The emergence of gravity: a 2D toy model

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Chiral effective theory

The chiral lagrangian is a non-renormalizable theory describing accurately pion physics at low energies.

It contains a (infinite) number of operators organized according to the number of derivatives

$$\mathcal{L} = f_\pi^2 \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger + \alpha_1 \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger + \alpha_2 \mathrm{Tr}\, \partial_\mu U \partial_\nu U^\dagger \partial^\mu U \partial^\nu U^\dagger + \ldots$$

$$U = \exp i\tilde{\pi}/f_\pi$$

$$\mathcal{L} = \mathcal{O}(p^2) + \mathcal{O}(p^4) + \mathcal{O}(p^6) + \ldots$$

Pions are the Goldstone bosons associated to the (global) symmetry breaking pattern of QCD

$$SU(2)_L \times SU(2)_R \to SU(2)_V$$

Locality, symmetry and relevance (in the RG sense) are the only guiding principles to construct $\mathcal{L}$.

The effective lagrangian still has the *full* symmetry

$$U \to L U R^\dagger$$

Loops

$$A_{N^\pi}(p_i) = \int \frac{d^4k}{(2\pi)^4} (\frac{1}{f_\pi})^{N^\pi} (k^2)^{N_V} (\frac{1}{k^2})^{N_P}$$

Consider e.g. $\pi\pi \to \pi\pi$ scattering. $N_\pi = 4$, $N_V = 2$ and $N_P = 2$

$$A_{N^\pi} \sim \frac{1}{16\pi^2 f_\pi^2} p^4$$

This counting works to all orders and IR divergences are absent (Weinberg)

At each order in perturbation theory the divergences that arise can be eliminated by redefining the coefficients in the higher order operators, e.g.

$$\alpha_i \to \alpha_i + \frac{c_i}{\epsilon}$$

Also logarithmic non-local terms necessarily appear. For instance (in a two-point function) they appear in the combination

$$\frac{1}{\epsilon} + \log \frac{-p^2}{\mu^2}$$

Unitarity

The cut provided by the log is absolutely required by unitarity. Let us separate

$$S = I + iT.$$

The identity corresponds to having no interaction at all.

Unitarity implies

$$S^\dagger S = I = I + i(T - T^\dagger) + T^\dagger T.$$

$$i(T - T^\dagger) = -T^\dagger T.$$

Loops are essential, even for effective theories. There is no such thing as a ‘classical effective theory’.

Chiral counting

The lowest-order, tree level contribution is $\sim \frac{p^2}{f_\pi^2}$

The one-loop chiral corrections is $\sim \frac{p^4}{16\pi^2 f_\pi^4}$

$\Rightarrow$ The counting parameter in the loop (chiral) expansion is

$$\frac{p^2}{16\pi^2 f_\pi^2}$$

Each chiral loop gives an additional power of p^2

$\mathcal{O}(p^{2n})$ counts as p^{2n}

Soft breaking terms: $\mathrm{Tr}\ \mu m(U + U^\dagger)$

$\Rightarrow\ m$ counts as p^2.

All coefficients in the chiral lagrangian are nominally of $\mathcal{O}(N_c)$.

Loops are automatically suppressed by powers of N_c, but enhanced by logs.

The gravity analogy

Einstein-Hilbert action shares several aspects with the chiral lagrangian (non-renormalizable, dimension two, massless quantum,...)

$$\mathcal{L} = M_P^2 \sqrt{-g}\mathcal{R} + \mathcal{L}_{matter}$$

$$\kappa^2 \equiv \frac{2}{M_P^2} = 32\pi G$$

M_P will play a role very similar to f_π

$\mathcal{R}$ contains two derivatives of the dynamical variable $g_{\mu\nu}$

$$\mathcal{R}_{\mu\nu} = \partial_\nu \Gamma^\alpha_{\mu\alpha} - \partial_\alpha \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} - \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu}$$

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\rho}\left(\partial_\beta g_{\rho\alpha} + \partial_\alpha g_{\rho\beta} - \partial_\rho g_{\alpha\beta}\right)$$

$$\mathcal{R} \sim \partial\partial g$$

In the chiral language, the Einstein-Hilbert action is $\mathcal{O}(p^2)$ (i.e. most relevant).

