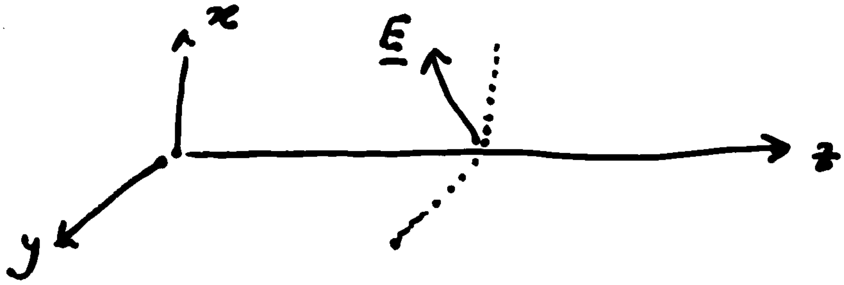
Pancharatnam, Bargmann and Berry Phases - A Retrospective

1983 Berry discovery of Geometric Phase in QM of adiabatic evolution

Generalisations, Applications — Precursors - S. Pancharatnam, V. Bargmann

S. Pancharatnam (1956) Interference phenomena in classical polarization optics



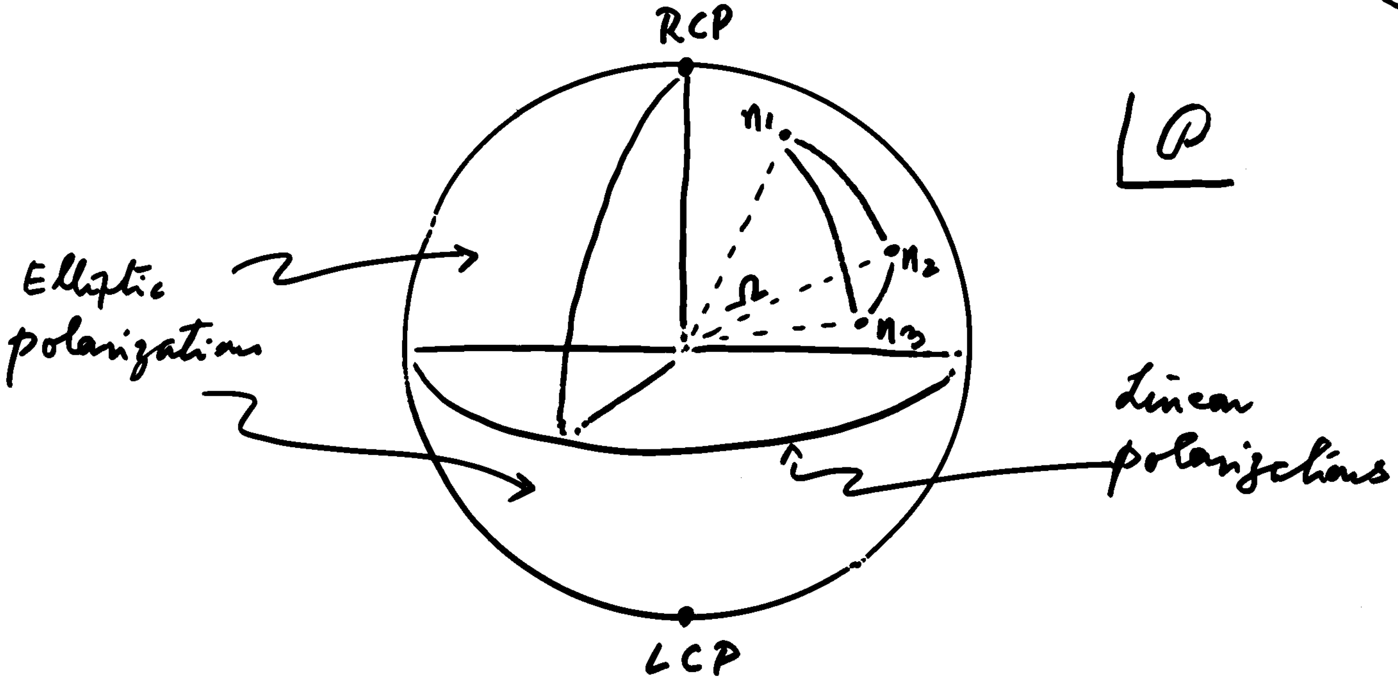
$\omega, \underline{k}$ fixed

$$\underline{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} \in \mathcal{H}, \text{ 2 dim. Hilbert space}$$

