Nuclear Spectra from Skyrmions

R. Battye, NSM,
P. Sutcliffe, S.W. Wood

Skyrmions arise in effective pion field theory with soliton solutions.

Integer topological charge B is baryon number.

For each B there is a static solution of minimal energy. These solutions have interesting shapes and symmetries.

Quantize the rotational and isorotational motion to obtain quantum states of definite spin and isospin.

Recent calculations give static masses and moments of inertia tensors for B up to 12. Hence spectra.

2. The Skyrme Model and its Solutions

Skyrme model involves four fields subject to a constraint:

$$\sigma(x) \;,\; \pi_1(x) \;,\; \pi_2(x) \;,\; \pi_3(x)$$

subject to

$$\sigma^2 + \pi_1^2 + \pi_2^2 + \pi_3^2 = 1 \,. \qquad (3\text{-sphere})$$

(SO(4) chiral symmetry)

One defines the Skyrme field

$$U(x) = \begin{pmatrix} \sigma(x) + i\pi_3(x) & i\pi_1(x) + \pi_2(x) \\ i\pi_1(x) - \pi_2(x) & \sigma(x) - i\pi_3(x) \end{pmatrix}$$

$$\in SU(2) \,.$$

Useful to define

$$R_\mu = (\partial_\mu U) U^{-1} \quad \in \text{Lie Alg.}(SU(2)) \,.$$

$(\mu = 0, 1, 2, 3)$

Lagrangian (in Skyrme energy and length units)

$$L = \int d^3x \left\{ -\frac{1}{2} \mathrm{Tr}(R_\mu R^\mu) + \frac{1}{16} \mathrm{Tr}\left([R_\mu, R_\nu][R^\mu, R^\nu] \right) + m^2 \,\mathrm{Tr}(U - 1) \right\} .$$

Boundary condition $U \to 1$ as $|\underline{x}| \to \infty$.

Energy for Time-independent fields is

$$E = \int_{\mathbb{R}^3} d^3x \left\{ -\frac{1}{2}\mathrm{Tr}(R_i R_i) - \frac{1}{16}\mathrm{Tr}([R_i, R_j][R_i, R_j]) - m^2\,\mathrm{Tr}(U - 1) \right\}$$

$$(i, j = 1, 2, 3)$$

So far, this is a field theory of light, interacting pions.
But it also has <u>Solitons</u>, called <u>Skyrmions</u>.

<u>Topology</u> Field $U(\underline{x})$ is a map

$$U : \mathbb{R}^3 \longrightarrow S^3$$

This has a <u>Degree</u> or winding number

$$B = -\frac{1}{24\pi^2}\int_{\mathbb{R}^3} \varepsilon_{ijk}\,\mathrm{Tr}(R_i R_j R_k)\, d^3x$$

This is an <u>Integer</u> and <u>Conserved in Time</u>

B is identified with <u>Baryon Number</u>

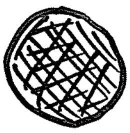
(Nuclear Phys: B = Atomic Number = #Protons + #Neutrons)

<u>Skyrmions</u> are the static field configurations of minimal energy <u>for each B</u>. Energy increases roughly linearly with B.

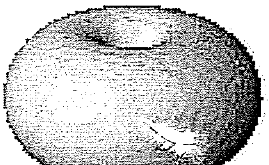
Skyrmions B = 2 - 8

Kopeliovich + Stern, Verbaarschot, N.S.
Braaten, Townsend + Carson
Battye + Sutcliffe

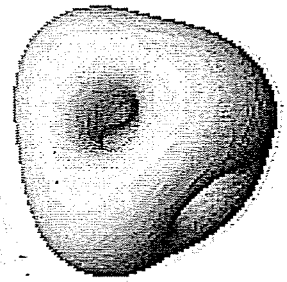
Shown is a surface of constant baryon density.
These look very like certain symmetric BPS monopoles.
$m_\pi = 0$ here.



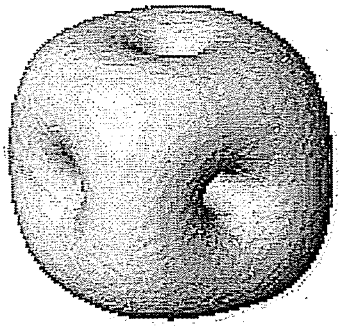
$B=1$



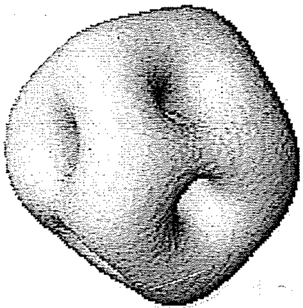
a: $B=2$



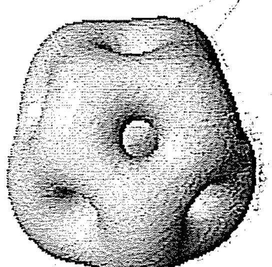
b: $B=3$



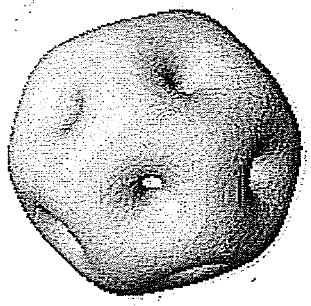
c: $B=4$



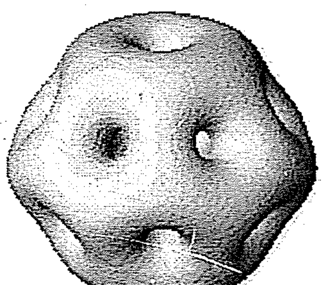
d: $B=5$



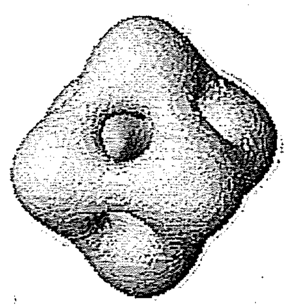
e: $B=6$



f: $B=7$



g: $B=8$



h: $B=5$ (saddle point)

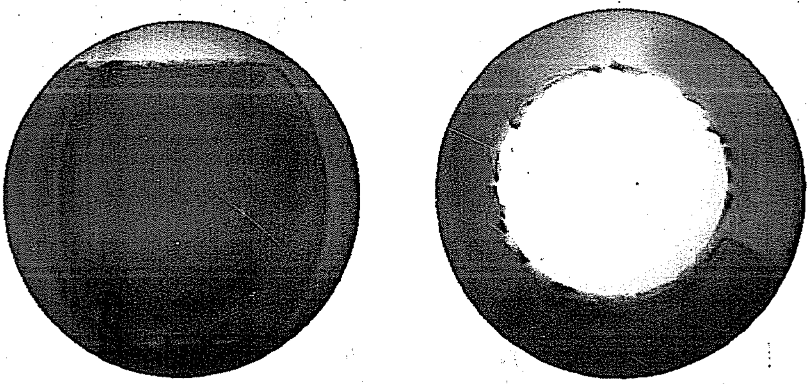
These figures are obtained using the Rational Map approximation. The true solutions have very similar shapes, and slightly lower ($\sim 1\%$) energy.

The surfaces of constant energy density look rather similar, too.

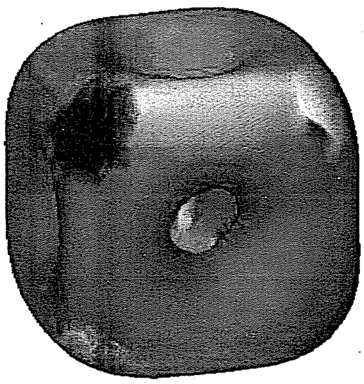
B is the topological charge = Baryon Number

Skyrmions $B = 1, 4, 8$ (R. Battye, NSM, P. Sutcliffe)