Why Einstein-Hilbert

Arguably, these considerations alone, in particular relevance in the RG sense (and not renormalizability) are the ones that single out Einstein-Hilbert action (in front e.g. of $\mathcal{R}^2$).

Einstein-Hilbert action has all the ingredients for being an effective theory describing the long distance properties of some unknown dyamics

Are gravitons just Goldstone bosons of some broken symmetry?

Quantum corrections in gravity

Analogous to the weak field expansion in pion physics

$$U = I + i\frac{\pi(x)}{f_\pi} + ...$$

one writes

$$\begin{aligned} g_{\mu\nu} &\equiv \eta_{\mu\nu} + \kappa h_{\mu\nu} \\ g^{\mu\nu} &= \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda} h_\lambda^{\ \nu} + \ldots \end{aligned}$$

so $\kappa \leftrightarrow \frac{1}{f_\pi}$

Curvatures:

$$\begin{aligned} R_{\mu\nu} &= \frac{\kappa}{2}\left[\partial_\mu\partial_\nu h^\lambda_{\ \lambda} + \partial_\lambda\partial^\lambda h_{\mu\nu} - \partial_\mu\partial_\lambda h^\lambda_{\ \nu} - \partial_\lambda\partial_\nu h^\lambda_{\ \mu}\right] + \mathcal{O}(h^2) \\ R &= \kappa\left[\Box h^\lambda_{\ \lambda} - \partial_\mu\partial_\nu h^{\mu\nu}\right] + \mathcal{O}(h^2) \end{aligned}$$

indices are raised and lowered with $\eta_{\mu\nu}$. This can be done around any fixed background space time metric.

Gauge fixing and field equations

Green functions do not exist without a gauge choice and it is most convenient to use harmonic gauge

$$\partial^\lambda h_{\mu\lambda} = \frac{1}{2}\partial_\mu h^\lambda_{\ \lambda}$$

The field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}, \qquad \sqrt{g}T^{\mu\nu} \equiv -2\frac{\delta}{\delta g_{\mu\nu}}\left(\sqrt{g}\mathcal{L}_m\right)$$

reduce in this gauge to

$$\Box h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^\lambda_{\ \lambda}\right)$$

The momentum space propagator is relatively simple in this gauge. Around Minkowski:

$$iD_{\mu\nu\alpha\beta} = \frac{i}{q^2 + i\epsilon}P_{\mu\nu,\alpha\beta} \qquad P_{\mu\nu,\alpha\beta} \equiv \frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right]$$

In addition one needs to include the gauge-fixing and ghost part

$$\mathcal{L}_{gf} = \sqrt{\bar{g}}\left(D^\nu h_{\mu\nu} - \frac{1}{2}D_\mu h^\lambda_{\ \lambda}\right)\left(D_\sigma h^{\mu\sigma} - \frac{1}{2}D^\mu h^\sigma_{\ \sigma}\right), \mathcal{L}_{gh} = \sqrt{\bar{g}}\eta^{*\mu}\left[D_\lambda D^\lambda \bar{g}_{\mu\nu} - R_{\mu\nu}\right]\eta^\nu$$

It is plain that perturbative calculations in quantum gravity are manifestly difficult.

