One Loop Energy of Short Strings on $AdS_5 \times S^5$

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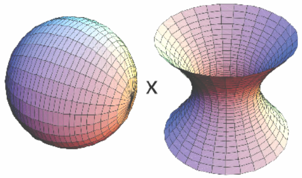
Outline

The very beginning: $AdS_5 \times S^5$ - $\mathcal{N} = 4$ SYM

Maldacena's conjecture states the equivalence $II\ B$ string on $AdS_5 \times S^5$ - $\mathcal{N} = 4$ SYM - Provided the Gauge/String coupling relation $\frac{4\pi\lambda}{N_c} \equiv g_s$.



$\lambda = N_c g^2_{SYM}$

First hint: symmetries

Isometries of $AdS_5 \times S^5$: $SO(4,2) \times SO(6)$

bosonic part of $PSU(2,2|4)$ $\mathcal{N} = 4$ SYM

States on both sides of the duality labelled by the eigenvalues of the Casimir $SO(4,2) \times SO(6)$: $[E = \Delta, S_1, S_2, J_1, J_2, J_3]$ (first three $SO(4,2)$, the others $SO(6)$)

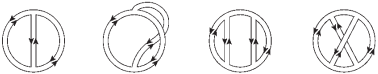
The Basic Prediction of the conjecture: Energy of a string state $E \Leftrightarrow \Delta$ eingenvalue of the Dil. operator (scaling dimensions) for the dual gauge O in $\mathcal{N} = 4$ SYM

$$E(g_s, \frac{R^2}{\alpha'}) = \Delta(\lambda, \frac{1}{N_c})$$

Weak-Strong duality ... Difficult to test!

Large N_c Limit, BPS, almost BPS

First simplification: Large N_c limit ($\frac{4\pi\lambda}{N_c} \equiv g_s$) $N_c \to \infty$, $g_s \to 0$, free String theory (the topology of the worldsheet is a cylinder) and planar gauge theory (only single trace gauge invariant operators)



$\sim N_c^3, \quad N_c, \quad N_c^4, \quad N_c^2$

Still difficult ... we have some perturbative understanding of the two sides in opposite regimes

- String theory $\sqrt{\lambda} = \frac{R^2}{\alpha'} = (\text{NL } \sigma\text{-model coupling})^{-1} \gg 1$
- Gauge theory $\lambda = g_{YM}^2 N_c \ll 1$

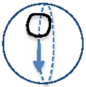
An exception: BPS states ($\mathrm{Tr}Z^L$): susy protected, trivial λ dependence, easy to check the correspondence

The Next step: Not BPS but Near to BPS ...

Almost BPS and Far from BPS

BMN or diluted limit: The idea is to take some state with large charge "J"- almost BPS with impurity: the relevant quantity is an "effective" coupling $\lambda' = \lambda/J^2$; $\sigma-$model corrections are suppressed.

$$\mathrm{Tr}(\underbrace{Z\ldots Z{\color{red}X}Z\ldots Z}_{J})$$



Dual String state almost pointlike, rotating along a big circle with large angular momentum J in S^5

$$J, N_c \to \infty \quad \frac{J}{N} \text{ fixed} \quad \lambda' = \frac{\lambda}{J^2} \text{ fixed} \quad E - J \text{ and } \Delta - J \text{ fixed}$$

BMN suggests: Simple solutions in $AdS_5 \times S^5$ duals of "long" SYM

Generalise to far from BPS states: (more "impurities"), but still some large charge

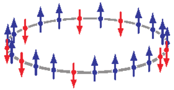
More complete dictionary between "simple" gauge theory operators and "simple", macroscopic solitonic string solutions.

If both string energies and scaling dimensions can be doubly expanded in λ/J^2 and $1/J$, we have a chance to compute both expansion and compare!

How to compute? Bethe Ansatz & Integrability

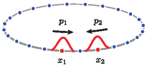
Δ: class. dim. Δ_0 + anomalous $\Delta(g)$ ... diagonalisation $\mathcal{D}$?

Planar, 1-loop $\mathcal{D}$ on $\mathfrak{so}(6)$ single trace operators $\equiv$ (generalised) Spin chain Hamiltonian $\mathcal{H}$



$\longleftrightarrow$ Tr($ZZWZZWZWZZWZZZZWWZZZZWZW$)

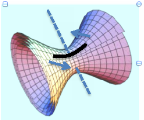
Simple Integrable system: Algebraic Bethe Ansatz to solve! Based on the analysis of the scattering of elementary excitations along the chain. **Integrable system $\equiv$ factorizable S-matrix**

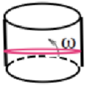


$S(2 \to 2) \Rightarrow S(n \to n)$

Fully generalised to the complete $\mathfrak{psu}(2,2|4)$ algebra
All loop **Asymptotic** Bethe Ansatz Equations
Integrability links the two sides of AdS/CFT and test it!

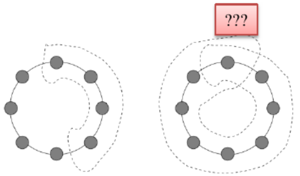
$$\mathrm{Tr}\,\Phi D_+^S \Phi \quad \xleftrightarrow{0<\lambda<\infty}$$





fast rotation

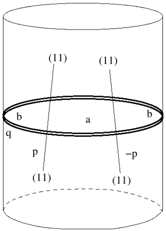
Wrapping Corrections



ABA equations are correct only for L large!
Finite size corrections associated to the "wrapping interactions": the interaction range increases withe the loop order ... at some point (g^{2L}) interaction "overlaps". Bethe ansatz is scattering based ... needs asymptotic states

Exp. suppressed with the size of the chain

Gauge/spin chain description ⇔ String theory/$\sigma-$model



QFT: virtual particles propagating and scattering around the cylinder:

Using Lüscher formulae ⇒ LO & NLO (g) finite size corrections.

Applied to AdS/CFT: wrapping for Konishi

Very hard to generalize to generic states (S-matrix, bound states ...)

TBA, Y and T

Thermodynamical Bethe Ansatz

On the “string side”: The spectrum of the theory at finite volume can be computed through thermodynamic quantities in a “mirror model” with large volume techniques at finite temperature

Trick: Double Wick rotation $\Rightarrow$ thermodynamics $\Rightarrow$ Put the theory on torus $R \gg L$ $\Rightarrow$ Find the free energy in the infinite volume but finite temperature $\Rightarrow$ Switch the meaninig of time and space directions on the torus $\Rightarrow$ interpret the free energy as the ground state in finite volume $L = \frac{1}{T}$

$$\mathcal{Z} = \sum_n e^{-E_n(R)T} = \sum_n e^{-E_n^{\text{mirror}}(T)R}$$

AdS/CFT: The mirror theory is not equivalent to the original one (light cone gauge!)

