{(k)}, \Phi, T)$

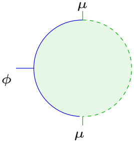
All terms in $S_{inst}(\mathcal{M}_{(k)}, \Phi, T)$ can be computed from strings diagrams:

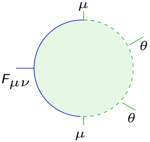
$S(\mathcal{M}_{(k)})$:



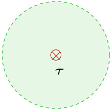


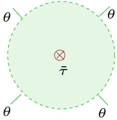
$S(\mathcal{M}_{(k)}, \Phi)$:





$S(\mathcal{M}_{(k)}, T)$:





$S_{inst}(\mathcal{M}_{(k)}, \Phi, T)$

The relevant term in our discussion is $S(\mathcal{M}_{(k)}, T)$:

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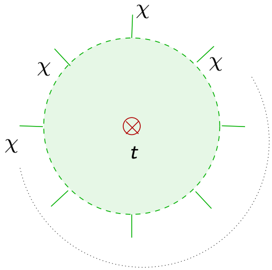

$$S(\mathcal{M}_{(k)}, T) = \qquad -\mathrm{i}\pi \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \,\mathrm{Tr}(\chi^{\ell})\, (\mathrm{i}\bar{p})^{\ell}\; T$$

The Non-perturbative Prepotential

The non-perturbative prepotential is:

$$F_{np}(\Phi, T) = \sum_k \int d\widehat{\mathcal{M}}_{(k)} \mathrm{e}^{-S_{inst}(\mathcal{M}_{(k)}, \Phi, T)}$$

$$= \cdots + \mathrm{i}\pi T \sum_k \int d\widehat{\mathcal{M}}_{(k)} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (\mathrm{i}\bar{p})^{\ell} \mathrm{Tr}\, \chi^{\ell}\, \mathrm{e}^{-S_{inst}(\mathcal{M}_{(k)}, \Phi)}$$

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Nekrasov, 2002; Flume+Poghossian, 2002; Bruzzo et al, 2003; ...

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but in fact their result is already known, since they are related to the quantum expectation values of chiral correlators:

$$\sum_k \int d\widehat{\mathcal{M}}_{(k)} \mathrm{e}^{-S_{inst}(\mathcal{M}_{(k)}, \Phi)} \mathrm{Tr}\, \chi^\ell \propto \langle \mathrm{Tr}\, \phi^{\ell+2} \rangle_{inst}$$

For instance:

$$\langle \mathrm{Tr}\, \phi^4 \rangle = 2a^4 + 6\Lambda^4 - \frac{9}{8}\frac{\Lambda^8}{a^4} + \frac{7}{8}\frac{\Lambda^{12}}{a^8} + \cdots$$

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so we get

$$F_{np}(\Phi, T) = -\mathrm{i}\pi\, T \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (\mathrm{i}\bar{p})^{\ell-2} \langle\, \mathrm{Tr}\, \phi^{\ell}\, \rangle_{inst}$$

The Exact Current J

We can now compute the complete superpotential:

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$$F(\Phi, T) = -i\pi\, T \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (i\bar{p})^{\ell-2} \langle \operatorname{Tr} \phi^{\ell} \rangle \ ,$$

that depends on the complete quantum correlators $\langle \operatorname{Tr} \phi^{\ell} \rangle$ and obtain the exact current J:

$$J = \frac{\bar{p}^2}{\pi} \frac{\delta F(\Phi, T)}{\delta T}\Big|_{T=0, \Phi=\langle\phi\rangle} = i \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (i\bar{p})^{\ell} \langle \operatorname{Tr} \phi^{\ell} \rangle$$

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$$\mathrm{i}\pi\, t = \mathrm{i}\pi\, t_0 - 2\left\langle \operatorname{Tr}\log\frac{z-\phi}{\mu}\right\rangle = -2\left\langle \operatorname{Tr}\log\frac{z-\phi}{\Lambda}\right\rangle$$

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The exact solution has the same form of the classical one

$$\mathrm{i}\pi\, t = -2\,\mathrm{Tr}\,\log\frac{z-\langle\phi\rangle}{\Lambda}$$

with the classical expectation values replaced by the quantum ones!

Closed Expression for the t Profile

The complete solution for the t field can be expressed in a closed form exploiting the information about the gauge theory of the source branes contained in the Seiberg-Witten curve.

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In the simple SU(2) case, when the curve is a torus

$$y^2 = (z^2 - u)^2 - 4\Lambda^4$$

where $u = \frac{1}{2}\langle \mathrm{Tr}\,\phi^2 \rangle$ and Λ is the dynamically generated scale of the effective theory, it can be shown that

$$\mathrm{i}\pi\, t = \log \frac{(z^2 - u) \; - \; \sqrt{(z^2 - u)^2 - 4\Lambda^4}}{(z^2 - u) \; + \; \sqrt{(z^2 - u)^2 - 4\Lambda^4}} \; .$$

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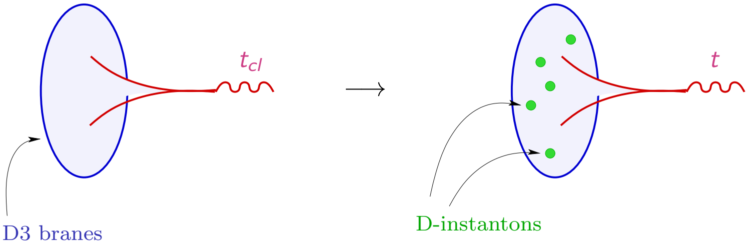
Similar expressions hold for the SU(N) cases.

