The Cardy-Verlinde equation and the gravitational collapse

Cosimo Stornaiolo
INFN -- Napoli

G. Maiella and C. Stornaiolo
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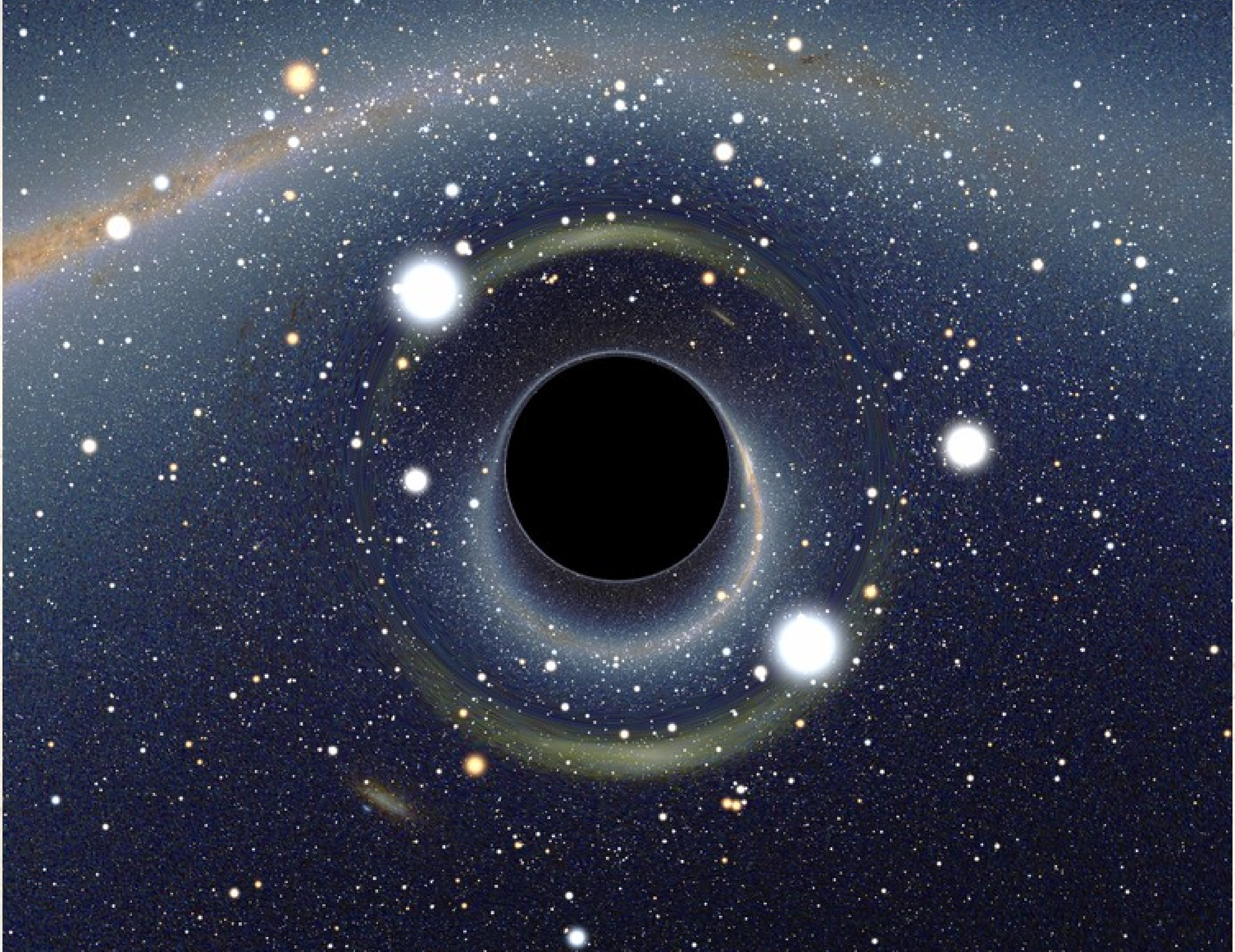



So you created everything.
Yes.
Including black holes.
Yes.
Which will eventually swallow up everything.
Yes.
Including you.
I'm working on that.
S. Harris

Black Hole Thermodynamics

The laws of black hole mechanics are expressed in geometrized units.

The Zeroth Law

The horizon has constant surface gravity κ for a stationary black hole.

$$\kappa = \frac{1}{4M}$$

The First Law

We have

$$dM = \frac{\kappa}{8\Pi} dA + \Omega dJ + \Phi dQ$$

where M is the mass, A is the horizon area, Ω is the angular velocity, J is the angular momentum, Φ is the electrostatic potential, κ is the surface gravity and Q is the electric charge.