Amplitude $\underline{E}$ → Intensity $I = \underline{E}^\dagger \underline{E}$

Amplitude $\underline{E}$ → Polarization state $\sim \hat{n} = \frac{1}{I}\, \underline{E}^\dagger \underline{\sigma}\, \underline{E} \in S^2$

$I, \hat{n}$ invariant under $\underline{E} \to e^{i\alpha} \underline{E}$



Coherent superposition of two amplitudes $E = E^{(1)} + E^{(2)}$

Total intensity has interference term:

$$I = E^\dagger E = I^{(1)} + I^{(2)} + 2\,\mathrm{Re}\; E^{(1)\dagger} E^{(2)}$$

I = maximum <=> maximum constructive interference

<=> $E^{(1)\dagger} E^{(2)}$ real positive

<=> 'in phase' in Pancharatnam sense,

an amplitude level definition.

Transitivity question

$E^{(1)\dagger} E^{(2)}$, $E^{(2)\dagger} E^{(3)}$ real positive : what about $E^{(1)\dagger} E^{(3)}$?

Single complex numbers $\alpha, \beta, \gamma \ldots$: transitivity holds:

$$\alpha^* \beta,\ \beta^* \gamma = \text{real positive} \Rightarrow \alpha^* \gamma = \frac{\alpha^* \beta\, \beta^* \gamma}{|\beta|^2} = \text{real positive too.}$$

But for two component complex electric field amplitudes – no transitivity!

$$\arg E^{(1)\dagger} E^{(3)} = \frac{1}{2}\Omega,$$

Ω = Solid angle or area of spherical triangle $\hat{n}_1\, \hat{n}_2\, \hat{n}_3$ on P.

'In phase' concept is at amplitude level, measure of lack of transitivity is calculable on P.

Bargmann (1964)

Pure states of a QM system described by vectors $\psi, \phi, \ldots$ in a complex Hilbert space $\mathcal{H}$. Probability interpretation $\rightarrow$ limit ourselves to normalised or unit vectors:

$\mathcal{H}$ = space of vectors $\psi, \psi', \phi, \phi' \ldots$, inner product $(\psi', \psi) \rightarrow$ unit sphere $\mathcal{B} = \{\psi \in \mathcal{H} \mid \|\psi\|^2 = (\psi, \psi) = 1\} \subset \mathcal{H}$.

But: ψ and $e^{i\alpha}\psi$ give same pure state, so go to ray space $\mathcal{R}$:

$\mathcal{H}$: complex vector space of all $\psi, \psi', \ldots$

$$\downarrow$$

$\mathcal{B} = \{\psi \in \mathcal{H} \mid \|\psi\| = 1\} \subset \mathcal{H}$

$$\downarrow$$

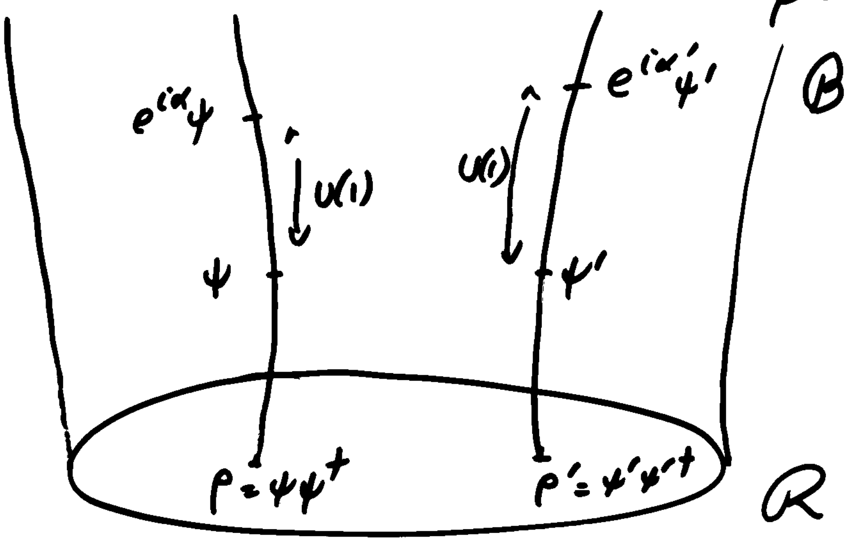
$\mathcal{R}$: space of unit rays $= \{\rho = \psi\psi^\dagger \mid \psi \in \mathcal{B}\}$