Y/T-system

For integrable models TBA equations are related to universal Y-system

Set of functional equations for functions $Y_{a,s}(u)$

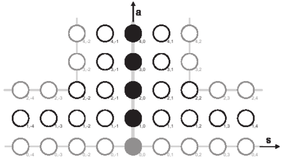
$$\frac{Y^+_{a,s} Y^-_{a,s}}{Y_{a+1,s} Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

Y-system $\equiv$ Hirota bilinear equation

$$T^+_{a,s} T^-_{a,s} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}$$

$$Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}$$

$T_{a,s}(u)$ defined on



“T” lattice for $\mathfrak{psu}(2,2|4)$

Solution of the Y-system $\Rightarrow E_{gs}$ (can be extended to excited states!)

$$E = \sum_j \epsilon_1(u_{4,j}) + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \frac{\partial \epsilon^*_a}{\partial u} \log\left(1 + Y^*_{a,0}(u)\right)$$

... is the problem solved?

Not so easy ... Y-system in principle describes the anomalous dimensions of any physical operator at any coupling ... but

Very difficult to extract infomation from the TBA/Y-system

Weak Coupling

- Konishi LO and NLO Wrapping from Y (OK with Luscher, and explicit diagrammatic)
- 4-loop $\Delta_w^{(8)} = 324 + 864\zeta(3) - 1440\zeta(5)$ + 5-loop order $\Delta_w^{(10)} = -11340 + 2592\zeta(3) - 5184\zeta(3)^2 - 11520\zeta(5) + 30240\zeta(7)$
- Very few other results for the β-deformed SYM, and some orbifold

Strong Coupling

Strong coupling Y-system seems very hard to handle ...

Only few numerical results ... Given a short operator $(\mathrm{Tr}\,\Phi\mathcal{D}_+^S\Phi)$ ... which is the dual string state? How can we compute Δ at strong coupling?

back to earth: semiclassical strings

What can we do in the strong coupling limit?

The quantisation of the Mestaev-Tseytlin string action still is a hard (and unsolved) problem ... but classical string solutions are known!

This is already non trivial! The integrability of the classical model is the key!

Idea: Short operators ⟺ Short Strings

... Take a classical solution, and try semiclassical quantisation

- Not rigorous, without pretending generality
- The hope is that of providing some "useful" hint on the structure of anomalous dimension at strong coupling analysing some "simple" classical solution

Two methods: **Using "standard" QFT techniques:**

- Not "explicitly" based on classical integrability of the string sigma model ... (but integrability will appear! at the one-loop level integrable finite gap Lamé equation)

Algebraic curve: Alternative approach, more formal, integrability based

Semiclassical Expansion

- Semiclassical strings are defined in the limit of "large" charges
- The charge essentially measures the length of the string:
  e.g. rotating string, the rotation (angular momentum) balances the contracting effect of the string tension
- We will consider simple cases, only one charge different from zero

We want short strings $\Rightarrow$ Small charges!

We may expand at large λ with $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ fixed, (the semiclassical string limit) and re-expand *then* in the limit $\mathcal{J} \ll 1$, i.e. $J \ll \sqrt{\lambda}$

$$\mathcal{E}_k = \sqrt{\mathcal{J}}\left[a_{0,k} + a_{1,k}\mathcal{J} + a_{2,k}\mathcal{J}^2 + \dots\right], \qquad k = \text{loop order}$$

$$E = \sqrt{\sqrt{\lambda}\,J}\,\left[a_{0,0} + \frac{a_{1,0}J + a_{0,1}}{\sqrt{\lambda}} + \frac{a_{2,0}J^2 + a_{1,1}J + a_{0,2}}{(\sqrt{\lambda})^2} + \dots\right]$$

$\sim$ near flat space expansion $E(\sqrt{\lambda}, \mathcal{J}) = 2\sqrt{n-1}\,\lambda^{1/4} + \sum_{k=0}^{\infty} \frac{b_k}{(\lambda^{1/4})^k}$

The same structure! Not rigorous ... but promising

Set up: Metsaev-Tseytlin Action I

For quadratic fluctuations we can consider the reduced action

$$I = -\frac{\sqrt{\lambda}}{2\pi} \int d^2\xi \,[\mathcal{L}_B + \mathcal{L}_F]\,, \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'}\,.$$

The bosonic Lagrangian is

$$\mathcal{L}_B = \frac{1}{2}\sqrt{-g}\, g^{ab}\left[G^{(AdS_5)}_{mn}(X)\,\partial_a X^m \partial_b X^n + G^{(S^5)}_{m'n'}(Y)\,\partial_a Y^{m'}\partial_b Y^{n'}\right].$$

We take $\xi^a = (\tau, \sigma)$ with periodicity in σ. The metric on $AdS_5 \times S^5$ is as usual

$$\begin{aligned} ds^2_{AdS_5} &= d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\,(d\theta^2 + \cos^2\theta\, d\phi_1^2 + \sin^2\theta\, d\phi_2^2), \\ ds^2_{S^5} &= d\gamma^2 + \cos^2\gamma\, d\varphi_3^2 + \sin^2\gamma\,(d\psi^2 + \cos^2\psi\, d\varphi_1^2 + \sin^2\psi\, d\varphi_2^2). \end{aligned}$$

The fermionic Lagrangian is

$$\mathcal{L}_F = i\,(\sqrt{-g}\, g^{ab}\,\delta^{IJ} - \varepsilon^{ab}\, s^{IJ})\,\bar{\theta}^I \rho_a\, D_a \theta^J + o(\theta^4)$$

where $I, J = 1, 2$ are the indices of the spacetime fermions, $s^{IJ} = \mathrm{diag}(1, -1)$.