The Second Law

The horizon area is, assuming the weak energy condition, a non-decreasing function of time,

$$\frac{dA}{dt} \geq 0$$

This "law" was superseded by Hawking's discovery that black holes radiate, which causes both the black hole's mass and the area of its horizon to decrease over time.

The Third Law

It is not possible to form a black hole with vanishing surface gravity. κ = 0 is not possible to achieve.

Black Hole temperature

The four laws of black hole mechanics suggest that one should identify the surface gravity of a black hole with temperature and the area of the event horizon with entropy, at least up to some multiplicative constants.

If one only considers black holes classically, then they have zero temperature and, by the no hair theorem, zero entropy, and the laws of black hole mechanics remain an analogy.

However, when quantum mechanical effects are taken into account, one finds that black holes emit thermal radiation (Hawking radiation) at temperature

$$T_H = \frac{\kappa}{2\pi}$$

$$T_H = \frac{\hbar c^3}{8\pi G M k_b}$$

Black Hole entropy

Similarly we can interpret the area law of a Black Hole as an entropy if we divide this area by a fundamental area

$$S_{black\ hole} = \frac{1}{4}\frac{A}{\ell_P^2}$$

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}}$$

Extended second principle

Generalized second law introduced as total entropy = black hole entropy + outside entropy

Bekenstein-Hawking Entropy bound

Bekenstein argued that black holes are maximum entropy objects—that they have more entropy than anything else in the same volume. In a sphere of radius *R*, the entropy in a relativistic gas increases as the energy increases. The only limit is gravitational; when there is too much energy the gas collapses into a black hole. Bekenstein used this to put an upper bound on the entropy in a region of space, and the bound was proportional to the area of the region. He concluded that the black hole entropy is directly proportional to the area of the event horizon.

Bekenstein Entropy Bound

In physics, the **Bekenstein bound** is an upper limit on the entropy *S*, or information *I*, that can be contained within a given finite region of space which has a finite amount of energy—or conversely, the maximum amount of information required to perfectly describe a given physical system down to the quantum level. It implies that the information of a physical system, or the information necessary to perfectly describe that system, must be finite if the region of space and the energy is finite. In computer science, this implies that there is a maximum information-processing rate for a physical system that has a finite size and energy, and that a Turing machine, with its unbounded memory, is not physically possible unless it has an unbounded size or energy. The universal form of the bound was originally found by Jacob Bekenstein as the inequality

$$S \leq \frac{2\pi k_b R E}{\hbar c}$$

Holographic principle

The holographic principle is based on the idea that for a given volume V the state of maximal entropy is given by the largest black hole that fits inside V.

This implies that the maximal entropy in any region scales with the radius *squared*, and not cubed as might be expected. In the case of a black hole, the insight was that the description of all the objects which have fallen into the hole, can be entirely contained in surface fluctuations of the event horizon.

Conformal field theory in 2D

- Let us consider a field theory in a 2d spacetime
- Generally covariant i.e. Invariant under diffeomorphisms
- Locally scale invariant i.e. Under Weyl transformations
- We can describe it by complex coordinates

$z, \bar{z}$

- And consider the symmetries of the theory under holomorphic and antiholomorphic diffeomorphisms

Conformal Field Theory in 2D

- These diffeomorphims are generated by "Virasoro generators"
- Which satisfy the commutator relations

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}$$

- Where L_{-1} , L_0 , L_1 form an invariant subalgebra
- If we represent this operators as

$$L_n = z^{n+1}\frac{\partial}{\partial z} \bar{L}_n = \bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$$

Conformal field theory in 2D

We can think of L_0 as the generator of the scale transformations. Cardy discovered that the density of states satisfied asymptotically the following relation

$$\rho(\Delta,\bar{\Delta}) \sim 2\pi\left\{\sqrt{\frac{c_{eff.}\Delta}{6}}+\sqrt{\frac{\bar{c}_{eff.}\bar{\Delta}}{6}}\right\}$$

$$c_{eff} = c - 24\Delta_0 \quad \bar{c}_{eff} = \bar{c} - 24\bar{\Delta}_0$$

Conformal field theory

With the words of Carlip:"A typical black hole is neither two-dimensional nor conformally invariant, ofcourse, so this result may at first seem irrelevant. But there is a sense in which black holes become approximately two-dimensional and conformal near the horizon. For fields in a black hole background, for instance, excitations in the r–t plane become so blue shifted relative to transverse excitations and dimensionful quantities that an effective two-dimensional conformal description becomes possible."

Cardy formula for entropy in CFT

In 1986 J. Cardy showed, using modular invariance, that entropy of a of a 1+1 dimensional CFT is given by the expression

$$S = 2\pi\sqrt{\tfrac{c}{6}\left(L_0 - \tfrac{c}{24}\right)} \qquad (1)$$

$$L_0 = ER$$

$\frac{c}{24}$ is the shift of energy due to the Casimir energy.