U(1)-bundle

Projection $\pi : \mathcal{B} \rightarrow \mathcal{R} : \psi \in \mathcal{B} \longrightarrow \pi(\psi) = \rho_\psi \in \mathcal{R}$.

$\mathcal{R} \longleftarrow$ one to one $\longrightarrow$ pure physical states

$\rho_1, \rho_2 \in \mathcal{R}$: $\mathrm{Tr}(\rho_1 \rho_2) = |(\psi_1, \psi_2)|^2 =$ quantum mechanical probability



<u>Wigner Theorem on Symmetry Operations in QM</u>

Symmetry = map $\mathcal{J} : \mathcal{R} \to \mathcal{R}$ preserving probabilities

$\rho_1' = \mathcal{J}(\rho_1), \rho_2' = \mathcal{J}(\rho_2)$: $\mathrm{Tr}(\rho_1' \rho_2') = \mathrm{Tr}(\mathcal{J}(\rho_1)\mathcal{J}(\rho_2)) = \mathrm{Tr}(\rho_1 \rho_2)$

<u>Theorem</u> Such $\mathcal{J}$ on $\mathcal{R}$ can be lifted to T acting on $\mathcal{B}$ (and $\mathcal{H}$):

$$\rho = \psi\psi^\dagger : \quad \mathcal{J}(\rho) = T\psi\,(T\psi)^\dagger = T\rho T^\dagger ,$$

T <u>either</u> linear unitary <u>or</u> antilinear antiunitary

$$\begin{tikzcd}
\psi_1, \psi_2 \in \mathcal{B} \arrow[rr, "T = \text{lift of } \mathcal{J}"] \arrow[d, "\text{unique}"] & & \psi_1', \psi_2' \in \mathcal{B} \\
\rho_1, \rho_2 \in \mathcal{R} \arrow[rr, "\mathcal{J}", "\text{unique}"'] & & \rho_1' = \mathcal{J}(\rho_1),\ \rho_2' = \mathcal{J}(\rho_2) \in \mathcal{R} \arrow[u, "\text{nonunique}"']
\end{tikzcd}$$

<u>Bargmann</u> recipe to determine option, given only $\mathcal{J}$:

$$\psi_1, \psi_2, \psi_3 \in \mathcal{B} \rightarrow \text{BI} \quad \Delta_3(\psi_1, \psi_2, \psi_3) = (\psi_1, \psi_2)(\psi_2, \psi_3)(\psi_3, \psi_1)$$
$$= \mathrm{Tr}\,(\rho_1\, \rho_2\, \rho_3)$$

Properties: (i) For $\dim \mathcal{H} \geq 2$, complex in general

(ii) Cyclic symmetry under $\psi_1 \rightarrow \psi_2 \rightarrow \psi_3 \rightarrow \psi_1$

(iii) Invariant under independent phase changes in ψ_1, ψ_2, ψ_3

Under a symmetry $\mathcal{J}$:

$$\Delta_3(\psi_1, \psi_2, \psi_3) = \mathrm{Tr}\,(\rho_1 \rho_2 \rho_3) \xrightarrow{\mathcal{J}} \mathrm{Tr}\,(\mathcal{J}(\rho_1)\, \mathcal{J}(\rho_2)\, \mathcal{J}(\rho_3))$$