Divergences

The following two results are well known

$$\mathcal{L}^{(div)}_{1loop} = -\frac{1}{16\pi^2\epsilon}\left\{\frac{1}{120}\bar{R}^2 + \frac{7}{20}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}\right\}$$

(t Hooft and Veltman)

$$\mathcal{L}^{(div)}_{2loop} = -\frac{209\kappa^2}{5760(16\pi^2)}\frac{1}{\epsilon}\bar{R}^{\alpha\beta}{}_{\gamma\delta}\bar{R}^{\gamma\delta}{}_{\eta\sigma}\bar{R}^{\eta\sigma}{}_{\alpha\beta}$$

(Goroff and Sagnotti)

It is less well appreciated that the two results are on a different footing. The result of 't Hooft and Veltman

– is gauge dependent

– vanishes when the field equation in empty space are used

– gives a net divergence when $T_{\mu\nu} \neq 0$, but the result is, in principle, incomplete.

The one-loop counterterms computed by 't Hooft and Veltman are largely irrelevant from the point of view of effective lagrangians (they vanish on shell).

de Sitter space-time

In de Sitter space

$$S = \frac{1}{16\pi G}\int dx\sqrt{-g}(\mathcal{R} - 2\Lambda)$$

$$\Gamma^{div}_{eff} = -\frac{1}{16\pi^2\epsilon}\int dx\sqrt{-g}[c_1\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + c_2\Lambda^2 + c_3\mathcal{R}\Lambda + c_4\mathcal{R}^2].$$

The constants c_i are actually gauge dependent and only a combination of them is gauge invariant.

Using the equations of motion (in absence of matter) $\mathcal{R}_{\mu\nu} = g_{\mu\nu}\Lambda$, the previous equation reduces to the (gauge-invariant) on-shell expression

$$\Gamma^{div}_{eff} = \frac{1}{16\pi^2\epsilon}\int dx\sqrt{-g}\frac{29}{5}\Lambda^2.$$

If we set $\Lambda = 0$ above, we get the well-known 't Hooft and Veltman divergence

$$\Gamma^{div}_{eff} = -\frac{1}{16\pi^2\epsilon}\int dx\sqrt{-g}[\frac{7}{20}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \frac{1}{120}\mathcal{R}^2].$$

Counterterms and power counting

Exactly as the chiral lagrangian Einstein-Hilbert requires an infinite number of counterterms

$$\mathcal{L} = M_P^2\sqrt{-g}\mathcal{R} + \alpha_1\sqrt{-g}\mathcal{R}^2 + \alpha_2\sqrt{-g}(\mathcal{R}_{\mu\nu})^2 + \alpha_3\sqrt{-g}(\mathcal{R}_{\mu\nu\alpha\beta})^2 + \ldots$$

The divergences can be absorbed by redefining the coefficients just as before

$$\alpha_i \to \alpha_i + \frac{c_i}{\epsilon}$$

The expansion parameter is a tiny number in normal circumstances

$$p^2/16\pi M_P^2$$

or

$$\nabla^2/16\pi^2 M_P^2, \qquad \mathcal{R}/16\pi^2 M_P^2$$

The most effective of all effective actions!!

Why we need genuine loop effects

Consider

$$\mathcal{L} = \frac{2}{\kappa^2} R + cR^2 + (matter)$$

The equation of motion is

$$\Box h + \kappa^2 c^2 \Box\Box h = (8\pi G T)$$

The Green function for this equation has the form

$$\begin{aligned} G(x) &= \int \frac{d^4 q}{(2\pi)^4} \frac{e^{iq\cdot x}}{q^2 + \kappa^2 c q^4} \\ &= \int \frac{d^4 q}{(2\pi)^4} \left[\frac{1}{q^2} - \frac{1}{q^2 + 1/\kappa^2 c} \right] e^{-iq\cdot x} \end{aligned}$$

Leading to a correction to Newton's law

$$V(r) = -G m_1 m_2 \left[\frac{1}{r} - \frac{e^{-r/\sqrt{\kappa^2 c}}}{r} \right]$$

Experimental bounds indicate $c < 10^{74}$. If c was a reasonable number there would be no effect on any observable physics at terrestrial scales.