Aiming to extend this result to any dimension, Verlinde noticed,
by using the holographic principle and the ADS/CFT correspondence,
that there is a striking similarity between
the Cardy formula for entropy and the first of the FLRW equations

He observed that making in equation (1) the following substitutions

$$2\pi L_0 \Rightarrow \frac{2\pi}{n} ER \quad (1)$$

$$2\pi \frac{c}{12} \Rightarrow (n-1)\frac{V}{4GR}$$

$$S \Rightarrow (n-1)\frac{HV}{4G}$$

One obtains the first of the Friemann cosmological equations for a closed n+1 dimensional universe

$$H^2 = \frac{16\pi G}{n(n-1)}\frac{E}{V} - \frac{1}{R^2} \qquad (2)$$

$$\dot{H} = -\frac{8\pi G}{n-1}\left(\frac{E}{V} + p\right) + \frac{1}{R^2}$$

In other words the two terms in the right-hand side of equation (2) are proportional to the Bekenstein and the Bekenstein-Hawking entropy bounds, then the left-hand side must also be interpreted as the total entropy. Comparison with the Cardy formula confirms this interpretation

Gravitational collapse of a homogeneous star

This model describes the spherically symmetric gravitational collapse of a homogeneous star.
Indeed from the Einstein equations are
(see Weinberg Gravitation and Cosmology pag. 342)

$$-2k - \ddot{R}(t)R(t) - 2\dot{R}^2(t) = -4\pi G\varrho(0)R^{-1}(t)$$

$$\ddot{R}(t)R(t) = -\tfrac{4\pi G}{3}\varrho(0)R^{-1}(t)$$

$$\Downarrow$$

$$\dot{R}^2(t) = -k + \tfrac{8\pi G}{3}\varrho(0)R^{-1}(t).$$

which coincides with equation (2) in 3+1 dimensions.

Since $k = \frac{R_S}{R_0} \leq 1$

Verlinde's substitutions are slightly modified in the following way

$$\frac{2\pi}{n\sqrt{k}} \frac{ER}{\ell_p^2} \equiv \frac{S_B}{\sqrt{k}}$$

$$2\pi \frac{c}{12} \Rightarrow (n-1) \frac{\sqrt{k}V}{4GR\ell_p^2} \equiv (n-1)\sqrt{k}S_{BH}$$

$$S \Rightarrow (n-1) \frac{HV}{4G\ell_p^2} \equiv (n-1)S_H$$

Notations

$$\ell_p = (\hbar G/c^3)^{1/2}$$

$$R_S = \frac{2GM}{c^2} = \frac{2G}{c^2} \frac{4\pi\varrho}{3} R^3 = \frac{8\pi G}{3c^4} \frac{E}{V} R^3$$

Entropy relations

S_B is the Bekenstein bound

S_{BH} is the Bekenstein – Hawking bound

S_H is the total entropy (H is for Hubble)

$$S_{BH} = \left(\frac{R}{R_S}\right) S_B$$

$$4S_H^2 + \left(\frac{S_B}{\sqrt{k}} - 2\sqrt{k}S_{BH}\right)^2 = \frac{S_B^2}{k}$$

$$\Downarrow$$

$$S_H^2 = S_{BH}\left(S_B - kS_{BH}\right)$$

$$\Downarrow$$

$$S_H(R) = S_{BH}(R)\sqrt{R_S\left(\frac{1}{R} - \frac{1}{R_0}\right)}$$

$$\Downarrow$$

$$S_H(R_S) = S_{BH}(R_S)\sqrt{1 - \frac{R_S}{R_0}} \qquad (3)$$

This means that when a horizon forms the total entropy takes the value given by equation (3) which is less than the area bound, in the following we show how the bound is saturated.

Application of the Cardy-Verlinde equation to the gravitational collapse

After the whole mass has formed the event horizon, the collapse continues under the Schwarzschild horizon by increasing the density ρ we want to describe how similarly entropy increases too.

We observe that there is relation between the Schwarzshild radius and the density at the time of the black hole formation (see Chandrasekhar 1964, Stornaiolo 2001)

