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# Stability in generalized local and non local Modified Gravity

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Contents:

# Current acceleration of visible Universe

Recent astrophysical data are in agreement with a universe in current phase of accelerated expansion, in contrast with the predictions of pure Einstein gravity in FRW space-time. Most part of energy contents (roughly 75%) in the universe is due to mysterious entity with negative pressure: Dark Energy Issue.

The simplest explanation:

- $\Lambda CDM$: Einstein gravity plus a small positive cosmological constant and Cold Dark Matter

$$\Lambda_{de} \equiv \Lambda = constant \qquad \Longrightarrow \qquad \frac{p_{de}}{\rho_{de}} = w_{de} = -1$$

suffers from the coincidence problem: why now has it this value? and the cosmological constant issue: $\Lambda_{ob}/\Lambda_{th} \simeq 10^{-120}$, very huge fine tuning, ( assuming supersymmetry again $\Lambda_{ob}/\Lambda_{th} \simeq 10^{-60}$).

♣ Alternative explanations

- Modification of gravity on large scale: Dvali-Gabadze-Porrati extra dimensional brane-world model.

- scalar-tensor theories: dark energy associated with cosmological scalar fields (quintessence: $w_{de} > -1$, phantom: $w_{de} < -1$,...)

- modified gravity: $R \to R + f(R)$, and generalization, dark energy has a geometrical origin due to higher-order curvature terms in the action.

These models may mimic $\Lambda CDM$, but with an effective non constant $\Lambda_{eff}$:

$$w_{eff} = \frac{p_{eff}}{\rho_{eff}} = -1 - B(H, \dot{H}, ..) \qquad H(t) = \frac{\dot{a(t)}}{a(t)}$$

: Hubble parameter

## ♣ Observations

In the analysis of experimental data one usually assumes $\Lambda_{de} = constant$ and one obtains the constraint

$$0.98 < |w_{de}| < 1.1 \quad w_{de} \simeq -1.04$$

Thus, observational data does not discriminate between

- $\Lambda CDM$ model ($w_{eff} = -1$)
- quintenssence ($w_{eff} > -1$)
- phantom ($w_{eff} < -1$)

Modified gravity models may permit natural transitions between $w_{eff} > -1$ and $w_{eff} < -1$. Future observations might confirm such transitions.

# Modified Gravity as models for Dark Energy

We recall that $\Lambda CDM$ model is the simplest modification to account for recent cosmological acceleration but, it is worth investigating more general modifications. Possible motivations run from quantum corrections to string models:

$$R \longrightarrow R + f(R) = F(R) \quad \text{or} \quad F(R,G), \quad F(R,G,Q,..)$$

$G$, $Q$ curvature invariants. Not new idea and it has been used in the past by many authors, for example as models for inflation where $f(R) = aR^2$ is induced by quantum effects
conformal anomaly induced model (Starobinsky 80).
Recently their interest in cosmology was triggered by the model (Capozziello et al 03., Carrol et al 03.) proposed in order to describe the current acceleration, the modification being

$$f(R) = -\frac{\mu^4}{R}\,.$$

♣ Examples: Modified $F(R)$ and $F(R,G)$ gravity in Jordan and Einstein frames

To begin with, we remind that MG (for example $F(R)$ models) are conformally equivalent to Einstein's gravity, coupleld with a self-interacting scalar field: Einstein frame formulation.

We will consider only the Jordan frame: dynamics of gravity described by $F(R)$ or $F(R,G)$ with

Minimally coupled matter    Matter follows geodesics .

Because of the minimal coupling of radiation and matter the observations are typically interpreted in the Jordan frame.

# The de Sitter stability issue

The stability of the de Sitter solution, relevant for inflation and Dark energy, may be investigated in modify models, at least, in four ways:

- Inhomogeneous perturbation in gauge invariant formalism
- Perturbation of Eqs. of motion in the Jordan frame
- One-loop gravity calculation around de Sitter background
- Dynamical system approach in FRW space-time

The first one has been considered by Faraoni (2005). We shall briefly discuss the last three approaches. The second and last one, DSA, can (quite directly) be extended to more general modified models as well as other critical points besides de Sitter one.