Set up: Metsaev-Tseytlin Action II

D_M^{IJ} is the 10d covariant derivative appearing in the supergravity equations of motion in terms of the spin connection and RR 5-form

$$D_M^{IJ} = \left(\partial_M + \frac{1}{4}\omega_M^{AB}\Gamma_{AB}\right)\delta^{IJ} - \frac{1}{8\cdot 5!}F_{A_1\ldots A_5}\Gamma^{A_1\ldots A_5}\Gamma_M\,\varepsilon^{IJ}.$$

For 10d MW spinors, and using the specific form of F

$$\begin{aligned} D_a\theta^I &= \left(\mathrm{D}_a\delta^{IJ} - \frac{i}{2}\varepsilon^{IJ}\Gamma_*\rho_a\right)\theta^J, \\ \mathrm{D}_a &= \partial_a + \frac{1}{4}\partial_a X^M\omega_M^{AB}\Gamma_{AB}, \\ \Gamma_* &= i\Gamma_{01234}, \quad \Gamma_*^2 = 1. \end{aligned}$$

Fixing κ symmetry with $\theta^1 = \theta^2$ we have the further simplification

$$\begin{aligned} \mathscr{L}_F &= -2i\bar{\theta}D_F\theta, \\ D_F &= -\rho^a\mathrm{D}_a - \frac{i}{2}\varepsilon^{ab}\rho_a\Gamma_*\rho_b. \end{aligned}$$

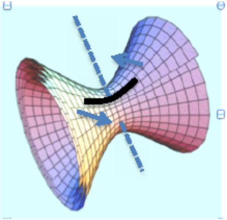
Example I: Rotating folded string in AdS_3

$$ds^2 = -\cosh^2\rho\ dt^2 + d\rho^2 + \sinh^2\rho\ d\phi^2$$

Classical closed string solution given by

$$t = \kappa\tau, \quad \phi = \omega\tau, \quad \rho = \rho(\sigma) = \rho(\sigma + 2\pi),$$

κ, ω are constant parameters. The equation of motion and its solution in conformal gauge



$$\rho'^2 = \kappa^2 \cosh^2\rho - w^2 \sinh^2\rho,$$

$$\sinh\rho(\sigma) = \frac{k}{\sqrt{1-k^2}}\,\mathrm{cn}(\omega\sigma + \mathbb{K}\,|k^2) \quad \rho'(\sigma) = \kappa\,\mathrm{sn}(\omega\sigma + \mathbb{K}\,|\,k^2)$$

ρ varies from 0 to its maximal value ρ_0: $\coth^2\rho_0 = \frac{\omega^2}{\kappa^2} \equiv 1 + \eta \equiv \frac{1}{k^2}$

Small spin or short string limit: $\rho_0 \to 0$, i.e. $\eta \to \infty$ or $k \to 0$ In the "short string" limit, when the string is rotating in the small central ($\rho = 0$) region of AdS_3, the spin is small and the parameter η is large

$$\mathcal{E}_0 = \sqrt{2\mathcal{S}}\left(1 + \frac{3}{8}\mathcal{S} + \ldots\right) \quad \mathcal{S} \ll 1.$$

1-loop corrections: Strategy

Leading quantum correction to the energy of this solution:

- expanding the action to quadratic order in fluctuations near the classical solution

$$\tilde{I} = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau \int_0^{2\pi} d\sigma \left(\tilde{\mathcal{L}}_B + \tilde{\mathcal{L}}_F\right)$$

- All the fluctuation operators have Lamé form:

$$\left[-\partial_x^2 + \; 2k^2\,\mathrm{sn}^2(x\,|\,k^2)\right] f(x) = \Lambda f(x) \qquad x \sim \sigma$$

semi-classical problem is governed by simple (finite-gap) operators.
Everything can be computed analytically in a closed form!

Integrability at work!!

- Semiclassical Quantization: "find a way to relate the eigenfrequences of the fluctuation aroud the classical solution to 1-Loop Energy"

Folded String: Stationary (simple) $\Rightarrow$ 2d effective action

The folded string is simple! rigid spinning string! solution is stationary, the coeff. in the fluct. Lagrangian do not depend on τ.

Switching to Euclidean time, the 1-loop correction $\Rightarrow$ 2d effective action Γ by dividing over the time interval ($t = \kappa\tau$)

$$E_1 = \frac{\Gamma}{\kappa \mathcal{T}}, \qquad \mathcal{T} \equiv \int d\tau \to \infty, \qquad \Gamma = -\ln Z$$

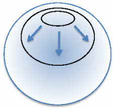
Z ratio of the fermionic and bosonic determinants.

Since the above rigid spinning string solution is stationary, (τ ind. fluctuation Lagrangian) the relevant 2-d functional determinants may be reduced to 1-d determinants

$$\ln\det[-\partial_\sigma^2 - \partial_\tau^2 + M^2(\sigma)] = \mathcal{T} \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \ \ln\det[-\partial_\sigma^2 + \Omega^2 + M^2(\sigma)]$$

Exact Solution Lamé eq. $\Rightarrow$ analytic expressions for the fluctuation determinants $\Rightarrow$ expansions in the small spin /short string limit.

Example II: Pulsating String in $R \times S^2$



$$ds^2 = -dt^2 + d\psi^2 + \sin^2\psi \, d\phi^2 \quad t = \kappa\tau, \quad \psi = \psi(\tau), \quad \phi = m\sigma \,,$$

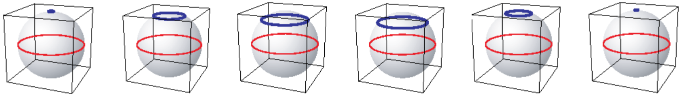
EQM and the conformal gauge constraint

$$\ddot{\psi} + m^2 \sin\psi\cos\psi = 0 \quad \dot{\psi}^2 + m^2 \sin^2\psi = \kappa^2$$

The classical solution $\sin\psi(\tau) = \frac{\kappa}{m}\,\mathrm{sn}\left(m\tau \,|\, \frac{\kappa^2}{m^2}\right), \quad |\sin\psi| \leq \sin\psi_0 = \frac{\kappa}{m}$

Energy and the oscillation number

$$\mathcal{E} = \frac{E}{\sqrt{\lambda}} = \kappa, \quad \mathcal{N} = \frac{N}{\sqrt{\lambda}} = \int_0^{2\pi} \frac{d\psi}{2\pi} \sqrt{\kappa^2 - m^2 \sin^2\psi}$$



Short string expansion of the classical energy

$$\mathcal{E}(\mathcal{N}) = \sqrt{2m\mathcal{N}} \left(1 - \frac{\mathcal{N}}{8m} - \frac{5\mathcal{N}^2}{128m^2} + \dots\right).$$

Almost the same strategy:

- Expand the action around the classical solution and find fluctuation operators
- Show that they are Lamé

$$\left[-\partial_x^2 + 2k^2 \,\mathrm{sn}^2(x\,|\,k^2) \right] f(x) = \Lambda f(x)$$

- Time-dependent case ($x \sim \tau$!) ... Quantisation of time-periodic solitons
- Bohr-Sommerfeld-Maslov semiclassical quantisation

  The 1-loop correction to their energy is determined in a more complicated way than just by summing characteristic frequencies! No more determinants! ... we will need "stability angles"
- Solve Lamé: "stability angles" for the pulsating string [and determinants (for the folded string)]
- Compute the energy and expand in the "short limit"

Quadratic fluctuation Lagrangian I

AdS_5 directions are represented by a free massless "ghost" field plus four free massive fields ($k = 1, 2, 3, 4;\ \partial_a \partial^a = -\partial_\tau^2 + \partial_\sigma^2$)

$$L_{AdS}^{(2)} = -\frac{1}{2}(\dot{\beta}^2 - \beta'^2) + \frac{1}{2}(\dot{y}_k^2 - y_k'^2 - \kappa^2 y_k y_k)$$

S^5 fluctuations $(\xi, \eta, z_1, z_2, z_3)$

$$L_S^{(2)} = \tfrac{1}{2}(\dot{\xi}^2 - \xi'^2 - M_\xi^2 \xi^2) + \tfrac{1}{2}(\dot{\eta}^2 - \eta'^2 - M_\eta^2 \eta^2) + \ m \cos\psi\,(\xi \eta' - \xi' \eta)$$

$$+ \tfrac{1}{2}(\dot{z}_i^2 - z_i'^2 - M^2 z_i^2)$$

where the background-dependent masses are

$$M^2 = \kappa^2 - 2m^2 \sin^2\psi, \quad M_\xi^2 = \kappa^2 + m^2 \cos(2\psi) \quad M_\eta^2 = m^2 \cos(2\psi)$$

Solving the Virasoro constraints one can show that the coupled system (ξ, η) is equivalent to a decoupled system of one massless mode + of the massive mode with the Lagrangian

$$L = \tfrac{1}{2}(\dot{g}^2 - g'^2 - \tilde{M}^2 g^2) \quad \tilde{M}^2 = \kappa^2\left(1 - \frac{2}{\sin^2\psi}\right)$$

Quadratic fluctuation Lagrangian II

Starting form the fermionic fluctuation Lagrangian

in the standard $\theta^1 = \theta^2$ kappa symmetry

$$\mathcal{L}_F = -2\,i\,\bar{\vartheta}\Big(-\rho^a\,D_a - \frac{i}{2}\varepsilon^{ab}\,\rho_a\,\Gamma_*\,\rho_b\Big)\,\vartheta\ ,$$

After some computations ...

$$D_F = \Gamma_0\partial_\tau - \Gamma_9\,\partial_\sigma + \Gamma_{079}\dot{\psi}\ .$$

We are interested in eigenvalues and determinant $\Rightarrow$ take the square of the simpler operator

$$\tilde{D}_F \equiv \Gamma_{09}\,D_F = \Gamma_9\partial_\tau - \Gamma_0\,\partial_\sigma - \Gamma_7\dot{\psi}\ .$$

Diagonalizing Γ_{97} (i.e. replacing it by $\pm i$) we get the following second order fermionic operator

$$\tilde{D}^2_{F\pm} = \partial^2_\tau - \partial^2_\sigma + M^2_\pm \qquad M^2_\pm = \dot{\psi}^2 \pm i\,\ddot{\psi}$$

Taking into account the specific form of the solution $\psi(\tau)$

$$M^2_\pm \sim k^2\,\mathrm{cn}^2(x\,|\,k^2) \mp i\,k\,\mathrm{sn}(x\,|\,k^2)\,\mathrm{dn}(x\,|\,k^2)$$

UV check

UV Check on the resulting fluctuation Lagrangian:

UV finiteness of the 1-loop partition function: In conformal gauge ⇔ sum of the effective mass-squared terms for bosons equals that for the fermions

$$
\begin{aligned}
AdS &: \quad 4 \times \kappa^2, \\
S^5 &: \quad 3 \times (\kappa^2 - 2m^2 \sin^2\psi), \\
&\quad\;\; 1 \times (m^2 \cos(2\psi) - m^2 \cos^2\psi), \\
&\quad\;\; 1 \times (\kappa^2 + m^2 \cos(2\psi) - m^2 \cos^2\psi), \\
F &: \quad -8 \times (\kappa^2 - m^2 \sin^2\psi)
\end{aligned}
$$

... indeed sums to zero.

$$
\begin{aligned}
AdS &: \quad 4 \times \kappa^2, \\
S^5 &: \quad 3 \times (\kappa^2 - 2m^2 \sin^2\psi), \\
&\quad\;\; 1 \times \kappa^2 \left(1 - \frac{2}{\sin^2\psi}\right), \\
F &: \quad -8 \times (\kappa^2 - m^2 \sin^2\psi)
\end{aligned}
$$

In static gauge: Sum proportional to the Euler density of the induced metric ... this is proportional to the Euler number which vanishes for the cylinder topology under discussion.

and the sum is $2m^2 \sin^2\psi - 2\frac{\kappa^2}{\sin^2\psi} = \sqrt{-g}\, R^{(2)}$

✓ Expand the action around the classical solution and find fluctuation operators

- Show that they are Lamé

$$\left[-\partial_x^2 + 2k^2 \,\mathrm{sn}^2(x \,|\, k^2) \right] f(x) = \Lambda f(x)$$

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Lamé form of fluctuation operators

S^5 modes z_i with mass $M^2 = \kappa^2 - 2m^2 \sin^2\psi$: $\quad O_I = -\partial_\tau^2 + 2m^2 \sin^2\psi - \kappa^2 - n^2$

$$O_I = m^2 \left[-\partial_x^2 + \; 2k^2 \,\mathrm{sn}^2(x\,|\,k^2) - \Lambda \right], \qquad x = m\tau, \quad k^2 = \frac{\kappa^2}{m^2}, \quad \Lambda = \frac{\kappa^2+n^2}{m^2}$$

S^5 mode with mass $\tilde{M}^2 = \kappa^2(1 - \frac{2}{\sin^2\psi})$: $\quad O_{II} = m^2 \left[-\partial_x^2 + \; 2\,\mathrm{ns}^2(x\,|\,k^2) - \Lambda \right]$