$$M^2 \propto R_S^2 = \frac{3c^2}{8\pi G}\frac{1}{\rho_S}$$

Formation of black holes under the horizon

a smaller black hole with mass M(1) forms when the density takes the value

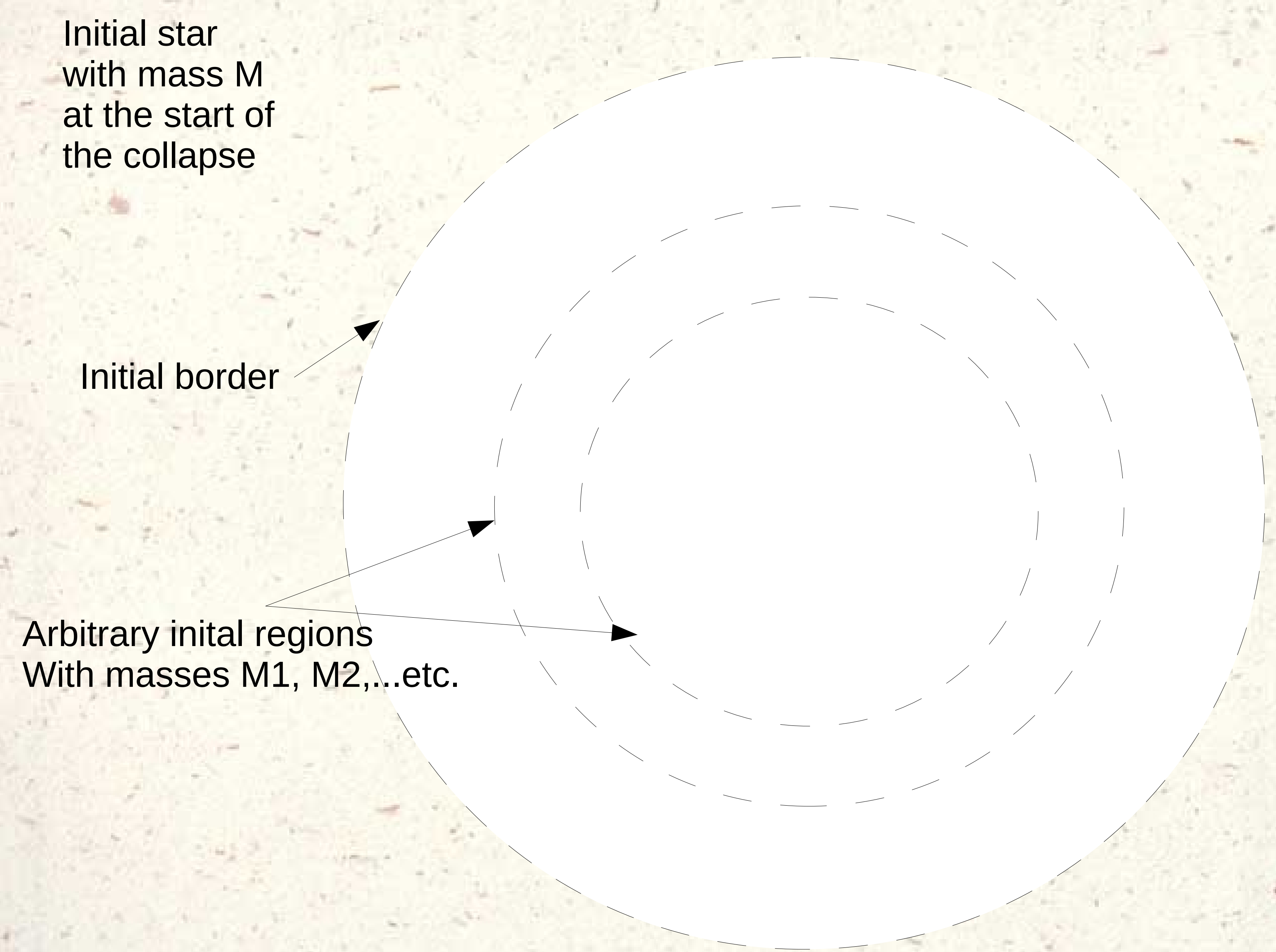
$$\rho^{(1)} = \frac{3c^6}{32\pi G^3 M^{(1)\,2}}$$