♣ dS stability of $F(R)$ and $F(R,G)$ models in the Jordan frame

Starting point: the trace of the equations of motion, which is not dynamical in Einstein gravity $R=-\kappa^2 T$, but for a general $f(R)$ model, is

$$3\Box f'(R)-2f(R)+Rf'(R)-R=\kappa^2 T\,.$$

The new non trivial extra dof : Scalaron: $1+f'(R)=e^{-\chi}$ propagates. Requiring $R=R_0=Cst$, one has de Sitter existence condition

$$R_0+2f(R_0)-R_0f'(R_0)=0\,,\qquad \text{in vacuum.}$$

Stability issue:
perturbation around dS: $R=R_0+\delta R$, with

$$\delta R=-\frac{1+f'(R_0)}{f''(R_0)}\delta\chi\,,\qquad f''(R_0)\neq 0$$

one arrives at Scalaron perturbation Eq.

$$\Box\delta\chi - M^2\delta\chi = -\frac{\kappa^2}{6(1+f'(R_0))}T\,.$$

Scalaron effective mass

$$M^2 \equiv \frac{1}{3}\left(\frac{1+f'(R_0)}{f''(R_0)} - R_0\right)\,.$$

If $M^2 < 0$, tachyon and instability. Thus $M^2 > 0$ one has stability and the related condition reads

$$M^2 > 0 \qquad \Longrightarrow \qquad \frac{1+f'(R_0)}{R_0 f''(R_0)} > 1\,.$$

$M^2$ has to be very large in order to pass both the local and the astronomical tests and $1+f'(R) > 0$, in order to have a positive effective Newton constant.

In a similar way, consider as an example $F(R,G)$ models, the trace of related EoMs

$$3\nabla^2 F'_R - 4G^{\mu\nu}\nabla_\mu\nabla_\nu F'_G - 2F + RF'_R + 2GF'_G = \kappa^2 T\,.$$

Requiring $R = R_0 = Cts$ and $G = G_0 = Cts$, one has de Sitter existence condition in vacuum

$$2F_0 - R_0 F'_{R_0} - 2G_0 F' G_{R_0} = 0\,.$$

Perturbing around dS space, namely $R = R_0 + \delta R$, and with $G = G_0 + \delta G$, observing that $\delta G = \frac{R_0}{3}\delta R$, one arrives at the perturbation Eq.

$$-\nabla^2 \delta G + M^2 \delta G = 0\,,$$

in which the scalaron effective mass reads

$$M^2 = \frac{R_0}{3}\left(\frac{9F'_{R_0}}{R_0[9F''_{R_0R_0} + 6R_0F''_{R_0G_0} + R_0^2 F''_{G_0G_0}]} - 1\right)\,.$$

Thus, if $M^2 > 0$, one has stability of the dS solution and the related condition reads

$$\frac{9F'_{R_0}}{R_0[9F''_{R_0R_0} + 6R_0F''_{R_0G_0} + R_0^2F''_{G_0G_0}]} > 1\,.$$

which generalizes the previous dS stability condition related to $F(R)$ models.

Example: $F(R) = R + f(G)$, the dS stability condition is

$$\frac{9}{R_0^3 f''_{G_0G_0}} > 1\,.$$

Similarly for more general $F(R, G, P, ..)$ models.

## ♣ One-loop $F(R)$ quantum gravity partition function

An approach in the spirit of effective theories: one has not to worry about the renormalizzability, but one has to consider the non-renormalizzability arising as a low-energy/long-distance approximationm of an underlying unknown fundamental theory.

This approach has a general validity and it can be applied in QED.

If we apply this method to a generic $F(R)$ model, one has to make a further approximation and limit himself to one-loop order, generalizing the seminal work by Fradkin and Tseytlin 83, who considered Einstein gravity with a cosmological constant