$\mathrm{ns}(z\,|\,k^2) = k\,\mathrm{sn}(z + i\mathbb{K}'\,|\,k^2)$

$$O_{II} = m^2 \left[-\partial_x^2 + \; 2\,k^2\,\mathrm{sn}^2(x\,|\,k^2) - \Lambda \right], \qquad x \equiv m\tau + i\mathbb{K}' \quad , \quad k = \frac{\kappa}{m} \quad , \quad \Lambda = \frac{\kappa^2+n^2}{m^2}$$

The fermion op. with the mass $M_\pm^2 = \psi'^2 \pm i\psi''$: $\quad O_{III}^\pm = -\partial_\tau^2 - \psi'^2 \mp i\psi'' - n^2.$

This operator is non-hermitian, but is PT-symmetric and has a real spectrum

Does not look like the standard single-gap Lamé operator ... rescaling of x and a Gauss/Landen/Jacobi Transformations

$$O_{III}^\pm = \bar{m}_\pm^2 \left[-\partial_{\bar{x}}^2 + 2\,\bar{k}_\pm^2\,\mathrm{sn}^2(\bar{x}\,|\,\bar{k}_\pm^2) - \Lambda \right], \bar{x} \equiv \bar{m}_\pm\tau + \frac{1}{2}\mathbb{K}(\bar{k}_\pm^2)$$

$$\bar{k}_\pm^2 = \pm 4 \frac{\frac{i\kappa}{m}\sqrt{1 - \frac{\kappa^2}{m^2}}}{\left(\sqrt{1 - \frac{\kappa^2}{m^2}} \pm \frac{i\kappa}{m}\right)^2}, \quad \Lambda = \frac{n^2}{\bar{m}_\pm^2 + \bar{k}_\pm^2}, \quad \bar{m}_\pm = \frac{m}{2}\left(\sqrt{1 - \frac{\kappa^2}{m^2}} \pm i\frac{\kappa}{m}\right)$$

- ✓ Expand the action around the classical solution and find fluctuation operators
- ✓ Show that they are Lamé

$$\left[-\partial_x^2 + 2k^2\,\mathrm{sn}^2(x\,|\,k^2)\right] f(x) = \Lambda f(x)$$

- Time-dependent case ($x \sim \tau$!) ... Quantisation of time-periodic solitons
- Bohr-Sommerfeld-Maslov semiclassical quantisation

  The 1-loop correction to their energy is determined in a more complicated way than just by summing characteristic frequencies! No more determinants! ... we will need "stability angles"
- Solve Lamé: "stability angles" for the pulsating string [and determinants (for the folded string)]
- Compute the energy and expand in the "short limit"

Bohr-Sommerfeld-Maslov quantisation I

Semiclass. quant. of (class. integrable, time-periodic) Hamiltonian

Classical integrability

$\exists\ n$ functions $F_i, \in C(T^*X)$ such that:

- $dF_1 \wedge \cdots \wedge dF_n \neq 0$, almost everywhere,
- $\{F_i, F_j\} = 0$,
- $H = H(F_1, \ldots, F_n)$.

Semi-classical integrability

$\exists$ quantum extensions $\widehat{F}_i$ of F_i ($\widehat{F}_i \xrightarrow{\hbar \to 0} F_i$)

- $dF_1 \wedge \cdots \wedge dF_n \neq 0$, almost everywhere,
- $[\widehat{F}_i, \widehat{F}_j] = O(\hbar^3)$
- $\widehat{H} = H(\widehat{F}_1, \ldots, \widehat{F}_n) + O(\hbar^2)$

Define n-tori (Liouville tori), action variables I_i, angle variables φ_i (the coor. of the torus)

We want to solve the semiclassical problem

$$\widehat{F}_i \psi = f_i \psi + O(\hbar^2)$$

WKB-like solution $\exists$ iff BSM quantisation condition is satisfied

$$\frac{1}{2\pi\hbar} \int_{\gamma_i} p \cdot dq = N_i + \frac{\mu_i}{4} + O(\hbar), \qquad i = 1, \ldots, n\ ,$$

N_i action variables, $\{\gamma_i\}$ cycles of a Liouville torus, μ_i Maslov indices they generalise the familiar 1/2 in the standard WKB

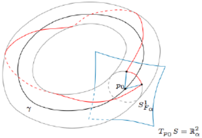
Bohr-Sommerfeld-Maslov quantisation II

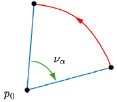
If the class. torus $p < n$ non trivial cycles $\Rightarrow$ change in the BSM q. condition

$$\frac{1}{2\pi\hbar}\int_{\gamma_k} p\cdot dq = N_k + \frac{\mu_k}{4} + \sum_{\alpha=p+1}^{n}\left(n_\alpha + \frac{1}{2}\right)\frac{\nu_\alpha^{(k)}}{2\pi} + O(\hbar)$$

The *stability angles* account the fluct. transverse to the codimension p invariant torus

$\nu_\alpha^{(k)}$ found studying small fluctuations ... they are nothing but the usual eigenfrequencies for fluctuations around static solitons





Superstring periodic $\mathcal{T}$ solutions: 1-dimensional integrable system with invariant tori embedded in the string phase space

$$\mathcal{E} = \mathcal{E}_{cl}(\mathcal{N}) + \frac{1}{2\sqrt{\lambda}}\frac{1}{\mathcal{T}}\sum_{\nu_s>0}\nu_s + O\left(\frac{1}{(\sqrt{\lambda})^2}\right)$$

Fluctuation operators are all of the single-gap Lamé form - “Schroedinger-like” periodic potential

Stability angles $\sim$ quasi momentum

Independent solutions $f_\pm(x)$

$$f_\pm(x+\mathcal{T}) = e^{\pm i\nu}\, f_\pm(x) \qquad \nu = p\mathcal{T}\ ,$$

Remarks on single-gap Lamé operators I

Consider the following eigenvalue problem for an ordinary differential operator with a periodic potential

$$\mathcal{O}f \equiv \left[-\partial_x^2 + V(x)\right] f(x) = \Lambda f(x) \ , \qquad V(x+L) = V(x)$$