Note that if $c \sim 1, \sqrt{\kappa^2 c} \sim 10^{-35} m$. The curvature is so small that R^2 terms are irrelevant at ordinary scales

Why we need genuine loop effects II

However using the full solution of the wave equation is *not* compatible with the effective lagrangian philosophy (higher orders in κ are sensitive to higher curvatures we have not considered). The leading behaviour of the correction is

$$\frac{e^{-r/\sqrt{\kappa^2 c}}}{r} \to 4\pi\kappa^2 c\delta^3(\vec{r})$$

$$\frac{1}{q^2+\kappa^2 c q^4} = \frac{1}{q^2} - \kappa^2 c + \cdots$$

Thus

$$V(r) = -Gm_1 M_2 \left[\frac{1}{r} + 128\pi^2 G c\delta^3(\vec{x})\right]$$

Totally unobservable, even as a matter of principle.

Of course, apart from the divergences there are finite pieces and *non-local* pieces since in DR we get at the one-loop level

$$\frac{1}{\epsilon} + \log\frac{-p^2}{\mu^2}$$

Or, in position space $\frac{1}{\epsilon} + \log\frac{\nabla^2}{\mu^2}, \qquad \nabla = \text{ covariant derivative.}$

Non-localities are due to the propagation of massless non-conformal modes, such as the graviton itself.

Quantum corrections to Newton law

Let us use 'chiral counting' arguments to derive the relevant quantum corrections to Newton law (up to a constant)

Propagator at tree level: $\frac{1}{p^2}$

One-loop corrections: $\frac{1}{p^2}(1 + A\frac{p^2}{M_P^2} + B\frac{p^2}{M_P^2}\log p^2)$

Consider the interaction with an static source ($p^0 = 0$) and let us Fourier transform

$$\int d^3x \exp(i\vec{p}\vec{x})\frac{1}{p^2} \sim \frac{1}{r} \qquad \int d^3x \exp(i\vec{p}\vec{x})1 \sim \delta(\vec{x})$$

$$\int d^3x \exp(i\vec{p}\vec{x}) \log p^2 \sim \frac{1}{r^3}$$

Thus the corrections are of the form

$$\frac{GMm}{r}(1 + C\frac{G\hbar}{r^2} + \ldots)$$

We note that

$$\left[\frac{Gm}{c^2}\right] = L, \qquad \left[\frac{G\hbar}{c^3}\right] = L^2$$

so C is a pure number.

The inclusion of matter

A long controversy regarding the value of C exist in the literature (Donoghue, Muzinich, Vokos, Hamber, Liu, Bellucci, Khriplovich, Kirilin, Holstein, Bjerrum-Bohr,...)

The commonly accepted result is obtained by considering the inclusion of *quantum* matter fields (a scalar field actually) and considering all type of loops

Feynman rules

$$\begin{aligned}
\tau_{\mu\nu} &= -\frac{i\kappa}{2}\left(p_\mu p'_\nu + p'_\mu p_\nu - g_{\mu\nu}[p\cdot p' - m^2]\right) \\
\tau_{\eta\lambda,\rho\sigma} &= \frac{i\kappa^2}{2}\Big\{ I_{\eta\lambda,\alpha\delta} I^{\delta}{}_{\beta,\rho\sigma}\left(p^\alpha p'^\beta + p'^\alpha p^\beta\right) \\
&\quad -\frac{1}{2}\left(\eta_{\eta\lambda} I_{\rho\sigma,\alpha\beta} + \eta_{\rho\sigma} I_{\eta\lambda,\alpha\beta}\right) p'^\alpha p^\beta \\
&\quad -\frac{1}{2}\left(I_{\eta\lambda,\rho\sigma} - \frac{1}{2}\eta_{\eta\lambda}\eta_{\rho\sigma}\right)[p\cdot p' - m^2]\Big\}
\end{aligned}$$

with

$$I_{\mu\nu,\alpha\beta} \equiv \frac{1}{2}[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}]$$