Here the main ideas of the $F(R)$ gravity one-loop calculation (Cognola, Elizalde, Nojiri, Odintsov,Z 05). Work in the Euclidean path integral formulation. Recall the dS existence condition $2F_0 = R_0 F'_0$. Assume that it is satisfied (on-shell codition). The small fluctuations around this dS instanton are: (Euclidean dS is $SO(4)$, maximally symmetric space)

$$g_{ij} = g_{0,ij} + h_{ij}\,, \qquad g^{ij} = g_0^{ij} - h^{ij} + h^{ik}h_k^j + \mathcal{O}(h^3)\,, \qquad h = g_0^{ij} h_{ij}\,,$$

and up to second order in $h_{ij}$

$$\frac{\sqrt{g}}{\sqrt{g_0}} = 1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{ij}h^{ij} + \mathcal{O}(h^3)$$

$$\begin{aligned} R \;=\; & R_0 - \frac{R_0}{4}\,h + \nabla_i\nabla_j h^{ij} - \Delta\, h \\ & + \frac{\hat{R}}{4}\,h^{jk}h_{jk} - \frac{1}{4}\,\nabla_i h \nabla^i h - \frac{1}{4}\,\nabla_k h_{ij}\nabla^k h^{ij} + \nabla_i h_k^i \nabla_j h^{jk} - \frac{1}{2}\,\nabla_j h_{ik}\nabla^i h^{jk}\,. \end{aligned}$$

Making use of the standard expansion of the tensor field $h_{ij}$ in irreducible components, namely

$$h_{ij} = h_{0ij} + \nabla_i \xi_j + \nabla_j \xi_i + \nabla_i \nabla_j \sigma + \frac{1}{4} g_{ij} (h - \Delta\, \sigma)\,,$$

where $\sigma$ is the scalar component, while $\xi_i$ and $h_{0ij}$ are the vector and tensor components with the properties

$$\nabla_i \xi^i = 0\,, \qquad \nabla_i h^i_{0j} = 0\,, \qquad h^i_{0i} = 0\,.$$

and making an expansion up to second order in the fields, one arrives at a very complicated Lagrangian density $L_2$, not reported here, describing Gaussian fluctuations around dS space. In order to quantise the model described by $L_2$, one has to add gauge fixing and related Fadeev-Popov ghost contributions.

Then, the computation of Euclidean one-loop partition function reduces to the computations of functional determinants. The functional determinants are divergent and may be regularized by zeta-function regularization.
Simplest example: $\lambda\phi^4$ scalar field. With $\phi = \Phi_0 + \eta$, the one-loop fluctuation operator is ($\Phi_0$ background field)

$$A = -\Box + m^2 + \frac{\lambda}{2}\Phi_0^2$$

For gauge theories, $A$ is singular due to the gauge invariance and a gauge fixing+ ghost contributions are necessary. The one-loop quantum partition function $Z[A]$ ($S_0$ classical action)

$$Z[A] \simeq e^{-S_0} \int d[\eta]\, e^{-\frac{1}{2}\int d^4x \eta A \eta}$$

reduces to a Gaussian functional integral computable in terms of the real eigenvalues $\lambda_n, A\phi_n = \lambda_n\phi_n$. Since $\phi = \sum_n c_n\phi_n$ formal functional measure $d[\phi]$ is ($\mu$ arbitrary renormalization parameter)

$$d[\phi] = \prod_n \frac{dc_n}{\sqrt{\mu}}$$

One-loop quantum "prefactor"

$$Z_1[A] = \prod_n \frac{1}{\sqrt{\mu}} \int_{-\infty}^{\infty} dc_n e^{-\frac{1}{2}\lambda_n c_n^2} = \left[\det(\mu^{-2}A)\right]^{-1/2}$$

One-loop Euclidean Effective Action

$$\Gamma_E =: -\log Z = S_0 + \frac{1}{2}\log(\det \mu^{-2} A)$$

Functional determinants ? Recall Schwinger argument :
from $\log\det A = \mathrm{Tr}\log A$ one has $\delta \log\det A = \mathrm{Tr} A^{-1}\delta A$ thus

$$(\log\det A) = -\left(\int_0^{\infty} dt\, t^{-1} \mathrm{Tr}\, e^{-tA}\right)$$

For large $t$, no problem $A$ non negative, but for small $t$ the Heat Kernel expansion

$$\mathrm{Tr}\, e^{-tA} \simeq \sum_{r=0}^{\infty} A_r t^{r-2}$$

Trouble: the functional determinant is divergent a $t = 0$. Need for a regularization. Useful is dimensional regularization

$$t^{-1} \to \frac{t^{\varepsilon-1}}{\Gamma(1+\varepsilon)}$$

Regularized functional determinant with $\varepsilon$ large

$$\log \det A(\varepsilon) = -\int_0^\infty dt \, \frac{t^{\varepsilon-1}}{\Gamma(1+\varepsilon)} \mathrm{Tr}\, e^{-tA} = -\frac{\zeta(\varepsilon, A)}{\varepsilon}$$

where the generalized zeta function associated with $A$ (defined for $Res > 2$)

$$\zeta(s, A) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, \mathrm{Tr}\, e^{-tA}$$