Assume quasi-periodic boundary conditions

$$f(x+L) = e^{i\alpha} f(x), \qquad \alpha \in [0, 2\pi).$$

Floquet-Bloch theory: two independent solutions $f_\pm(x) = e^{\pm i p(\Lambda) x} \chi_\pm(x)$, where $\chi_\pm(x)$ are periodic, so that under translation through one period the solutions $f_\pm(x)$ change by a phase

$$f_\pm(x+L) = e^{\pm i p(\Lambda) L} \ f_\pm(x)$$

$p(\Lambda)$ is the *quasi-momentum*. The discriminant is $\Delta(\Lambda) = 2\cos(L\, p(\Lambda))$

Remarks on single-gap Lamé operators II

Quadratic fluctuation operators have "single-gap Lamé" form

$$\left[-\partial_x^2 + 2k^2\,\mathrm{sn}^2(x\,|\,k^2)\right] f(x) = \Lambda f(x) \quad ,$$

The two independent Bloch solutions

$$f_\pm(x) = \frac{H(x\pm\alpha)}{\Theta(x)}\, e^{\mp x Z(\alpha)} \;,$$

H, Θ, Z are the Jacobi Eta, Theta and Zeta functions
Spectral parameter $\alpha = \alpha(\Lambda)$: related to the eigenvalue Λ by the transcendental equation:

$$\mathrm{sn}(\alpha\,|\,k^2) = \sqrt{\frac{1+k^2-\Lambda}{k^2}} \;.$$

Periodicity properties of the Jacobi functions $\Rightarrow$ solutions $f_\pm(x)$ acquire a phase under a shift through one period $2\mathbb{K}$:

$$f_\pm(x+2\mathbb{K}) = -f_\pm(x)\, e^{\mp 2\mathbb{K} Z(\alpha)} \equiv f_\pm(x)\, e^{2i\mathbb{K} p(\alpha)} \;.$$

This defines the quasi-momentum as $p(\Lambda) = i\,Z(\alpha\,|\,k^2) + \frac{\pi}{2\mathbb{K}}$

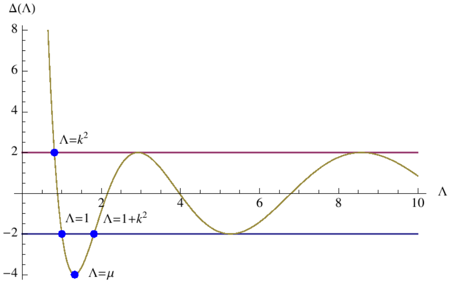
Remarks on single-gap Lamé operators III

Explicit expression for the quasi-momentum implies that we can write an explicit expression for the

- Functional determinant $\det O = \Delta(0) - 2\cos\alpha$
- Stability Angles (just rescaling by $\mathcal{T}$)

The periodic Lamé potential has the special property that its band spectrum has only a single gap - three band edges

$$\Lambda_1 = k^2 \ , \quad \Lambda_2 = 1 \ , \quad \Lambda_3 = 1 + k^2$$



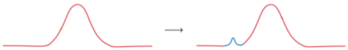
The periodic and the antiperiodic eigenvalues

$$\Delta(\Lambda) = \begin{cases} +2 & \text{(periodic)} \\ -2 & \text{(antiperiodic)} \end{cases}$$

The role of Integrability

Why everything is so simple (solvable Lamé)?

- The solution we considered are part of a more general class “finite gap” String Solutions, characterised by an “algebraic curve” $\Leftrightarrow$ set of cuts on a Riemann surface (Point where the Bethe roots condensate in the continuum limit)
- Integrable system: given a solitonic solution $\Leftrightarrow$ study the perturbated system by adding another “small soliton” (Bäcklund Transformation)



- Algebraic curve semiclassical quantization: deforming the cuts definining the algebraic curve (adding extra roots)

- Solution of the Lamé equation $\Rightarrow$ Baker Akhiezer function

- ✓ Expand the action around the classical solution and find fluctuation operators
- ✓ Show that they are Lamé

$$\left[-\partial_x^2 + 2k^2\,\mathrm{sn}^2(x\,|\,k^2)\right] f(x) = \Lambda f(x)$$

- ✓ Time-dependent case ($x \sim \tau$!) ... Quantisation of time-periodic solitons
- ✓ Bohr-Sommerfeld-Maslov semiclassical quantisation

  ...
- ✓ Solve Lamé
- Compute the energy and expand in the "short limit"

Pulsating String: Stability angles I

- $\mathcal{E} = \mathcal{E}_{cl}(\mathcal{N}) + \frac{1}{2\sqrt{\lambda}} \frac{1}{\mathcal{T}} \sum_{\nu_s>0} \nu_s + O(\frac{1}{(\sqrt{\lambda})^2})$
- The period of the problem is $\mathcal{T} = \frac{4\mathbb{K}}{m}$.
- We want the short string: small $\kappa \Leftrightarrow$ small semiclassical oscillation parameter $\mathcal{N}$
- compute exact stability angles (Lamé), expand, and then sum

---

4 massless AdS_5 fluctuations

$$\nu_{AdS_5} = 4\mathbb{K}\sqrt{k^2 + \frac{n^2}{m^2}} \qquad k \equiv \frac{\kappa}{m}$$

Expanding in small κ, i.e. in small k,

$$\begin{aligned} \nu_{AdS_5} &= \frac{2\pi n}{m} + k^2\left(\frac{\pi m}{n} + \frac{\pi n}{2m}\right) + \frac{\pi k^4\left(-8m^4+8m^2n^2+9n^4\right)}{32mn^3} \\ &+ \frac{\pi k^6\left(16m^6-8m^4n^2+18m^2n^4+25n^6\right)}{128mn^5} + \ldots \end{aligned}$$

Pulsating String: Stability angles II

S^5 bosonic fluctuations (Type I and II)

$$\nu_{S^5} = \pm 4\mathbb{K}\left(i\,\mathbb{Z}(\alpha\,|\,k^2) + \frac{\pi}{2\mathbb{K}}\right) \equiv \pm 4\mathbb{K}\, i\,\mathbb{Z}(\alpha\,|\,k^2)\,, \quad \mathrm{sn}(\alpha\,|\,k^2) = \sqrt{\frac{1+k^2-\Lambda}{k^2}} = \frac{1}{k}\sqrt{1-\frac{n^2}{m^2}}\,.$$

define $a = \sqrt{1-\frac{n^2}{m^2}}$

$$\mathbb{Z}(\mathrm{sn}^{-1}(\frac{a}{k}\,|\,k^2)\,|\,k^2) = i\int_1^{a/k}\frac{dt}{\sqrt{t^2-1}}\left(\sqrt{1-k^2t^2} - \frac{\mathbb{E}}{\mathbb{K}}\frac{1}{\sqrt{1-k^2t^2}}\right).$$