Realistic pion mass

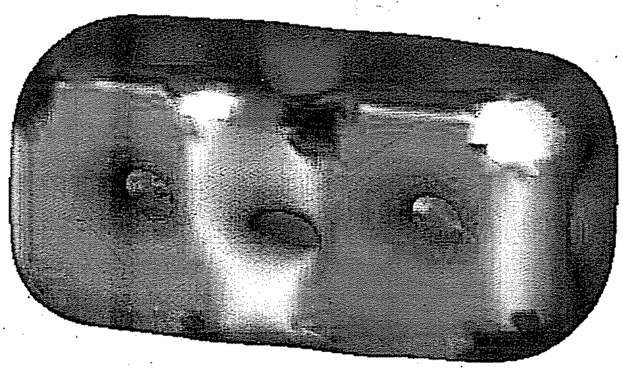


B = 1 Hedgehog

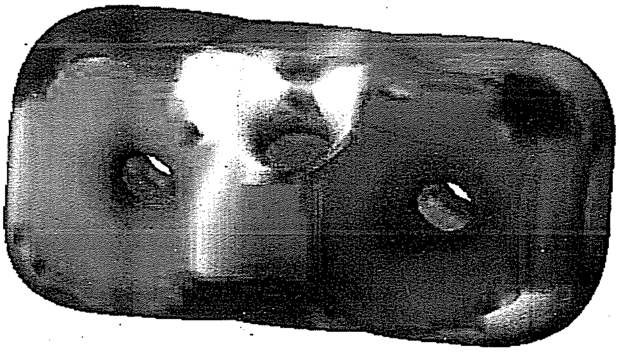


B = 4 Skyrmion

(Skyrmion α-particle
^{4}He(lium))



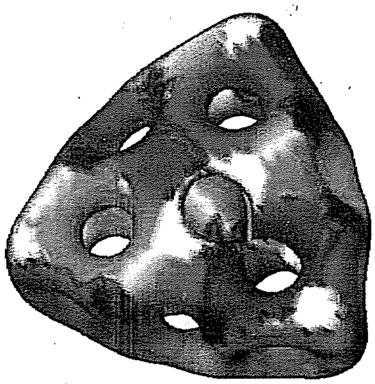
B = 8 Solutions



Most Stable Configuration: ^{8}Be(ryllium)

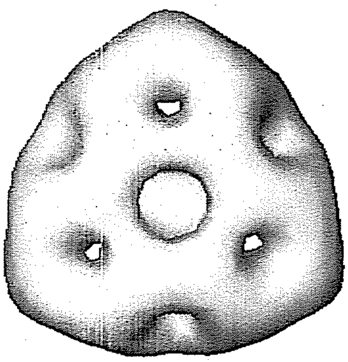
Colours show strength of pion fields π_1, π_2, π_3

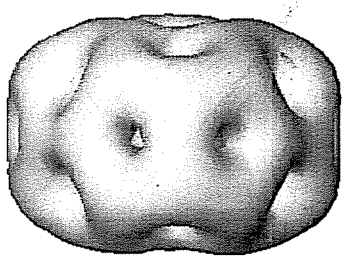
B=12



Low energy, attractive arrangement of three $B=4$ cubes. Symmetry D_{3h}.

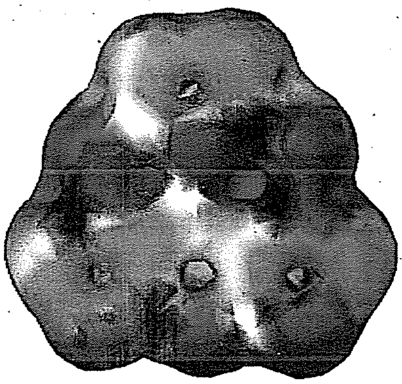
B=1+11





Relaxed double rat. map ansatz $R_1^{in} + R_{11}^{out}$

B=12, m=2



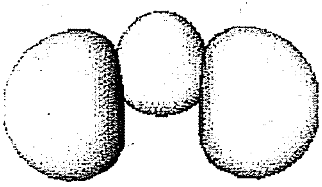
Stable solution with C_3-symmetry

Charge 3 SU(2) BPS Monopoles : Triangular Symmetry

(Sutcliffe)

Skyrmions scatter in a similar way.

Illustrates deformation of B = 3 Skyrmion into unit Skyrmions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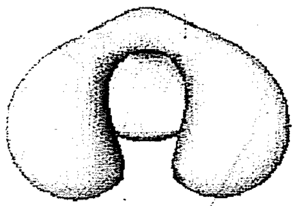
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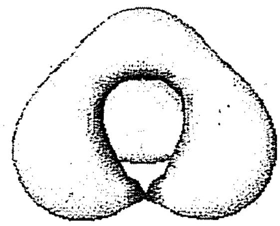
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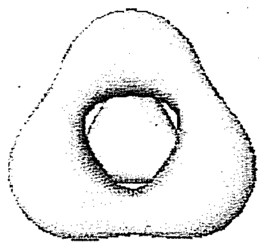
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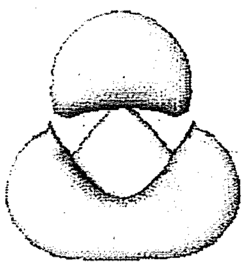
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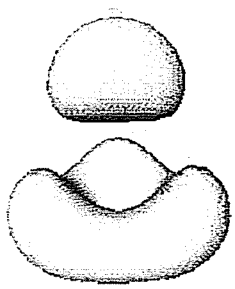
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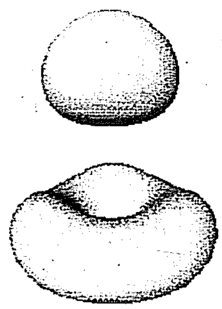
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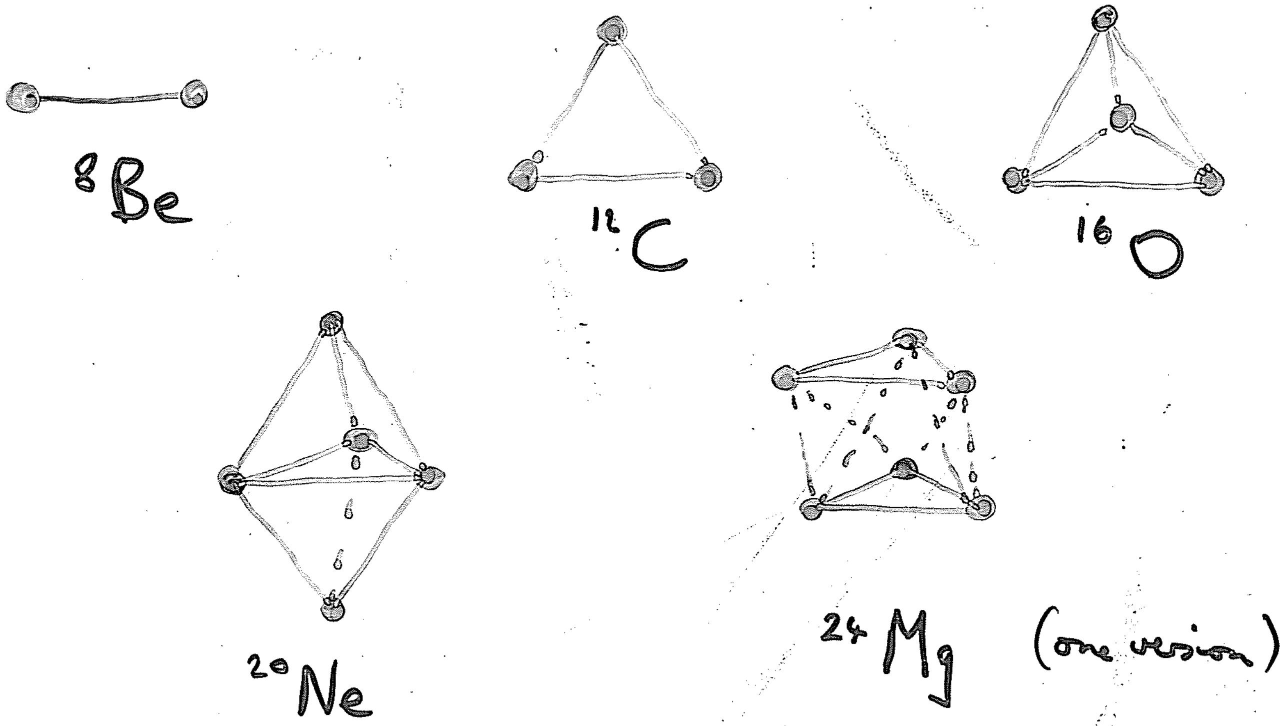
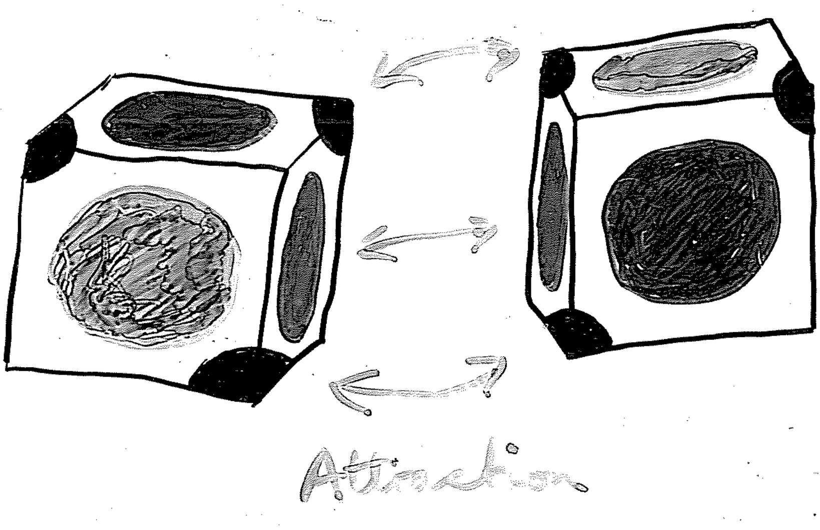
Constructing Solutions from $B = 4$ Cubes

Skyrme model analogue of the α-particle model



These molecules can each be in a spin 0^+ state, and higher spin states also possible.