To be able to remove the cutoff one make use of
Seeley Theorem:
The analytic continuation of $\zeta(s,A)$ in the whole complex space $s$ is regular at $s=0$.
Thus providing the zeta-function determinant (Ray-Singer 71, Hawking 1975)

$$\log\det A = -\zeta'(0,A)$$

Within dimensional regularization

$$\log\det A(\varepsilon) = -\frac{1}{\varepsilon}\zeta(0,A) - \zeta'(0,A) + O(\varepsilon)$$

The computable Seeley-de Witt coefficient $A_2 = \zeta(0,A)$ gives the ultraviolet divergence, $\zeta'(0,A)$ gives the finite contribution (in general, difficult computational task). In our case, Euclidean dS is $SO(4)$ and one knows the spectrum of Laplace like operators, and the analytic continuation of $\zeta(s,A)$ can be performed.

In the $F(R)$ model case, one has 3 finite contributions associated with tensor, vector and scalar decompositions. The evaluation requires a huge calculation and leads to on-shell one-loop effective action

$$\begin{aligned}\Gamma_{on-shell} &= \frac{24\pi F_0}{GR_0^2} + \frac{1}{2}\log\det\left[\ell^2\left(-\Delta_2 + \frac{R_0}{6}\right)\right] \\ &\quad -\frac{1}{2}\log\det\left[\ell^2\left(-\Delta_1 - \frac{R_0}{4}\right)\right] \\ &\quad +\frac{1}{2}\log\det\left[\ell^2\left(-\Delta_0 - \frac{R_0}{3} + \frac{2F_0}{3R_0F_0''}\right)\right].\end{aligned}$$

The last term is the modification with respect to Einstein theory. As a result, in the scalar sector, one has an effective mass $M^2 = \frac{1}{3}\left(\frac{2F_0}{R_0F_0''} - R_0\right)$. Stability requires $M^2 > 0$, in agreement with the previous scalaron analysis and gauge invariant formalism (Faraoni 2005).

# Stability: Dynamical system approach

(Ellis, Amendola, Tsusikawa, Dunsby, Troisi, and many others.)
Work in FRW spatial flat metric

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$$

Main idea: rewrite the generalized Einstein-Friedmann equations in an equivalent system of first order differential equations, introducing new dymamical variables $\Omega_i$:

$$\frac{d}{dt}\vec{\Omega}(t) = \vec{v}(\vec{\Omega}(t))\,, \qquad \vec{\Omega} \equiv \left(\frac{R}{6H^2}, \frac{R}{6H^2 F_R'}, \frac{\dot{F}_R}{HF_R}, \dots\right)$$

here the evolution parameter has been denoted by $t$ (typically $\ln a(t)$).
The critical (or fixed ) points are defined by

$$\frac{d}{dt}\vec{\Omega}_0 = 0 \qquad \Longrightarrow \qquad \vec{v}(\vec{\Omega}_0) = 0$$

## ♣ The stability Theorem

The key point is:
Hartman-Grobman theorem:
The orbit structure of a dynamical system in the neighbourhood of a hyperbolic fixed point is topologically equivalent to the orbit structure of the associated linearized dynamical system, defined by a stability matrix $M_0$.

Recall that a hyperbolic fixed point is such that its stability matrix $M_0$ does not have vanishing eingenvalues.

In other words the theorem states:
The flux of a dymanical system in a neighbourhood of a hyperbolic fixed point can be continuosly deformed to the flux of the related linearization.