The inclusion of matter II

$$\mathcal{L}_{RR} = \frac{1}{3849\pi^3 r^3}(42R_{\mu\nu}R^{\mu\nu} + R^2)$$

$$\mathcal{L}_{RT} = -\frac{\kappa}{8\pi^2 r^3}(3R_{\mu\nu}T^{\mu\nu} - 2RT)$$

$$\mathcal{L}_{TT} = \frac{\kappa^2}{60\pi r^3}T^2$$

Using the equation of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}$$

$$\Rightarrow \quad \mathcal{L}_{total} = -\frac{\kappa^2}{60\pi r^3}(138T_{\mu\nu}T^{\mu\nu} - 31T^2)$$

The final result is positive: gravity is more atractive at long distances

$$C = \frac{41}{10\pi}$$

What happens for *classical* matter, e.g. a cloud of dust, is in my view still an open problem.

Power counting in effective gravity

- 3-graviton coupling: $\sim \kappa q^2$
- 4-graviton coupling: $\sim \kappa^2 q^2$
- (On-shell) matter– 1-graviton coupling: $\sim \kappa m^2$
- (On-shell) matter– 2-graviton coupling: $\sim \kappa^2 m^2$
- Graviton propagator: $\sim \frac{1}{q^2}$
- Matter propagator $\sim \frac{1}{mq}$

If we iterate the 4-graviton vertex to produce a one loop diagram we obtain schematically

$$\mathcal{M}_{loop} \sim \kappa^4 \int \frac{d^4 l}{(2\pi)^4} \frac{(l-p_1)^2 (l-p_2^2)^2}{l^2 (l-q)^2}$$

If this loop integral is regularized dimensionally, which does not introduce powers of any new scale, the integral will be represented in terms of the exchanged momentum to the appropriate power. Thus we have

$$\mathcal{M}_{loop} \sim \kappa^4 q^4$$

Power counting in effective gravity II

When matter fields are included in loops the situation is more subtle The tree level result is

$$\mathcal{M}_{tree} = \kappa^2 \cdot \frac{m_1^2 m_2^2}{q^2}$$

Iterating this to form a loop

$$\mathcal{M}_{loop} \sim \kappa^4 m_1^4 m_2^4 \cdot \int d^4 l \cdot \frac{1}{m_1(l+p)} \cdot \frac{1}{m_2(l+p')} \cdot \frac{1}{(l+q')^2} \cdot \frac{1}{(l+q)^2}$$

which by the same reasoning is

$$\mathcal{M}_{loop} \sim \kappa^4 \cdot \frac{m_1^3 m_2^3}{q^2} \sim \kappa^2 \cdot \frac{m_1^2 m_2^2}{q^2} \cdot \kappa^2 m_1 m_2$$

Here the expansion parameter appears as $\kappa^2 m^2$ This issue has been studied by Donoghue

$$A_{(N_m, N_g)} \sim q^D$$

$$D = 2 - \frac{N_E^m}{2} + 2N_L - N_V^m + \sum_n (n-2) N_V^g[n] + \sum_l l \cdot N_V^m[l]$$

If we disregard matter vertices this is *identical* to Weinberg's result for chiral theories However it is dangerous the negative N_V^m term appearing in D. Although no general proof exists yet, Donoghue has been able to prove cancellation of the dangerous terms at the one-loop level except for the terms leading to $1/r$ corrections (classical, non-linear)

The use of equations of motion

In chiral lagrangians they allow to get rid of redundant operators

$$U\Box U^\dagger - (\Box U)U^\dagger = 0$$

$$\mathrm{Tr}\ U\Box U^\dagger \to 0$$

Notice that in gravity, the equation of motion mixes terms of different ‘chiral’ order

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} - g_{\mu\nu}\Lambda$$

For instance, it is incorrect to use

$$R_{\mu\nu} = g_{\mu\nu}\Lambda$$

in ’t Hooft and Veltman calculation. It just does not reproduce the de Sitter result.