In Skyrme model seek bound states of $B = 4$ cubes



Quantization

Complete quantization of the pion fields (fluctuating about the Skyrmion solutions) is not feasible.

Instead, quantize the collective motion:

1. Translations
2. Rotations
3. Isospace Rotations.

Spin and Isospin are unified — this distinguishes Skyrmion quantization from collective nuclear motion.

Need to calculate moment of inertia tensor.

Constraints on states arise from need to get spin $\frac{1}{2}$ / isospin $\frac{1}{2}$ nucleons.

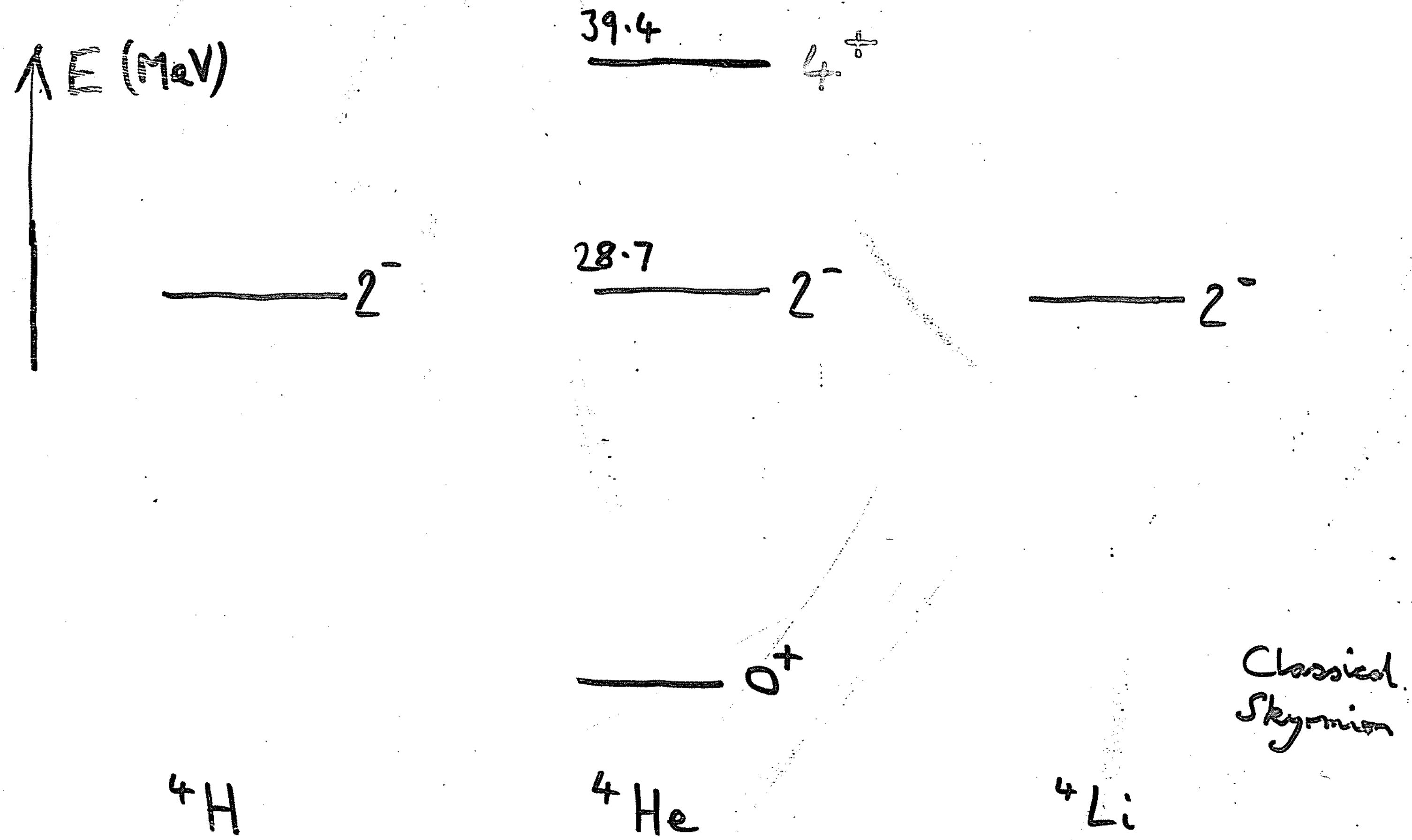
Each symmetry of Skyrmion acts as ± 1 on states.

(Rational map helps in calculating these signs.)
(S. Krusch)

This correlates the possible spin/isospin combinations.

Quantization for $B = 1, 2, 3$ well known. (Adkins, Nappi, Witten; Braaten, Carson)

$B = 4$ Skyrmion Levels (Irwin; Manko, Manton + Wood)



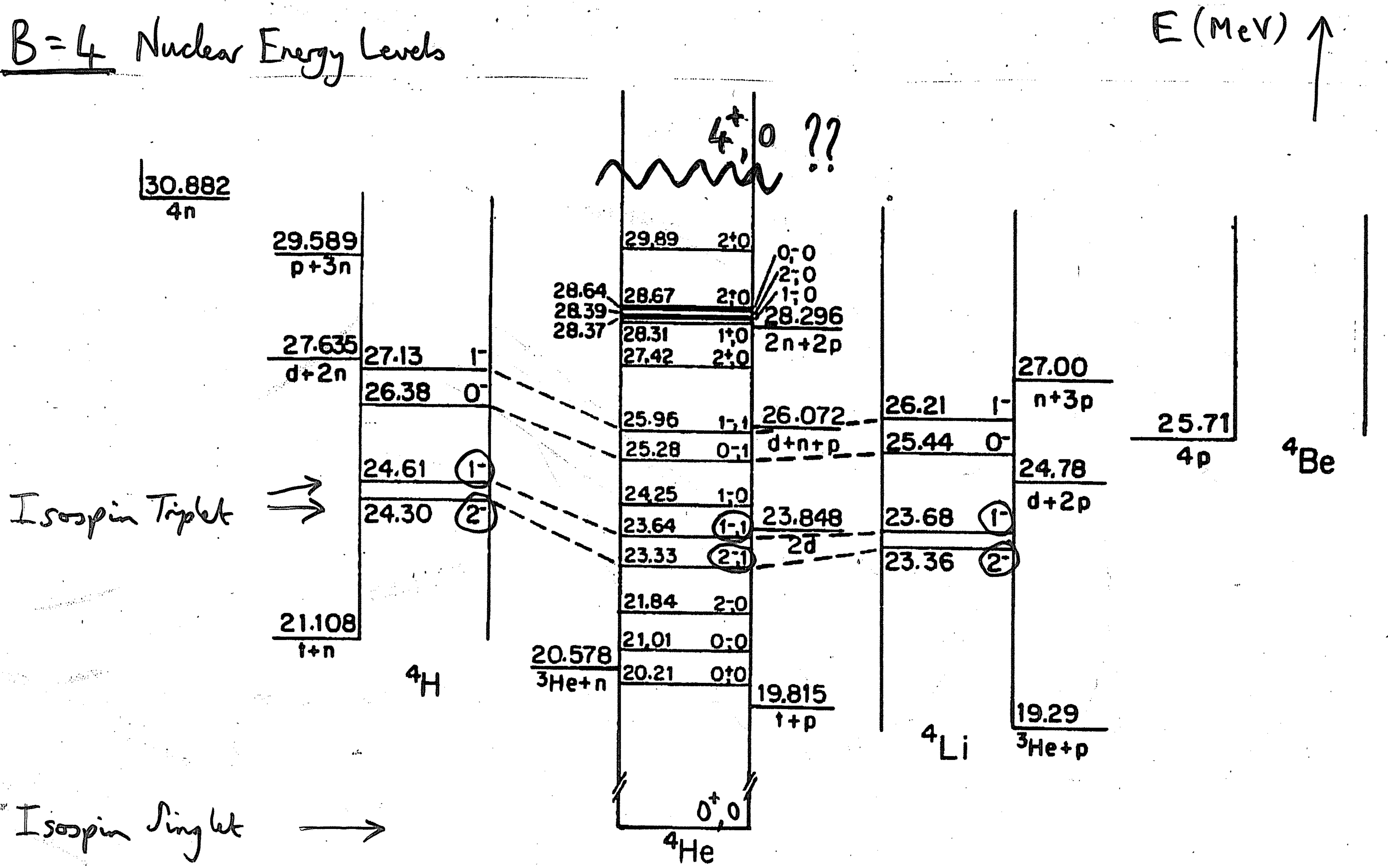
Note: No 2^+ state of ^4He around 10 MeV.