As a result:
in order to study the stability of the above non linear system of differential Eqs. at the critical points, it is sufficient to investigate the related linear system of differential Eqs.:

$$\frac{d}{dt}\delta\vec{\Omega}(t) = M_0 \delta\vec{\Omega}(t)\,, \quad M_0 \quad \text{stability matrix} \quad \text{evaluated at } \vec{\Omega}_0$$

The solution of the linearization is simple and reads

$$\vec{\delta}\Omega(t) = e^{(t-t_1)\,M_0}\vec{\delta}\Omega(t_1)$$

Stability: determined by the signs of the eigenvalues of $M_0$.
Thus: the non linear system is stable if all eingenvalues of the matrix $M_0$ have negative real parts.

♣ Stability for $F(R)$ models: the de Sitter case

Limiting to de Sitter case, neglecting matter and radiation, one may deal with a very simple autonomous system in the two unknown quantities $R$ and $H$

$$\dot{R} = \frac{1}{f''}\left(f'H + \frac{f - Rf'}{6H}\right),$$

$$\dot{H} = \frac{R}{6} - 2H^2 .$$

The critical points are defined by $\dot{R} = 0$ and $\dot{H} = 0$:

$$H_0^2 = \frac{R_0}{12}, \qquad 2f_0 = R_0 f_0' .$$

The linearized system around de Sitter critical point simply reads

$$\dot{\delta R} = H_0 \delta R - \frac{4f_0}{f_0''}\delta H ,$$

$$\dot{\delta H} = \frac{\delta R}{6} - 4H_0 \delta H .$$

The eigenvalues of the stability matrix depend on

$$\beta_0 = \frac{1 + f'(R_0)}{R_0 f''(R_0)}$$

The stability condition associated with the de Sitter critical point requires negative eingenvalues of stability matrix

$$1 < \beta_0 \qquad \Longrightarrow \qquad \frac{1 + f'(R_0)}{R_0 f''(R_0)} > 1\,.$$

In agreement with scalaron perturbation analysis and one-loop de Sitter calculation.

In the matter-radiation sector, where $\rho$ is non vanishing, other critical points may exist, but their analytical determination, in realistic cases, could become problematic, and numerical analysis is required.

## ♣ Stability for $F(R,G)$ models: the de Sitter case

Again limiting to de Sitter case, for the sake of simplicity let us consider a modified model $R+f(G)$, The related autonomous system in the two unknown quantities $G$ and $H$ reads

$$\dot{G} = \frac{1}{24 f''_G H^3}\Big((G f'_G - f) - 6H^2\Big)$$

$$\dot{H} = \frac{G}{24H^2} - H^2 \,.$$

The critical points are defined by $\dot{G}=0$ and $\dot{H}=0$ Thus, we have the solutions

$$H_0^4 = \frac{G_0}{24}\,, \qquad 2G_0 f'_0 - f_0 = 6H_0^2\,.$$

The linearized system around de Sitter critical point simply reads

$$\dot{\delta G} = H_0 \delta G - \frac{1}{2H_0^2 f_0''} \delta H \,,$$

$$\dot{\delta H} = \frac{\delta G}{24 H_0^2} - 4 H_0 \delta H \,.$$

As a result, one can read off the stability matrix and the stability condition is

$$\frac{9}{R_0^3 f_0''} > 1 \,.$$

in agreement with the perturbation approach in Jordan frame.

## ♣ Stability for $F(R)$ models in presence of matter

For the sake of simplicity we consider only the $F(R)$ models. In presence of matter, the new variables may be defined by

$$\Omega_R = \frac{R}{6H^2}\,, \quad \Omega_F = -\frac{f(R) - Rf'(R)}{6H^2(1 + f'(R))}\,, \quad \Omega_\rho = \frac{\chi\rho}{3H^2(1 + f'(R))}\,,$$

Dynamical system equivalent to Einstein-Friedman Eqs. reads

$$\frac{d}{d\alpha}\Omega_R = 2\Omega_R(2 - \Omega_R)\Omega_R - \beta\,(1 - \Omega_F - \Omega_\rho)$$

$$\frac{d}{d\alpha}\Omega_F = 2\Omega_F(2 - \Omega_R) + (\Omega_F - \Omega_R)(1 - \Omega_F - \Omega_\rho)$$

$$\frac{d}{d\alpha}\Omega_\rho = [2(2 - \Omega_R) - 3(w + 1) + 1 - \Omega_F - \Omega_\rho]\Omega_\rho\,,$$