Gravity as a Goldstone phenomenon

We have given arguments why the Einstein-Hilbert action could be viewed as an effective one

– Dimensionful coupling constant ($M_{pl} \sim f_\pi$)

– Derivative couplings ($\sqrt{-g}\mathcal{R} \sim g\partial\partial g$)

– Action based on RG criteria of relevance, not on renormalizability (unlike Yang-Mills)

– Power counting anologous to ChPT

– Massless quanta ($\pi \leftrightarrow g_{\mu\nu}$)

– Obvious global symmetry to be broken ($GL(D) \subset Diff$)

As an entertainment we shall investigate a formulation inspired as much as possible in the chiral symmetry breaking of QCD

– No *a priori* metric, only affine connection is needed (parallelism)

– Lagrangian is manifestly independent of the metric

– Breaking is triggered by fermion condensate

Chiral Symmetry Breaking

A successful model for QCD is the so-called chiral quark model. Consider the matter part lagrangian of QCD with massless quarks (2 flavours)

$$\mathcal{L} = i\bar{\psi}\,\partial\!\!\!/\psi = i\bar{\psi}_L\,\partial\!\!\!/\psi_L + i\bar{\psi}_R\,\partial\!\!\!/\psi_R$$

This theory has a global $SU(2) \times SU(2)$ symmetry that forbids a mass term M

However after chiral symmetry breaking pions appear and they must be included in the effective theory. Then it is possible to add the following term

$$-M\bar{\psi}_L U \psi_R - M\bar{\psi}_R U^\dagger \psi_L$$

invariant under the full global symmetry

$$\psi_L \to L\psi_L, \qquad \psi_R \to R\psi_R, \qquad U \to LUR^\dagger$$

Chiral symmetry breaking is characterized by the presence of a fermion condensate

$$< \bar{\psi}\psi > \neq 0$$

To determine whether the condensate is zero or not one is to solve a ‘gap’-like equation in some modelization of QCD, or on the lattice.

Integrating out the fermions reproduces the chiral effective lagrangian.

Spontaneous $GL(D)$ breaking

There is only one possible term bilinear in fermions that is invariant under Lorentz $\times$ *Diff*

$$\bar{\psi}_a \gamma^a \nabla_\mu \psi^\mu$$

To define ∇ we only need an affine connection

$$\nabla_\mu \psi^\mu = \partial_\mu \psi^\mu + i\omega_\mu^{ab} \sigma_{ab} \psi^\mu + \Gamma^\nu_{\mu\nu} \psi^\mu$$

Note that no metric is needed at all to define the action if we assume that ψ^μ behaves as a contravariant spinorial vector density under *Diff*, i.e. $\sim \sqrt{g}\psi$.

We would like to find a non trivial condensate

$$< \bar{\psi}_a \psi^\mu > \neq 0, \qquad SO(D)_L \times GL(D)_R \to SO(D)_V$$

We have to include some dynamics to trigger symmetry breaking

$$S_I = i \int d^4x ((\bar{\psi}_a \psi^\mu + \bar{\psi}^\mu \psi_a) B^a_\mu + c \det(B^a_\mu))$$

Note that the interaction one also behaves as a density thanks to one of the Levi-Civita symbol hidden in the determinant of B.

Note that the lagrangian is not hermitean (= to euclidean fermion mass)

Equations of motion et al

The equations of motion (in 2D) show that

$$< \bar{\psi}_a \psi^\mu + h.c. > \sim \epsilon_{ab} \epsilon^{\mu\nu} B^b_\nu$$

We conjecture $B^a_\mu \sim e^a_\mu$.