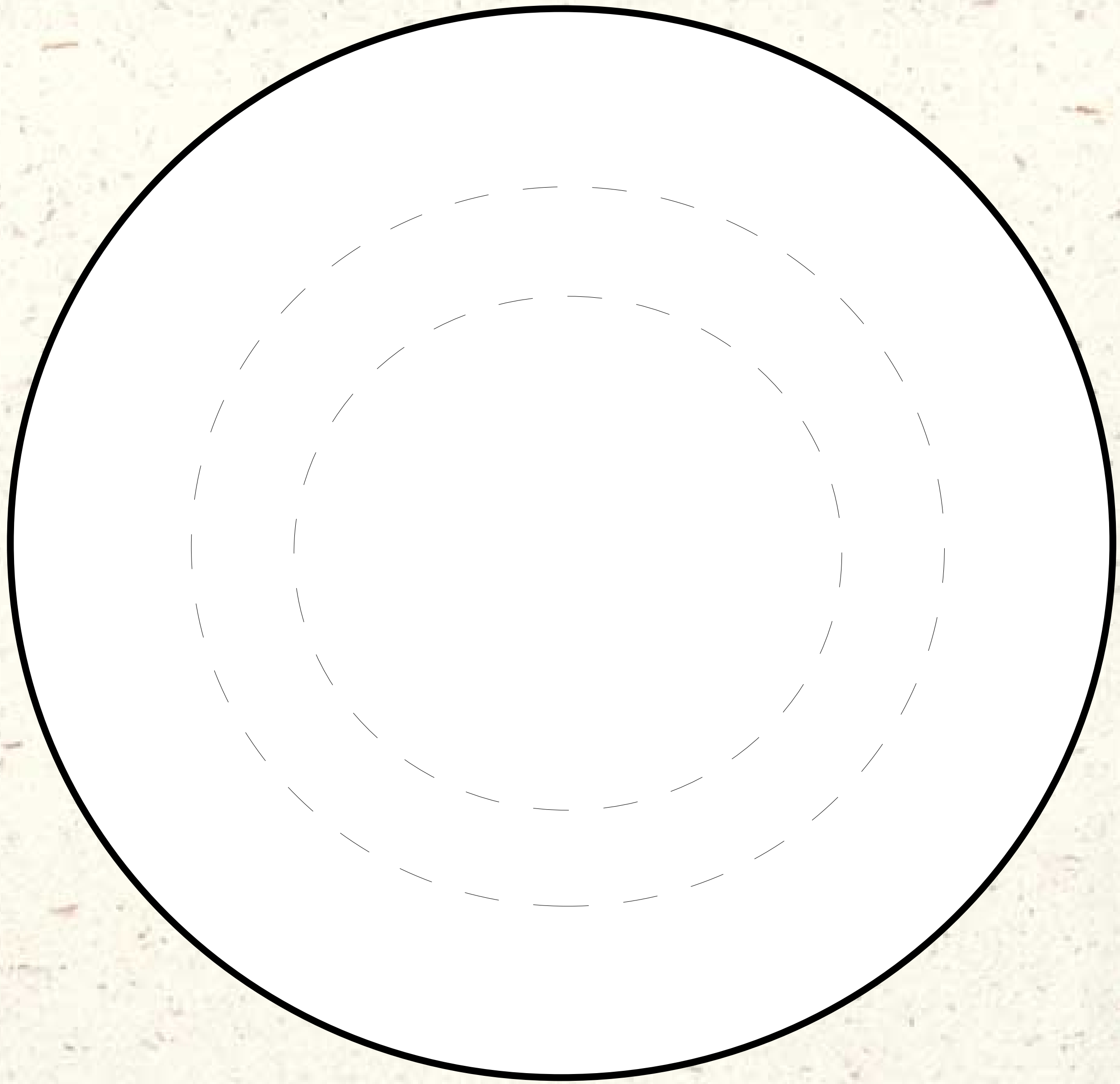
according to equation (3) with entropy

$$S_H(R_S^{(1)}) = S_{BH}(R_S^{(1)})\sqrt{1-\frac{R_S^{(1)}}{R_0^{(1)}}}$$

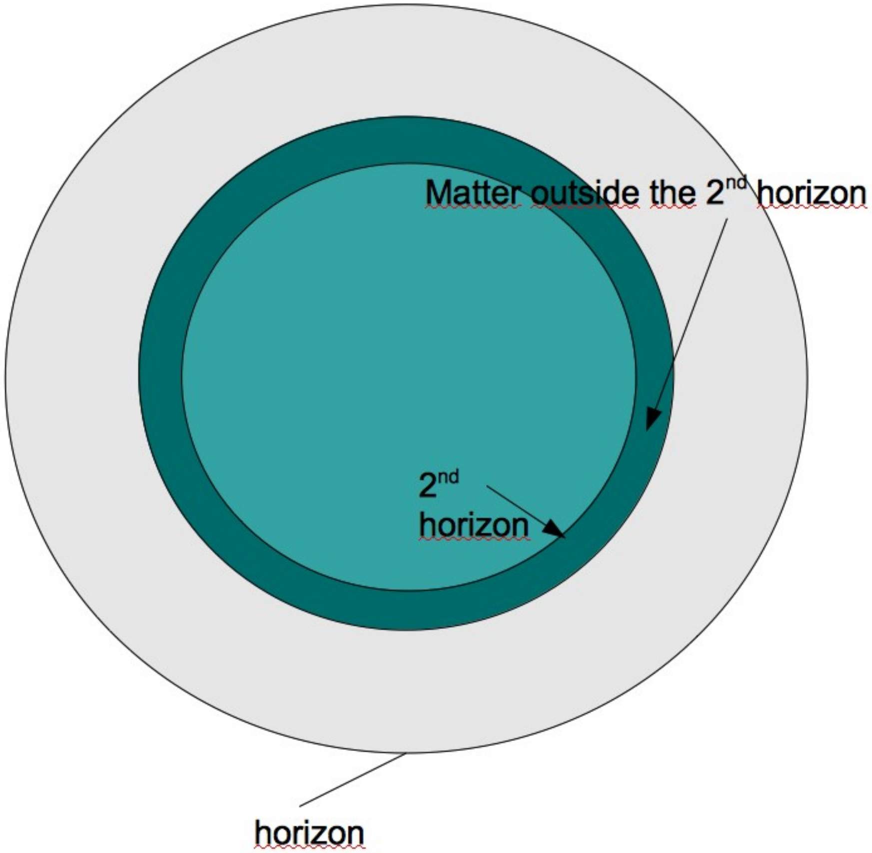
afterwards the black hole (1) is accreted by the residual matter, its horizon grows to the Schwarzschild radius R_{S}, and finally the entropy of the total black hole with mass M grows up to the value