The two basic integrals are

$$\int_1^{a/k}\frac{dt}{\sqrt{t^2-1}}\sqrt{1-k^2t^2} = i\,(\mathbb{E}\left(\arcsin\frac{a}{k}\,|\,k^2) - \mathbb{E}\right)\,,$$

$$\int_1^{a/k}\frac{dt}{\sqrt{t^2-1}}\frac{1}{\sqrt{1-k^2t^2}} = i\left(\mathbb{F}(\arcsin\frac{a}{k}\,|\,k^2) - \mathbb{K}\right).$$

In order to expand at small k, we use the transformation

$$\mathbb{E}(\arcsin\frac{a}{k}\,|\,k^2) = \frac{\mathbb{E}}{\mathbb{K}}\mathbb{F}(\arcsin\frac{a}{k}\,|\,k^2) + i\sqrt{1-a^2}\sqrt{1-\frac{k^2}{a^2}}\left(\frac{\Pi(a^2\,|\,k^2)}{\mathbb{K}} - 1\right).$$

The final result is remarkably simple: all incomplete elliptic integrals simplify.

$$\mathbb{Z}(\mathrm{sn}^{-1}(\frac{a}{k}\,|\,k^2)\,|\,k^2) = i\sqrt{1-a^2}\sqrt{1-\frac{k^2}{a^2}}\left(1-\frac{\Pi(a^2\,|\,k^2)}{\mathbb{K}}\right).$$

Pulsating String: Stability angles III

$$\nu_{S^5} = -4\,i\,\mathbb{K}\,\mathbb{Z}(\mathrm{sn}^{-1}(\frac{a}{k}\,|\,k^2)\,|\,k^2) \quad = \quad \frac{2\pi n}{m} + \frac{\pi k^2 n}{2m} + \frac{\pi k^4 n(13m^2 - 9n^2)}{32m(m-n)(m+n)} + \frac{\pi k^6 n(45m^4 - 62m^2n^2 + 25n^4)}{128m(m-n)^2(m+n)^2} + \ldots$$

The singularity at $n = m$ is only apparent, since it happens at $a = 0$ where our derivation cannot be applied; the above expression is just zero at that point

---

Fermionic fluctuations

$$\nu_F = \pm 4\,i\,\mathbb{K}\left[\frac{1}{2}\,\mathbb{Z}(\alpha(\beta)\,|\,k^2) + i\,\sqrt{\beta}\sqrt{1 + \frac{16\,\beta\,k^2}{(1-4\beta)^2}}\,\right], \quad \alpha(\beta) = \mathrm{cn}^{-1}\left(-\frac{1+4\beta}{1-4\beta}\,|\,k^2\right), \quad \beta = \frac{n^2}{m^2}$$

$$\nu_F = \frac{2\pi n}{m} + \frac{\pi n(3m^2 + 4n^2)}{2m(2n-m)(m+2n)}k^2 - \frac{\pi n(15m^6 - 276m^4n^2 - 304m^2n^4 + 576n^6)}{32m(m-2n)^3(m+2n)^3}k^4 - \frac{\pi n(35m^{10} - 780m^8n^2 + 9696m^6n^4 + 9856m^4n^6 - 28928m^2n^8 + 25600n^{10})}{128m(m-2n)^5(m+2n)^5}k^6 + \ldots$$

... and the Sum

$$\mathcal{E}_1 = \frac{1}{2\mathcal{T}\kappa}\sum_{n=-\infty}^{\infty}\nu_n = 2+\kappa(1-4\log 2)+\frac{1}{8}\kappa^3\Big(3\zeta_3+1+4\log 2\Big)$$
$$+\frac{1}{4}\kappa^5\Big(-\frac{63\zeta_3}{16}-\frac{15\zeta_5}{16}+\frac{7}{32}+\log 2\Big)+O(\kappa^7)$$

Check: The sum over n is convergent

... remember that we can organise the short string expansion of the energy as

$$E = E\Big(\frac{N}{\sqrt{\lambda}},\sqrt{\lambda}\Big) = \sqrt{\lambda}\,\mathcal{E}_0(\mathcal{N})+\mathcal{E}_1(\mathcal{N})+\frac{1}{\sqrt{\lambda}}\mathcal{E}_2(\mathcal{N})+\dots\ ,$$
$$\mathcal{E}_k = \sqrt{2\mathcal{N}}\,\Big(a_{0k}+a_{1k}\,\mathcal{N}+a_{2k}\,\mathcal{N}^2+\dots\Big)+c_{0k}+c_{1k}\,\mathcal{N}+\dots\ .$$

we thus find that for the pulsating string in $\mathbb{R}\times S^2$

$$E_1\equiv\mathcal{E}_1 = 2+\sqrt{2\mathcal{N}}\Big[1-4\log 2+\Big(\frac{3}{2}\log 2+\frac{3}{4}\zeta_3+\frac{1}{8}\Big)\mathcal{N}$$
$$+\Big(\frac{25}{32}\log 2-\frac{135}{32}\zeta_3-\frac{15}{16}\zeta_5+\frac{11}{128}\Big)\mathcal{N}^2+\dots\Big].$$

and finally ...

... can be re-written in terms of N and the string tension λ as follows

$$E = \sqrt{2N\sqrt{\lambda}}\,\left(a_{00} + \frac{a_{10}N + a_{01}}{\sqrt{\lambda}} + \dots\right) + c_{01} + \dots$$

$$a_{00} = 1, \quad a_{10} = -\frac{1}{8}, \quad a_{01} = 1 - 4\log 2, \quad c_{01} = 2$$

Apply the same strategy to other, simple, classical solutions!