A 4-fermion interaction is induced (instantons?)

$$\epsilon_{\mu\nu} \epsilon^{ab} (\bar{\psi}_a \psi^\mu + \bar{\psi}^\mu \psi_a)(\bar{\psi}_b \psi^\nu + \bar{\psi}^\nu \psi_b)$$

Fermions e-o-m:

$$\gamma^a \nabla_\mu \psi^\mu + B^a_\mu \psi^\mu = 0$$

$$\gamma^a \nabla_\mu \psi_a + B^a_\mu \psi_a = 0.$$

Energy-momentum tensor (traceless if $w_\mu = 0$):

$$T^\mu_\nu = i\bar{\psi}^\mu \gamma^a \partial_\nu \psi_a + i\bar{\psi}_a \gamma^a \partial_\nu \psi^\mu - \delta^\mu_\nu L.$$

Propagator and renormalizability

We shall consider the above model in $D = 2$ for simplicity.

Note the peculiar 'free' kinetic term $\gamma^a \otimes k_\mu$

$$M = \begin{pmatrix} iB_{11} & k_1 & iB_{12} & k_2 \\ k_1 & iB_{11} & k_2 & iB_{12} \\ iB_{21} & -ik_1 & iB_{22} & -ik_2 \\ ik_1 & iB_{21} & ik_2 & iB_{22} \end{pmatrix}$$

The field B^a_μ shall develop a v.e.v. that we conventionally take to be

$$\langle B^a_\mu \rangle = M \delta^a_\mu .$$

The scale M plays then the role of a dynamically generated mass for the fermions (not unlikely the 'constituent mass' in chiral dynamics)

$$\Delta(k)_{ij} = \begin{pmatrix} iM & k_1 & 0 & k_2 \\ k_1 & iM & k_2 & 0 \\ 0 & -ik_1 & iM & -ik_2 \\ ik_1 & 0 & ik_2 & iM \end{pmatrix} .$$

Propagator and renormalizability II

The propagator of the fermion field can be written (in any number of dimensions) as

$$\Delta^{-1}(k)_{ij} = \frac{-i}{M}(\delta_{ij} - \frac{\gamma_i(\not{k} - iM)k_j}{k^2 + M^2}).$$

Naively, because the coupling constant c is dimensionless in 2D, we would expect the model to be renormalizable. However this expectation is jeopardized by the behaviour of the propagator. Indeed the diagonalization of the kinetic term (plus induced mass) gives as eigenvaules: M (twice), $k + M$ and $k - M$. Therefore the propagator does not behave, in general, as $1/k$ and therefore the usual counting rules do not apply.

The model proposed does not contain a metric and the number of counterterms that one can write is extremely limited. In fact the metric will be generated *after* the breaking, but the counterterms of a field theory do not depend on whether there is spontaneous breaking of a global symmetry or not.

The gap equation

Let us first consider the case where there is no connection at all ($w_\mu(x) = 0$). We can then use homogeneity and isotropy arguments to look for constant solutions of the gap equation associated to

$$V_{eff} = ic\det(B^a_\mu) + 2\int \frac{d^n k}{(2\pi)^n} \mathrm{Tr}\left(\log(\gamma^a k_\mu + iB^a_\mu)\right)$$

The extremum of V_{eff} are found from

$$cn\epsilon_{aa_2\dots a_n}\epsilon^{\mu\mu_2\dots\mu_n} B^{a_2}_{\mu_2}\dots B^{a_n}_{\mu_n} + 2\mathrm{tr}\int \frac{d^n k}{(2\pi)^n}(\gamma\otimes k + iB)^{-1}|^\mu_a = 0$$

Notice that the equations are invariant under the permutation

$$B_{ij} \to B_{\sigma(i)\sigma(j)}, k_i \to k_{\sigma(i)}, \sigma\epsilon S_2$$

The ‘gap equation’ to solve for constant values of B_{ij} is (for $D = 2$)

$$cB_{ij} - \frac{1}{2\pi}B_{ij}\log\frac{\det B}{\mu} = 0 \Rightarrow \det B \neq 0$$

A logarithmic divergence has been absorbed in c. The solution for the dynamical mass is

$$M = \mu e^{\pi c(\mu)}, \quad V_{eff} = i\frac{\mu^2 e^{2\pi c(\mu)}}{2\pi}, \quad \mu\frac{dc}{d\mu} = -\frac{1}{2\pi}$$

This term is the induced cosmological constant. The i is related to the change of signature.