These states satisfy:

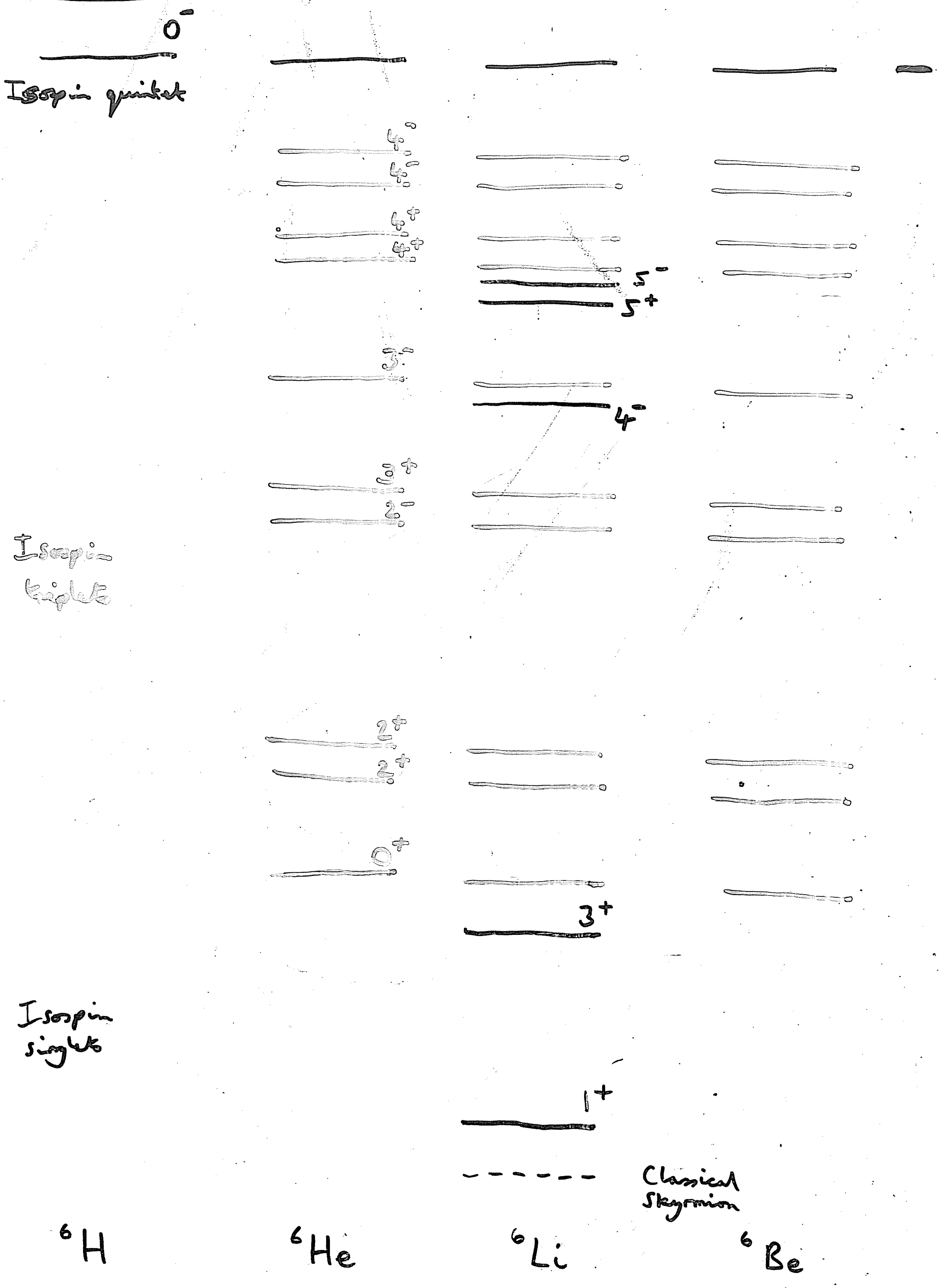
$$\left.\begin{aligned} e^{i\frac{\pi}{2}L_3}\, e^{i\pi K_1}\, |\Psi\rangle &= |\Psi\rangle \\ e^{i\frac{2\pi}{3\sqrt{3}}(L_1+L_2+L_3)}\, e^{i\frac{2\pi}{3}K_3}\, |\Psi\rangle &= |\Psi\rangle \end{aligned}\right\} \text{Cubic symmetry generators}$$

where $\underline{L}$ and $\underline{K}$ are <u>spin</u> and <u>isospin</u> operators.

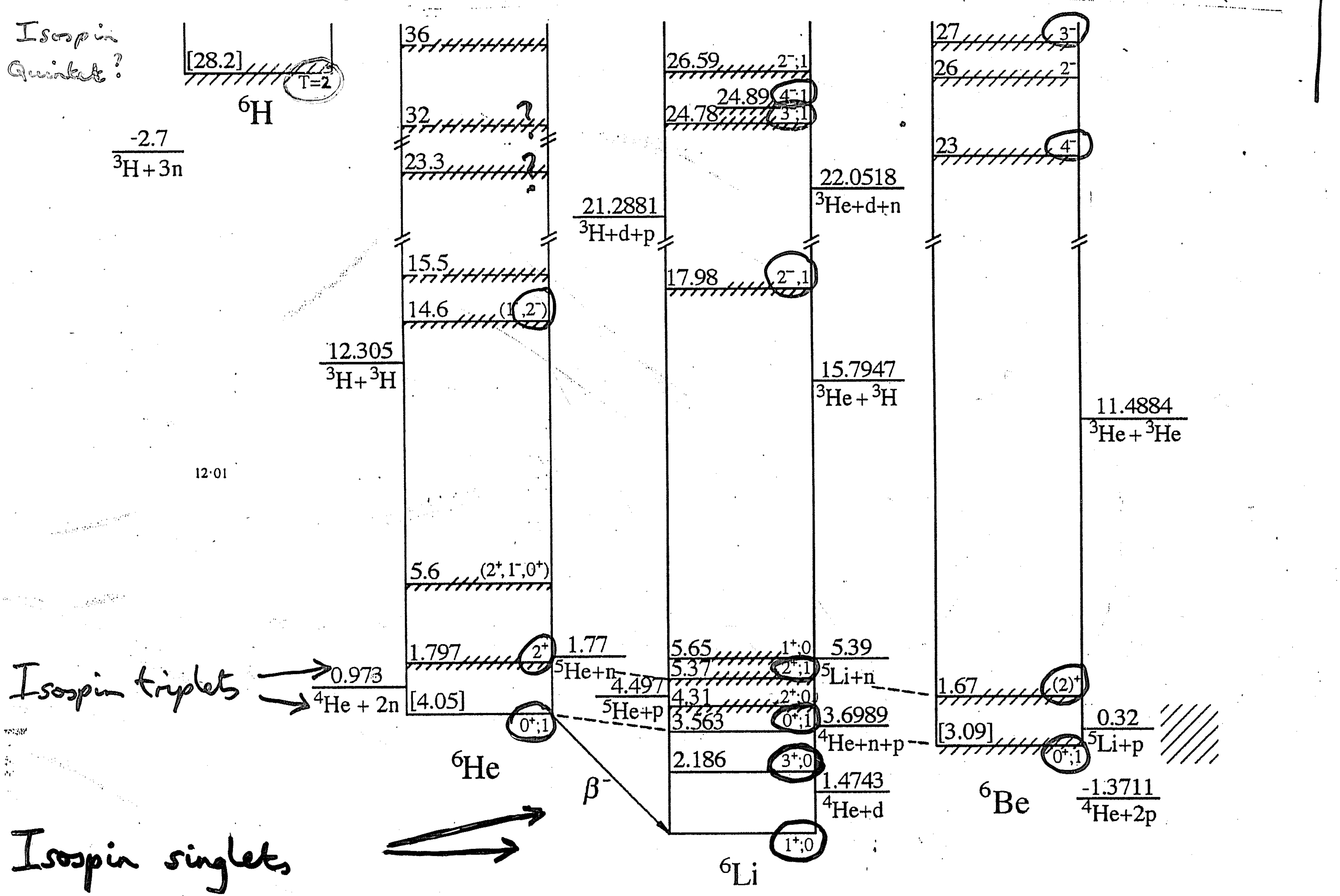
B=4 Nuclear Energy Levels



Energy Levels of Quantized $B = 6$ Skyrmion (Wood et al.)