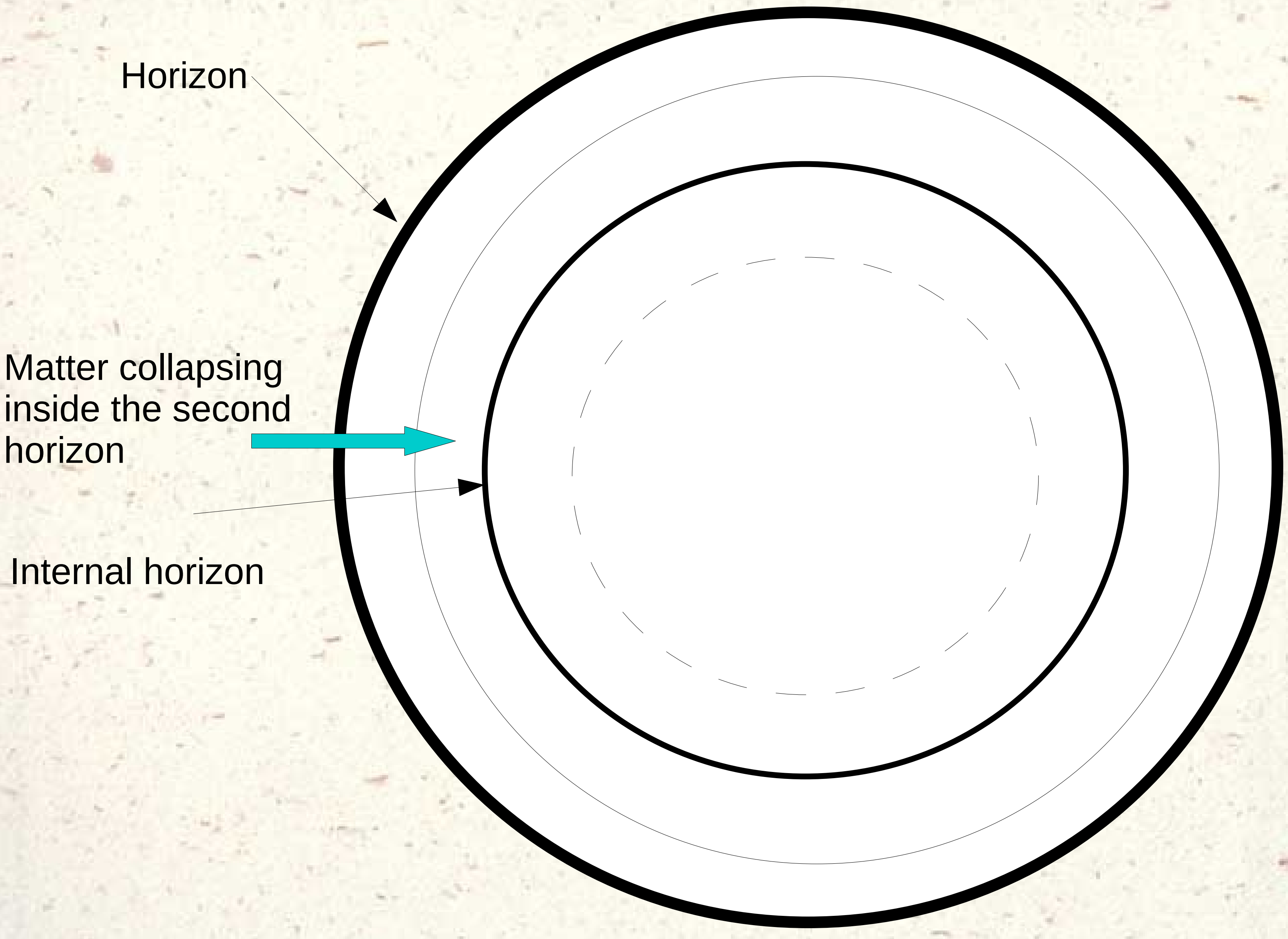
$$S_H(R_S) \geq S_{BH}(R_S)\sqrt{1-\frac{R_S^{(1)}}{R_0^{(1)}}}$$


Initial star
with mass M
at the start of
the collapse
Initial border
Arbitrary inital regions
With masses M1, M2,...etc.