$\Delta_3(\psi_1, \psi_2, \psi_3)$ unitary case

$\Delta_3(\psi_1, \psi_2, \psi_3)^*$ antiunitary case

<u>Instant</u> proof of nontransitivity of 'in phase' concept:

$$\Delta_3(E^{(1)}, E^{(2)}, E^{(3)}) = \underline{E^{(1)\dagger} E^{(2)}\, E^{(2)\dagger} E^{(3)}\, E^{(3)\dagger} E^{(1)}} = \text{complex !}$$

<u>Berry (1983)</u>

<u>1926</u> Schrödinger evolution with time dependent Hamiltonian:

$$i\hbar \frac{d\psi(t)}{dt} = H(t)\,\psi(t)$$

Instantaneous eigenvectors, eigenvalues

$$H(t)\,\psi_n(t) = E_n(t)\,\psi_n(t), \quad \{\psi_n(t)\} \text{ O. N. basis in } \mathcal{H}$$

Time independent case:

$$\psi_n(t) = e^{-iE_n t/\hbar}\,\psi_n, \quad H\psi_n = E_n\psi_n$$

1928 Born - Fock : Quantum adiabatic Theorem

No degeneracies, no level crossings.

$(\psi_n(t), \dot{\psi}_n(t)) = 0$ to limit phase freedoms

Approximate solution to Schrodinger equation:

$$\psi(0) = \psi_n(0) \to \psi(t) \simeq e^{-i\int_0^t dt' E_n(t')/\hbar}\,\psi_n(t),$$

provided $\langle \psi_m(t) | \frac{\partial H(t)}{\partial t} | \psi_n(t) \rangle \ll (E_m(t) - E_n(t))^2/\hbar$, $m \neq n$

1983 Berry added cyclic condition : $H(T) = H(0)$.

$$\psi(T) \simeq e^{i\varphi_{tot}}\,\psi(0),$$

φ_{tot} = total phase = $\varphi_{dyn} + \varphi_{geom}$,

φ_{dyn} = dynamical phase $= -\frac{1}{\hbar}\int_0^T dt\, E_n(t)$,

φ_{geom} = ray space quantity = geometric phase

1987 Aharonov - Anandan : no need for adiabaticity.

$$\psi(T) = e^{i\varphi_{tot}}\,\psi(0)\text{ , any cyclic solution:}$$

$$\varphi_{dyn}(t) = -\frac{1}{\hbar}\int_0^t dt'\,(\psi(t'), H(t')\psi(t')),$$

$$\varphi_{geom} = \varphi_{tot} - \varphi_{dyn}(T)$$

1988 Samuel- Bhandari : no need for cyclic condition.

(i) Ray space $\mathcal{R}$ has Riemannian metric $\to$ geodesics. Dim $\mathcal{H} = 2$, $\mathcal{R} = S^2$: arcs of great circles.

(ii) Any two 'nonorthogonal' points in $\mathcal{R} \to$ unique (shorter) geodesic connecting them.

(iii) Any unitary Schrodinger evolution $\to$ connect end points by geodesic $\to$ cyclic 'evolution' $\to$ geometric phase.

	Unitary	Adiabatic	Cyclic	Schr. Equation	GP
Born Fock	✓	✓	—	✓	Not recognised
Berry	✓	✓	✓	✓	Discovered
A - A	✓	—	✓	✓	Exists
S - B	✓	—	—	✓	Exists

1993 R. Simon - N.M. Kinematic approach.

Smooth parametrised curve $\mathcal{C}$ in $\mathcal{B}$:

$$\mathcal{C} = \{\psi(s) \in \mathcal{B} \mid s_1 \le s \le s_2\} \subset \mathcal{B}$$