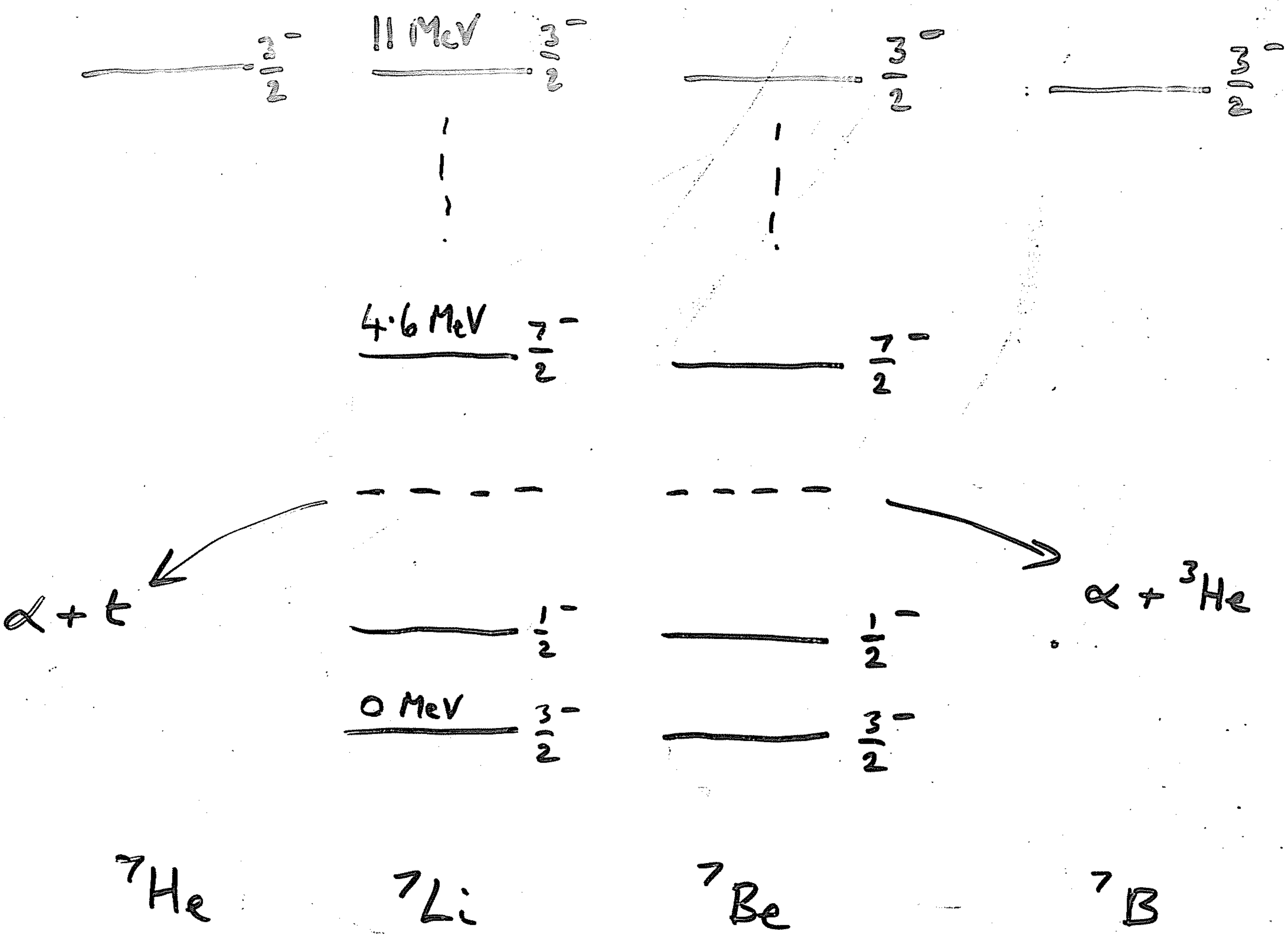
B = 6 Nuclear Energy Levels



B = 7 States (Irwin; ...)

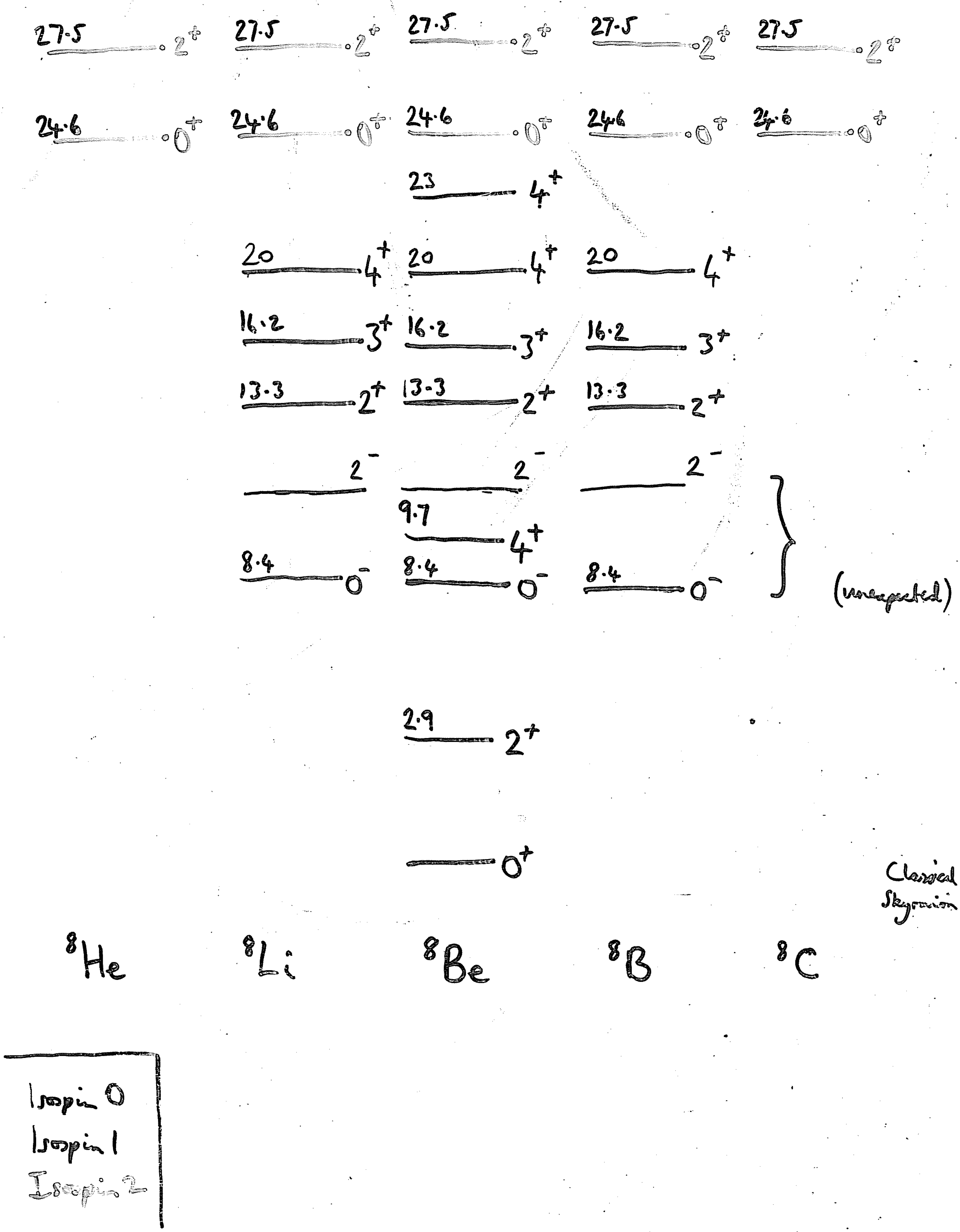
Dodecahedral B = 7 Skyrmion has ground state $J = \frac{7}{2}$, $I = \frac{1}{2}$.