Formation of a second horizon inside the black hole




Horizon
Matter collapsing inside the second horizon
Internal horizon

We have to think that this process is iterated until the singularity forms. and that at any increment n the final entropy grows. Indeed the ratio

$$\frac{R_S^{(n)}}{R_0^{(n)}} = \left(\frac{\rho_0}{\rho^{(n)}}\right)^{1/3}$$

where ρ_0 is the density of the spherical body at the starting time of the collapse.

$$n \to \infty \quad \Longrightarrow \quad \rho^{(n)} \to \infty$$

$$S_H(R_S) = \lim_{n\to\infty} S_{BH}(R_S)\sqrt{1 - \left(\frac{\rho_0}{\rho^{(n)}}\right)^{1/3}} = S_{BH}(R_S)$$

i.e. the Hubble entropy reaches the value of the area entropy.

We stress that the construction introduced here recalls the one used by (Balasubramanian,1999) in association with a renormalization group equation.

Actually,we prove that the saturation of the holographic bound follows from the evolution of the equation of state described by a renormalization group equation. Our aim is to show that these results have very far reaching consequences. In such a context we wish to analyze briefly the relation between entropy, c-theorem and entanglement. Let us start with a brief analysis of the c-theorem in 2d CFT2 by stressing its relation with the black hole entropy and to the generalized second principle of thermodynamics.

Phase transitions inside a black hole

Equation $S_H^2 = S_{BH}\left(S_B - kS_{BH}\right)$ which is the Cardy-Verlinde equation

can be obtained by multiplying equation

$$H^2 = \frac{16\pi G}{n(n-1)}\frac{E}{V} - \frac{1}{R^2} \qquad \frac{3V^2}{16\pi G R^2 \ell_p^4}$$

It can be written as $$S_H^2 \equiv \frac{3V^2 H^2(t)}{16\pi G \ell_p^4} = -\frac{3kV^2}{16\pi G R^2 \ell_p^4} + \frac{\varrho V^2}{2\ell_p^4}$$

The preceding results suggest to look for the equation of state such that

$$\frac{\varrho V^2}{2\ell_p^4} \to constant \quad \text{i.e.} \quad \rho \propto \frac{1}{R^6}$$

Generalized equations of state in cosmology

Such behavior is possible only if $P = \rho$

Since we are starting with a dust fluid $P = 0$

$$\dot{\rho} = -3\frac{\dot{R}}{R}(\rho + P) \longrightarrow P = \left(-\frac{1}{3}\frac{R}{\rho}\frac{d\rho}{dR} - 1\right)\rho$$

$$\gamma = -\frac{1}{3}\frac{R}{\rho}\frac{d\rho}{dR}$$

$$P = (\gamma(\rho) - 1)\,\rho \quad \text{or} \quad \mathrm{P} = (\gamma(\mathrm{R}) - 1)\,\rho$$

Relation between the equation of state and the scale or the density

$$\frac{c_s^2}{c^2} = \frac{dP}{d\rho}$$

$$\frac{c_s^2}{c^2} = \frac{dP}{d\rho} = \rho\frac{d\gamma}{d\rho} + \gamma(\rho) - 1$$

$$\frac{c_s^2}{c^2} = \frac{dP}{d\rho} = -\frac{1}{3}\frac{R}{\gamma}\frac{d\gamma}{dR} + \gamma(R) - 1$$

Both these equations take the form of renormalization group equations for the adiabatic index during the gravitational collapse.

Assigning the speed of sound as a function of r one is able solve the equation (*) which is a Bernoulli equation.

$$\gamma(R) = \frac{\exp\left(-3\int_{R_0}^{R} dR' F(R')/R'\right)}{-3\int_{R_0}^{R} 1/R' \exp\left(-3\int_{R_0}^{R'} dR'' F(R'')/R''\right) dR'}$$

Properties of the stiff matter

$$P = (\gamma - 1)\rho$$

$$\gamma = 2$$

$$\rho = \mathcal{A} R^{-6} \qquad \mathcal{A} = 18\frac{G^4 M^4}{c^8}$$

Stiff matter properties have been widely studied. In
W. Fischler and L. Susskind, Holography and cosmology,'' (1998)
T. Banks and W.Fischler,``M-theory observables for cosmological space-times,(2001)
it has been proved that in any FLRW with a homogeneous fluid and equation of state

$$P = \rho$$

the entropy contained within a sphere of radius equal to the particle horizon
scales with the area of the sphere.