Pulsating string in AdS_3

$$\sinh\rho(\tau) = \sqrt{R_+}\,\mathrm{cn}\big(x + \mathbb{K}(k^2)\,|\,k^2\big)\ ,$$

$$x = m\sqrt{R_+ - R_-}\ \tau\ \equiv w\,\tau\ , \quad R_\pm = \frac{-m \pm \sqrt{m^2 + 4\,\mathcal{E}_0^2}}{2m}$$

$$k^2 = \frac{R_+}{R_+ - R_-} = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + \left(\frac{2\mathcal{E}_0}{m}\right)^2}}\right).$$

$$N = \sqrt{\lambda}\ \mathcal{N}\ , \qquad \mathcal{N} = \frac{1}{2\pi}\oint d\rho\ \dot{\rho}$$

spinning folded string in $\mathbb{R}\times S^2$

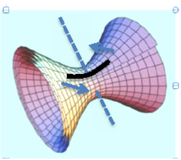
$$\sin\theta = \sqrt{q}\,\mathrm{sn}(w_{21}\,\sigma|q), \qquad \cos\theta = \mathrm{dn}(w_{21}\,\sigma|q),$$

$$q = \sin^2\theta_0 = \frac{\kappa^2 - w_1^2}{w_2^2 - w_1^2}, \qquad w_{21} = \sqrt{w_2^2 - w_1^2} = \frac{2}{\pi}\mathbb{K}(q).$$

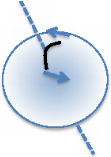
$$\mathcal{E}_0 = \kappa, \qquad \mathcal{J}_1 = \frac{w_1}{w_{21}}\ {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1, q\right),$$

$$\mathcal{J}_2 = \frac{w_2}{w_{21}}\ \frac{q}{2}\ {}_2F_1\left(\frac{1}{2}, \frac{3}{2}, 2, q\right). \quad \frac{\mathcal{J}_1}{w_1} + \frac{\mathcal{J}_2}{w_2} = 1.$$

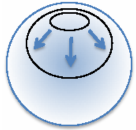
Konishi multiplet vs. semiclassical strings (?)



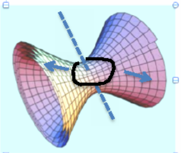
folded in AdS_3



folded in RxS^2



pulsating RxS^2



pulsating AdS_3

$$E_1 = 2 + \sqrt{2\mathcal{J}}\Big[2 - 4\log 2 + \Big(-\frac{1}{2} - \frac{3}{2}\log 2 + \frac{3}{4}\zeta_3\Big)\mathcal{J} + \Big(\frac{1}{64} - \frac{15}{32}\log 2 + \frac{51}{32}\zeta_3 - \frac{15}{16}\zeta_5\Big)\mathcal{J}^2 + \dots\Big],$$

$$E_1 = 1 + \sqrt{2\mathcal{S}}\Big[\frac{3}{2} - 4\log 2 + \Big(-\frac{23}{16} + \frac{3}{2}\log 2 + \frac{3}{4}\zeta_3\Big)\mathcal{S} + \Big(\frac{689}{256} - \frac{63}{32}\log 2 - \frac{15}{32}\zeta_3 - \frac{15}{16}\zeta_5\Big)\mathcal{S}^2 + \dots\Big],$$

$$E_1 = 2 + \sqrt{2\mathcal{N}}\Big[1 - 4\log 2 + \Big(\frac{1}{8} + \frac{3}{2}\log 2 + \frac{3}{4}\zeta_3\Big)\mathcal{N} + \Big(\frac{11}{128} + \frac{25}{32}\log 2 - \frac{135}{32}\zeta_3 - \frac{15}{16}\zeta_5\Big)\mathcal{N}^2 + \dots\Big],$$

$$E_1 = 1 + \sqrt{2\mathcal{N}}\Big[\frac{5}{2} - 4\log 2 + \Big(-\frac{37}{8} + \frac{5}{2}\log 2 + \frac{3}{4}\zeta_3\Big)\mathcal{N} + \Big(\frac{3915}{256} - \frac{231}{32}\log 2 - \frac{117}{32}\zeta_3 - \frac{15}{16}\zeta_5\Big)\mathcal{N}^2 + \dots\Big],$$

"Some" universality: Highest transcendentality terms are equal

Konishi multiplet vs. semiclassical strings (?)

Folded spinning string in $\mathbb{R} \times S^2$

$$E = \sqrt{2J\sqrt{\lambda}}\left(1 + \frac{\frac{1}{8}J + 2 - 4\log 2}{\sqrt{\lambda}} + \dots\right) + 2 + \dots$$

Folded spinning string in AdS_3

$$E = \sqrt{2S\sqrt{\lambda}}\left(1 + \frac{\frac{3}{8}S + \frac{3}{2} - 4\log 2}{\sqrt{\lambda}} + \dots\right) + 1 + \dots$$

Pulsating string in $\mathbb{R} \times S^2$

$$E = \sqrt{2N\sqrt{\lambda}}\left(1 + \frac{-\frac{1}{8}N + 1 - 4\log 2}{\sqrt{\lambda}} + \dots\right) + 2 + \dots$$

Pulsating string in AdS_3

$$E = \sqrt{2N\sqrt{\lambda}}\left(1 + \frac{\frac{5}{8}N + \frac{5}{2} - 4\log 2}{\sqrt{\lambda}} + \dots\right) + 1 + \dots,$$

Summary and comments

- Exact structure of one-loop correction to energy for a class of classical string solutions (simple elliptic functions).

  (Next in complexity to the simplest rational class, trigonometric functions)

- In all cases where there is only one charge/adiabatic invariant besides the energy the fluctuation operators can be decoupled and put into a single-gap Lamé type form.

- We have found the one-loop energies in the limit of small values of the semiclassical parameters small size of the string/ "near-flat" approximation

- The hope: This "short-string" limit may shed light on the structure of strong-coupling corrections to dimensions of "short" dual gauge theory operators for which the "wrapping" contributions are important.

Still to do ...

- The semiclassical approximation is based on assumption that $\sqrt{\lambda} \gg 1$ with semiclassical parameters like $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$, $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ or $\mathcal{N} = \frac{N}{\sqrt{\lambda}}$ fixed, so that S, J or N are formally large. Still, taking the "short-string" limit in which $\mathcal{S}, \mathcal{J}, \mathcal{N} \to 0$ one may conjecture that that limit "commutes" with large the $\sqrt{\lambda}$ limit ... is this true?

> Consider the Folded String case - Konishi operator:
> structure of the semiclassical result OK, coefficient?
> One would expect rational ... [Numerical by Gromov & C.]

- ... we start with class. solution with all charges = 0 but one ... tricky! Add an additional momentum in S^5!
- Comparison, step by step, with the algebraic curve
- Classical solutions with two charges different from zero ... it seems very difficult ... in principle, the nice Lamé equation can be generalized to a many component one solved by Baker Akhiezer function

Thank you!