Nuclear Spectrum :

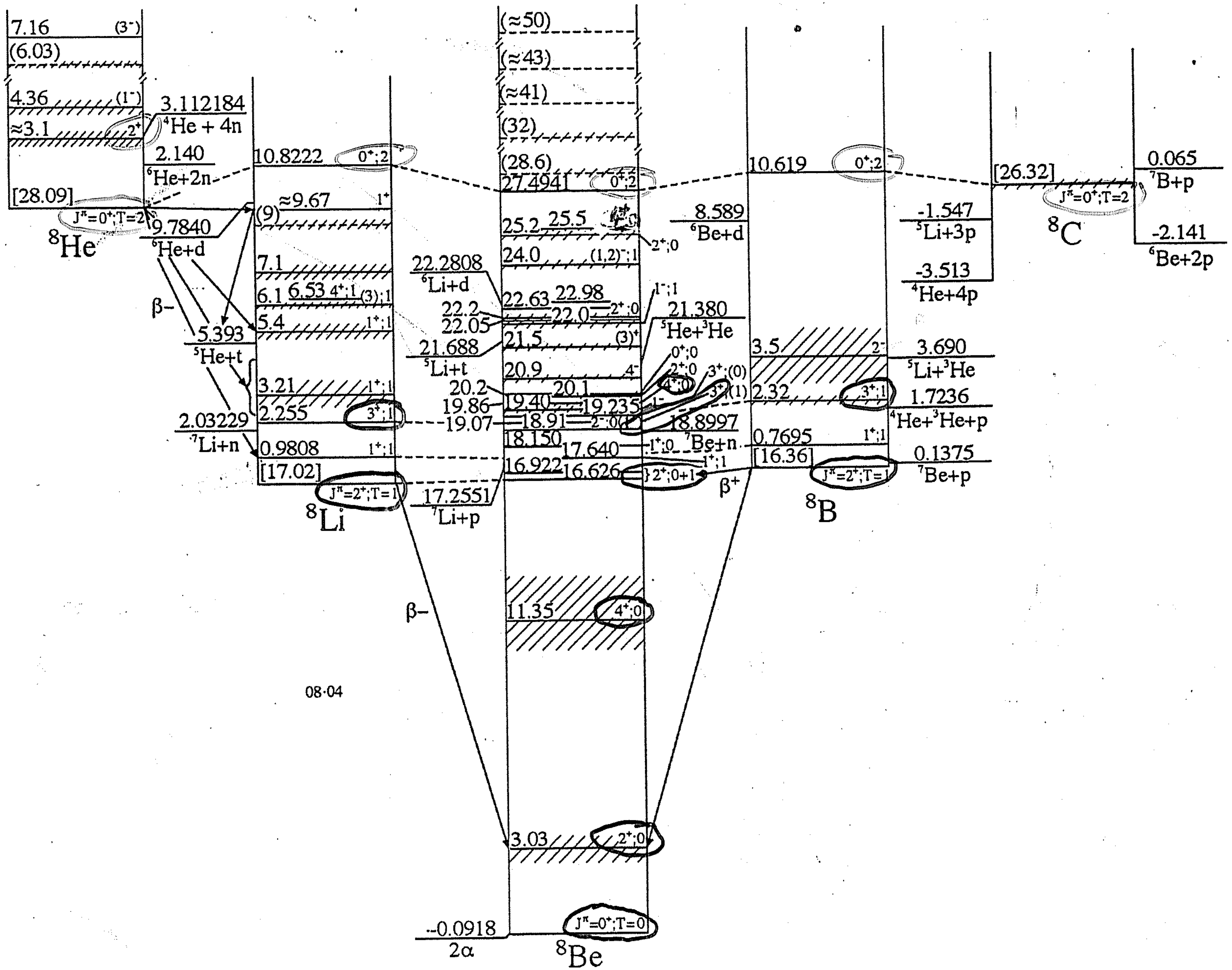


Need model including $(B=4)+(B=3)$ clusters to capture spectrum better. (Work in progress.)

B=8 Skyrmion Levels (Manko, Manton + Wood)



B = 8 Nuclear Energy Levels



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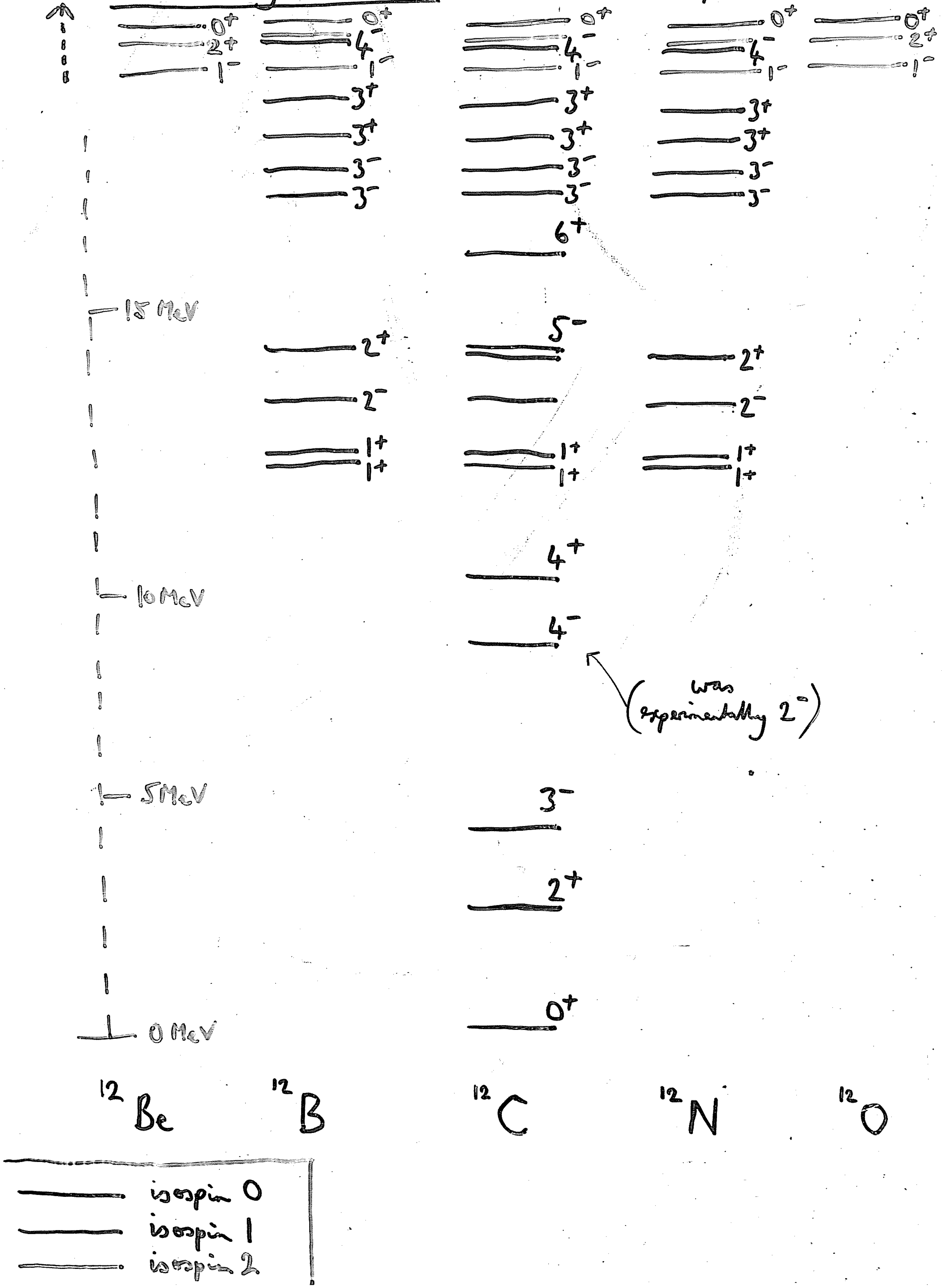
B = 12 Skyrmion Levels (Battye, NSM, Sutcliffe, Wood)