The t Profile

We have computed the exact t profile taking into account the full dynamics of the D3 brane theory, including the non-perturbative effects induced by the D-instantons:



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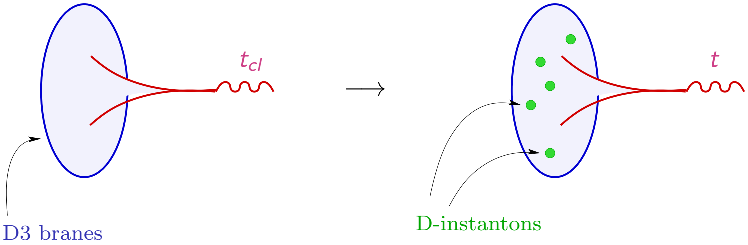

Going from $t_{cl}(z)$ to $t(z)$ the result is naturally expressed as function of $u = \frac{1}{2} \langle \mathrm{Tr}\, \phi^2 \rangle$ instead of $a^2 = \frac{1}{2} \mathrm{Tr}\, \langle \phi^2 \rangle$.

t as Gauge Coupling?

Can this profile be read as the exact gauge coupling of a dual SU(2) gauge theory?

$$\mathrm{i}\pi\, t = \log \frac{(z^2 - u) - \sqrt{(z^2-u)^2 - 4\Lambda^4}}{(z^2 - u) + \sqrt{(z^2-u)^2 - 4\Lambda^4}}$$

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- in the classical case the gauge coupling τ was identified with the classical solution of 2 D3 branes sitting at the origin: $a = 0$
- in the quantum case the equivalent choice is

$$u = a^2 + \frac{1}{2}\frac{\Lambda^4}{a^2} + \frac{5}{32}\frac{\Lambda^8}{a^6} + \frac{9}{64}\frac{\Lambda^{12}}{a^{10}} + \cdots = 0$$

$\Rightarrow$ Enhancon !!!

t as Gauge Coupling?

Can this profile be read as the exact gauge coupling of a dual SU(2) gauge theory?

$$\mathrm{i}\pi\, t = \log \frac{z^2 - \sqrt{z^4 - 4\Lambda^4}}{z^2 + \sqrt{z^4 - 4\Lambda^4}}$$

- in the classical case, the complex variable z was identified with the classical parameter of the Coulomb branch of the dual theory a while $\Lambda \equiv \Lambda$
- in the quantum case we have to identify z with the quantum parameter of the Coulomb branch of the dual theory $u^{1/2}$ and $\Lambda \equiv \Lambda$

t as Gauge Coupling?

Can this profile be read as the exact gauge coupling of a dual SU(2) gauge theory?

$$\mathrm{i}\pi\, t = \log \frac{u - \sqrt{u^2 - 4\Lambda^4}}{u + \sqrt{u^2 - 4\Lambda^4}}$$

- Using the Seiberg-Witten curve of an SU(2) theory one can verify that this expression does not coincide with the torus modular parameter, which is in fact the gauge coupling τ!

τ as Function of t

There is however a relation between τ and t!

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- Using again the Seiberg-Witten curve one can verify that t is the anharmonic ratio of the roots of the curve, which is known to be a particular modular function of τ:

$$\mathrm{i}\pi\, t = \log \frac{u - \sqrt{u^2 - 4\Lambda^4}}{u + \sqrt{u^2 - 4\Lambda^4}} = -16\,\frac{\eta(4\tau)^8}{\eta(\tau)^8}$$

where $\eta(\tau) = \mathrm{e}^{\mathrm{i}\pi\tau/24}\prod_n(1 - \mathrm{e}^{\mathrm{i}\pi n\tau})$ is the Dedekind theta-function.

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- This relation holds for all the SU(2) theories with an arbitrary matter content.
- Similar relations hold for SU(3) and SU(4) (and presumably for all the SU(N))!

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- t(z) still contains all the information about the gauge theory coupling, even if it does not directly coincide with it!

Conclusions and Outlooks

- D brane systems provide a very efficient set-up to describe gauge theories in a “stringy way”.
- Also non-perturbative effects can be explicity described in this set-up introducing instantonic branes.
- Gauge/Gravity correspondence can, at least in some cases, incorporate also the non perturbative features of the field theory.
- Possible developments:
  - Gravity $\Rightarrow$ compute the exact solution for the other gravitational fields.
  - Gauge $\Rightarrow$ study strong coupling regime of the gauge theories.