On the Palatini formalism

For non-zero connection ($w_\mu \neq 0$) the gap equation is not applicable and one needs to derive the full effective action. Then one would minimize the fiels B^a_μ as a function of w_μ.

Eventually we want to perform a double minimization with respect to B_μ and w_μ.

2D gravity is rather peculiar and indeed the condition

$$w^{ab}_\mu = e^a_\nu \partial_\mu E^{\nu b} + e^a_\nu E^{\sigma b} \Gamma^\nu_{\sigma\mu},$$

that holds in any number of dimensions does not follow in 2D from any variational principle. Einstein-Hilbert in 2D depends on w_μ only through the exterior derivative dw which is linear in the affine connection w_μ. In fact the scalar curvature term $\sqrt{g}R$ does not contain any coupling between $g_{\mu\nu}$ and w_μ. Adding higher derivatives does not help really as the Riemann tensor contains only an independent component that can be ultimately related to the scalar curvature.

If we are to reproduce 2D gravity, the metric should not be constrained by the connection.

The effective action

Heat kernel calculation:

$$D^a_\mu = \gamma^a(\partial_\mu + iw_\mu\sigma_3) + B^a_\mu.$$

We shall consider the expansion around a fixed background preserving $SO(D)$, but not the full symmetry group G; that is $B^a_\mu = M\delta^a_\mu$, where for vanishing affine connection M can be determined via the gap equation.

We decompose

$$B^a_\mu = \xi_L{}^a{}_b \bar{B}^b_\nu \xi_R^{-1\nu}{}_\mu,$$

where $\xi_L \in SO(D)$, $\xi_R \in GL(D)$. It is technically advantageous to absorb the matrices ξ_L and ξ_R in the fermion fields (in QCD this is the so-called 'constituent' quark basis

$$\mathcal{D}^b_\mu = \xi^{\dagger}_L{}^b{}_a \gamma^a(\partial_\rho + iw_\rho\sigma_3)\xi_R{}^\rho{}_\mu + \bar{B}^b_\mu.$$

Effective action:

$$W = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\text{tr}\left\langle x|e^{-tX}|x\right\rangle,$$

$$X_{\nu\mu} \equiv \mathcal{M}^\dagger\mathcal{M},$$

with

$$\mathcal{M} = i\mathcal{D}^b_\mu, \qquad \mathcal{M}^\dagger = i\mathcal{D}_{\nu b}$$

The effective action II

The heat kernel provides a covariant expansion, which is reassuring, but is difficulted by two problems:

Zero modes of the kinetic term

$$\psi_a \rightarrow \psi'_a = (\delta_a^b - \frac{1}{D}\gamma_a\gamma^b)\psi_b.$$

Another invariance of the free action is provided by redefining, in Fourier space,

$$\psi^\mu(k) \rightarrow \psi'^{\ \mu} = P_{\mu\nu}\psi^\nu(k),$$

where $k^\mu P_{\mu\nu} = 0$

A given order in the t expansion does not correspond to a given order in external fields or derivatives

It is better to use a diagramatic expansion in the external fields.

The effective action III

The calculation is performed in the conformal gauge with

$$B_\mu^a = M e^{-\sigma/2} \delta_\mu^a .$$

Then

$$S_{\text{eff}} = \int d^2x \left[\mu^2 \frac{e^{\tilde{c}}}{2\pi} e^{-\sigma} - \frac{1}{48\pi} \frac{\partial_\mu \sigma \partial_\mu \sigma}{2} + \frac{(W_{\mu\nu})^2}{24\pi M^2} - \frac{w^2}{2\pi} \right] + ...$$

There is no relation between metric and connection (Palatini in 2D)

Minimization wrt w_μ gives $w = 0$ at leading order in the $1/M^2$ expansion.

Dots correspond to higher curvatures.

All the expected features of $2D$ gravity are reproduced!