B = 12 Nuclear Energy Levels

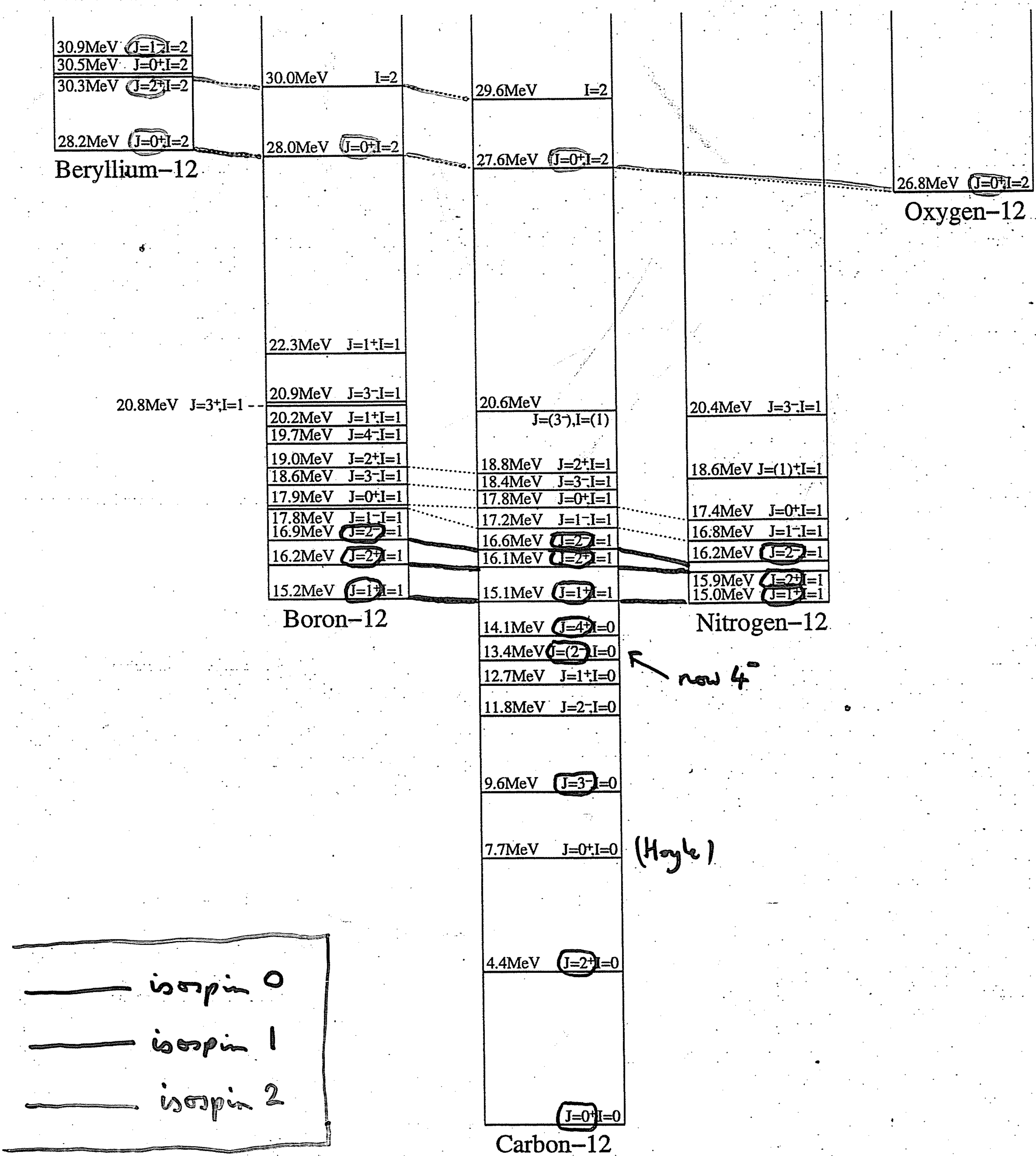
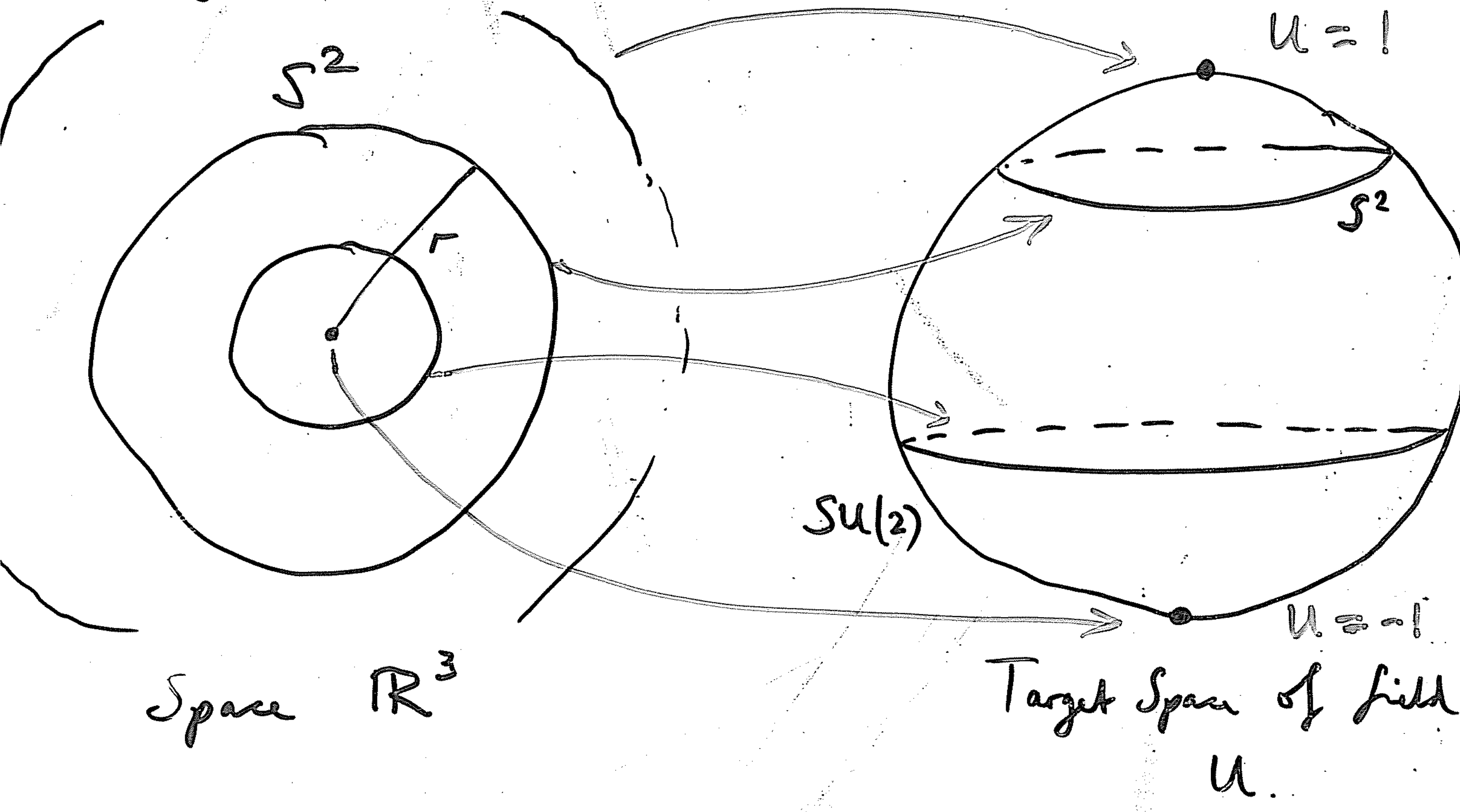


FIGURE III.8. Energy level diagram for nuclei of mass number 12.

Energy Minimizers (and Approximate Minimizers)



For moderate B (between 1 and 7), the Skyrmion has $U = -1$ at centre.

Approximately, a sphere S^2 in space is mapped to a sphere of latitude S^2 on $SU(2)$.

This mapping $S^2 \to S^2$ is independent of r, and is a rational map of degree B in stereographic coord. z.