Demand invariance under

$s \to s' = f(s)$, $\frac{df(s)}{ds} > 0$: reparametrisation

$\psi(s) \to \psi'(s) = e^{i\alpha(s)}\psi(s)$: local phase changes

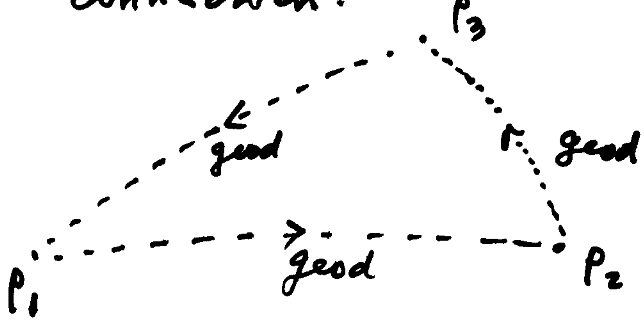
Essentially unique expression

$\varphi_{geom}[\mathcal{C}$, actually $C = \pi[\mathcal{C}]] = \varphi_{tot}[\mathcal{C}] - \varphi_{dyn}[\mathcal{C}]$,

$$\varphi_{tot}[\mathcal{C}] = \arg(\psi(s_1), \psi(s_2)),$$

$$\varphi_{dyn}[\mathcal{C}] = -\,\mathrm{Im}\int_{s_1}^{s_2} ds\,\left(\psi(s), \frac{d\psi(s)}{ds}\right)$$

BI - GP connection:



$\arg\Delta_3(\psi_1, \psi_2, \psi_3) = -\varphi_{geom}$ [geodesic triangle, vertices p_1, p_2, p_3]

$\dim \mathcal{H} = 2$, $\mathcal{R} = S^2 \longrightarrow$ Pancharatnam $\frac{1}{2}\Omega$ result

$\mathcal{R}$ has Riemannian metric; always even dimensional; has classical phase space structure. Then:

Geometric Phases ~ two dimensional symplectic areas.

Some further results

(i) GP's from unitary representations of Lie groups; SU(3).

(ii) GP's for mixed states – definition, interpretation

(iii) 3-level systems, behaviour near degeneracies

(iv) Guoy phase in classical optics → Bargmann invariant, Mp(2) geometric phase.

Null Phase Curves

Saw BI ⟷ GP connection via geodesics.

What is the most general connection?

$\gamma = \{\psi(s) \in \mathcal{B} \,|\, s_1 \leq s \leq s_2\}$ is a NPC $\Longleftrightarrow$

$\Delta_3(\psi(s), \psi(s'), \psi(s'')) =$ real positive, all s, s', s''

Properties

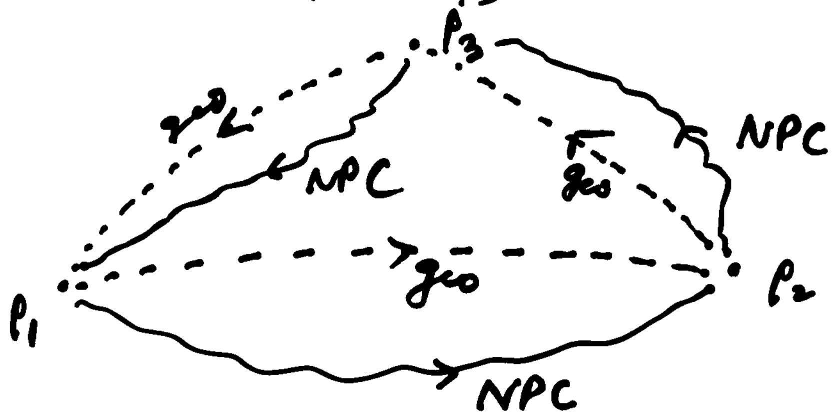
(a) $\dim \mathcal{H} = 2$, $\mathcal{R} = S^2$: NPC's coincide with geodesics.

(b) dim $\mathcal{H} \geq 3$: every geodesic is a NPC but not conversely

(c) dim $\mathcal{H} \geq 3$: given any $P_1, P_2 \in \mathcal{R}$, there are infinitely many NPC's connecting them, but only one geodesic

(d) NPC's are not solutions of any system of finite order ODE's; recall geodesics are solutions to second order ODE's.

(e) Most general NPC connecting $P_1, P_2 \in \mathcal{R}$ can be described explicitly, has nonlocal features



$\arg \Delta_3(\psi_1, \psi_2, \psi_3) = -\varphi_{geom}$ ['triangle' with vertices P_1, P_2, P_3 and sides any connecting NPC's].