Self gravitating fluid with a central black hole

Considering a self-gravitating fluid with a central black hole (see Zurek and Page(1984)) and when the Hawking radiation produced by a black hole is treated as a self-screening atmosphere(see t' Hooft, (1998)), it was proved that the ratio between the entropy S and the area Σ is for any equation of state

$$P = (\gamma - 1)\rho$$

$$\frac{S}{\Sigma} = \frac{\gamma}{7\,\gamma - 6}$$

it comes out that has the interesting property that this ratio is equal to ¼ when

$$\gamma = 2$$

Some dimensional considerations

So far we have considered the usual gravitational collapse in 3+1 dimensions, but it is easy to show that all the results obtained in the previous sections can be directly extended to any n+1 spacetime

$$H^2 = \frac{16\pi G}{n(n-1)}\frac{E}{V} - \frac{k}{R^2} \qquad \dot{\rho} + n\frac{\dot{R}}{R}(\rho + P) = 0$$

For n larger than 3 we find again the relation

$$\rho \propto \frac{1}{R^2}$$

indeed in P. Diaz, M. A. Per and A. Segui, ``A fluid of black holes at the beginning of the universe'' (2005) it was shown that at any dimension the number density

$$\mathcal{N} \sim R_s^n$$

combined with the relation for the mass $M \sim R_s^{n-2}$

Gives again this relation which we used to describe the gravitational collapse and the formation of the infinite sequence of black holes

Conclusions

We have analyzed the gravitational collapse from the point of view of the holographic principle and the generalized second principle.
we found by using the Cardy-Verlinde formula that the holographic principle Is self-consistent.
But it seems that this can happen only if there is a final phase transition to a fluid with equation of state

$$P = \rho$$

It follows that the thermodynamics at very high densities corresponds to the thermodynamics of a conformal theory in 1+1 dimensions.

We showed that these analysis, can be extended to any dimension n. This implies that the quantum processes at very high densities can be treated with the methods of conformal field theory at any dimension.

We found that the quantum effects implicit in the Cardy-Verlinde equation (and exact in 2 dimensions) "cover" the classical singularity, because the generation of an infinite sequence of internal concentrical black holes lead to a production of Hawking radiation at an increasing rate so that the collapse is accompanied by a phase transition (of quantum origin) of matter into radiation.

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Beyond the conclusions....

During this work we have found that the process of black hole formation involves the formation of an infinite sequence of smaller black holes. We just mentioned that this implies the production of Hawking radiation inside the black hole, which becomes more Intense when the black holes formed are smaller.

In the papers of Zurek and Page, and 't Hooft , a generalization of the Tolman-Oppenheimer-Volkoff equation was considered for a self gravitating fluid composed by the Hawking particles containing a central black hole.
The radiation "feels" a negative point mass at r=0.

Here we find a similar configuration, but in a dynamical evolution and out of thermodynamic equilibrium.

It is our purpose to analyze in deeper detail in a further paper.perhaps by a numerical analysis

Let us resume some points to be discussed

The results found in this work suggest that

- The equation of state for stiff matter, may has physical meaning at very high densities and at very high energies, very close to the classical singularity.

- Matter becomes stiff when gravitation is the prevailing interaction between the microscopical constituents of the fluid;

- There is a phase transition where all the matter is converted into Hawking radiation

- At very small scales the black holes change their nature because quantum effects can modify the effective gravitational lagrangian with new contributions quadratic terms in the scalar curvature and in the Ricci tensor;

- According to many theories torsion effects appear;

- Does singularity really forms?

Analogies with QFT

The formation and subsequent vanishing, by emission of Hawking radiation, of the black holes is a dynamical quantum process.
It looks very similar to the renormalization group in QFT.
In fact the collapse, classically would never end (i.e. at the centre of the "sphere" one finds a "singularity" but the quantum fluctuations of the vacuum become stronger and stronger when the system approaches the singularity.
In a pictorial way we can think that the fluctuations produce small black holes which screen the "would be" singularity at the center of the sphere.
One would recognize a similarity with the "atmosphere" introduced in Zurek and Page (1984) and 'tHooft (1997).
Both pictures remind us also of the methods of "imaging" used to find the solutions of electromagnetic static potentials.

An analogy with the Quantum Hall effect?

In such a context we notice a quite original similarity with the Quantum Hall Effect (QHE) [See e.g. G. Cristofano, D. Giuliano and G.Maiella, ``Effective Lagrangian, Casimir energy and conformal field theory description of the quantum Hall effect'' (1994).]
In fact the QHE at Laughlin fillings v=1/(2p+1) can be described by a CFT$_2$ with central charge c=1 .
It has been noted that the boundary (Casimir) effect induces the correct electric force of sign opposite to the repulsive force between electrons. That results on the ''confinement'' of them inside the cylinder (no spilling out!).
Naturally the trace anomaly of the tensor $T_{\mu\nu}$ there is equal to c/6 (as for the gravity) which gives raising to the Casimir energy and for the Black Hole in ADS$_3$ to the dissipative Hawking radiation [see G.Maiella,``Black Hole in ADS and Quantum Field Theory'' arXiv:0802.4177 [hep-th].]