The most symmetric rational map is favoured. E.G for $B = 4$,

$$R(z) = \frac{z^4 + 2\sqrt{3}\,i\,z^2 + 1}{z^4 - 2\sqrt{3}\,i\,z^2 + 1}$$

Energy Minimizing Rational Maps

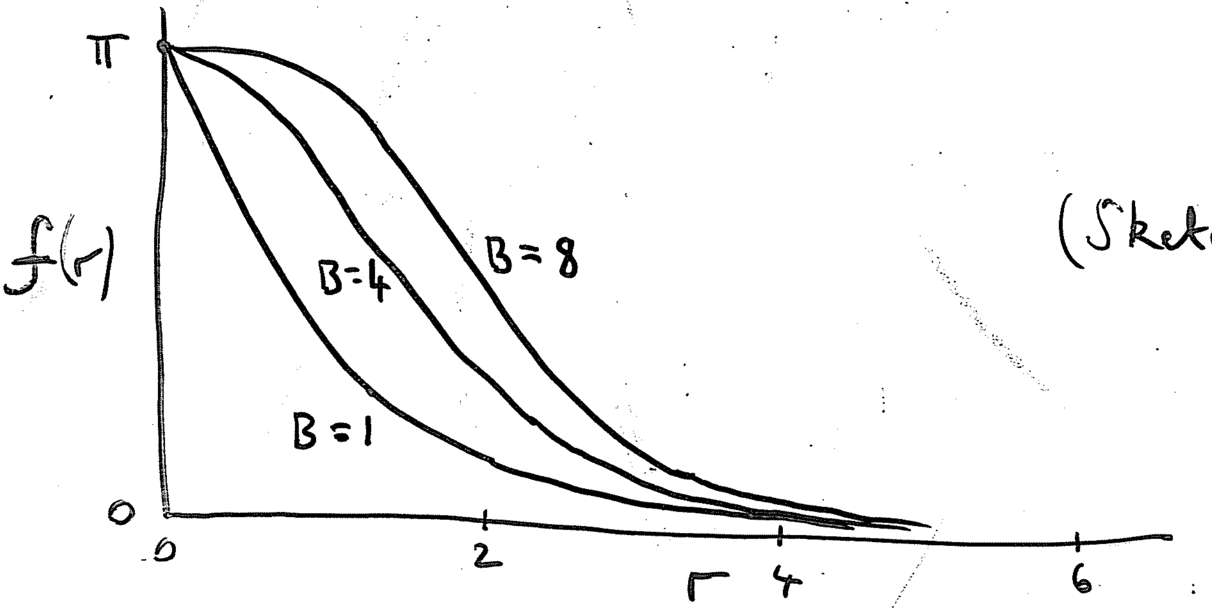
(Houghton, Manton, Sutcliffe)

B	$R(z)$		Angular part of energy $\mathcal{I}$	Symmetry
1	z		1	$O(3)$
2	z^2		5.81	$O(2) \times \mathbb{Z}_2$
3	$\dfrac{\sqrt{3}iz^2 - 1}{z(z^2 - \sqrt{3}i)}$		13.58	T_d
4	$\dfrac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}$		20.65	O_h
5	$\dfrac{z(z^4 + bz^2 + a)}{az^4 - bz^2 + 1}$	$a = 3.07$, $b = 3.94$	35.75	D_{2d}
6	$\dfrac{z^4 + a}{z^2(az^4 + 1)}$	$a = 0.16i$	50.76	D_{4d}
7	$\dfrac{z^5 - a}{z^2(az^5 + 1)}$	$a = \frac{1}{7}$	60.87	Y_h
8	$\dfrac{z^6 - a}{z^2(az^6 + 1)}$	$a = 0.14$	85.65	D_{6d}

These maps were found by assuming the symmetries, and then minimizing $\mathcal{I}$ with respect to one or two parameters (e.g. a and b for $B = 5$)

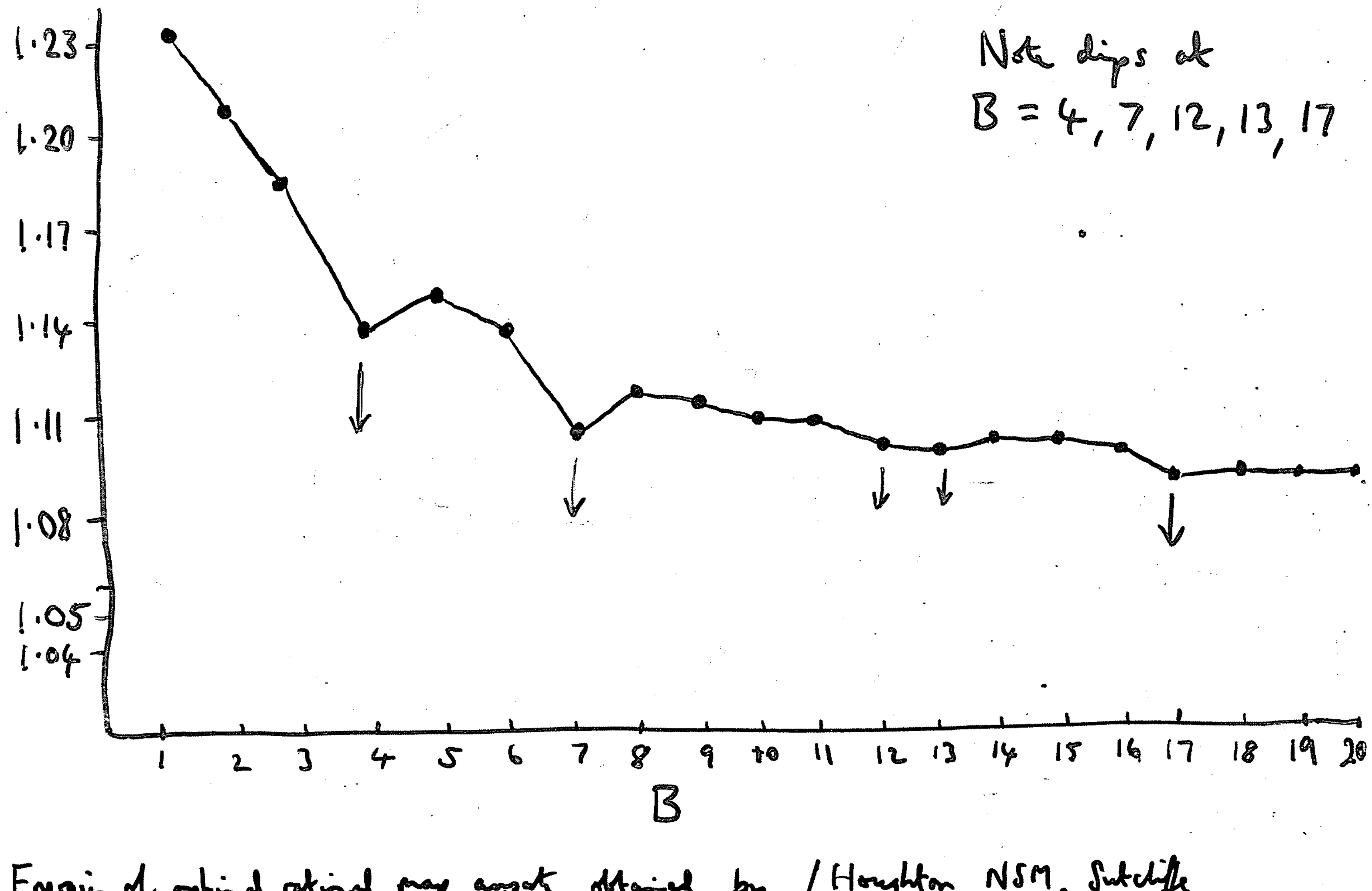
Battye and Sutcliffe have recently made a systematic search for maps that minimize $\mathcal{I}$, using simulated annealing. They have results up to $B = 20$, and confirm the results above.

Radial Profiles for $m = 0$



As B increases, the central "hole" where $U = -1$ gets bigger.

Energies (plot of $E / 12\pi^2 B$)



Energies of optimal rational map ansatz, obtained by { Houghton, NSM, Sutcliffe / Battye and Sutcliffe

<u>Conclusions</u>

- Skyrmions – a field theoretical model for nuclei. Gives structure to protons and neutrons and allows them to merge. Nucleon – nucleon potential energies from model. (NN force includes tensor force.)
- Unifies spin and isospin excitations.
  (e.g. ^{6}Li and ^{6}He have similar intrinsic structure)
- Topology underlies stability and spin / isospin selection rules.

- Further refinement needed – include vibrational modes, strange quarks. Ideally, join up with Chiral Perturbation Theory and QCD. (Model has some variants – some arise from String Theory.)

- Work completed ~~underway~~ to study ^{10}B ^{12}C etc.
  New ideas needed for ^{7}Li (in progress).

- More moduli would be desirable – use instantons and AdS/QCD.