here the evolution parameter is $\alpha(t) = \ln a(t)$ and $w = \frac{p}{\rho}$, and the function $\beta$ is

$$\beta(R) = \frac{1 + f'(R)}{Rf''(R)}\,.$$

There is also the quantity

$$\Omega_{\dot{F}} = -\frac{\dot{f}'(R)}{H(1 + f'(R))}\,,$$

which satisfies the constraint

$$\Omega_{\dot{F}} + \Omega_F + \Omega_\rho = 1\,.$$

Note that one has a complete autonomous system as soon as the quantity $\beta$ can be expressed as a function of $\Omega_i$. This requires the inversion of

$$\frac{Rf'(R) - f(R)}{R(1 + f'(R))} = \frac{\Omega_F}{\Omega_R} \qquad \Longrightarrow \qquad R = R(\frac{\Omega_F}{\Omega_R})$$

After this inversion, in principle, one has $\beta = \beta(\Omega_R, \Omega_F)$, and may close the above system. The possible problems are: non unique inversions, non trivial domains with divergent points, ect.

## ♣ Critical points in $F(R)$ models

The non linear algebraic system for critical points is

$$0 = 2\Omega_R(2-\Omega_R)\Omega_R) - \beta(1-\Omega_F-\Omega_\rho)\,,$$

$$0 = 2\Omega_F(2-\Omega_R) + (\Omega_F-\Omega_R)(1-\Omega_F-\Omega_\rho)$$

$$0 = [2(2-\Omega_R) - 3(w+1) + 1 - \Omega_F - \Omega_\rho]\Omega_\rho\,.$$

In vacuum $\rho = 0$, de Sitter critical point always exists:

$$\Omega_R = 2\,, \quad \Omega_F = 1\,, \quad \Omega_\rho = 0\,.$$

$$\Omega_R = 2\,, \qquad \Longrightarrow \qquad R_0 = 12H_0$$

and

$$\Omega_F = 1 \qquad \Longrightarrow \qquad R_0 = R_0 f'(R_0) - 2f(R_0)\,,$$

which coincides with the de Sitter existence condition. The linear system at de Sitter critical point $(2, 1, 0)$ is:

$$\frac{d}{d\alpha}\delta\Omega_R = -4\,\delta\Omega_R + 2\beta_0\,\delta\Omega_F + 2\beta_0\,\delta\Omega_\rho$$

$$\frac{d}{d\alpha}\delta\Omega_F = -2\,\delta\Omega_R + \delta\Omega_F + \delta\Omega_\rho$$

$$\frac{d}{d\alpha}\delta\Omega_\rho = 0\;\delta\Omega_R + 0\;\delta\Omega_F - 3\gamma\,\delta\Omega_\rho\,,$$

and one can read off the stability matrix $M_0$.

The eigenvalues of the stability matrix

$$\lambda_1 = -3\gamma\,, \quad \gamma > 0$$

$$\lambda_{2,3} = \frac{1}{2}\left(-3 \pm \sqrt{25 - 16\beta_0}\right)$$

The stability condition associated with the de Sitter critical point requires negative eingenvalues of stability matrix

$$1 < \beta_0 \qquad \Longrightarrow \qquad \frac{1 + f'(R_0)}{R_0 f''(R_0)} > 1\,.$$

In agreement with scalaron perturbation analysis and one-loop de Sitter calculation.

In the matter-radiation sector, where $\Omega_\rho$ is non vanishing, other critical points may exist, but their analytical determination, in realistic cases, could become problematic, since one has to know explicitly $\beta$ in order to close the system. In general, numerical analysis is necessary.

## ♣ Examples of modified $F(R)$ models: the viable models

They have recently been proposed (Hu-Sawicki, Starobinsky, Appleby-Battye, Nojiri-Odintsov, Capozziello-Tsujikawa, and others, 07-09)
Aim: try to describe a large part of history of the universe with a viable $F(R) = R + f(R)$, describing the current acceleration but also compatible with local stringent gravitational tests of Einstein gravity: $F(R) = R$.

Main Idea: disappearing of cosmological constant for low curvature and mimicking the $\Lambda CDM : f(R) = -2\Lambda$ model for high curvature:
Requirements:
a. $f(R) \to 0\,, \quad R \to 0\,,$ local tests
b. $f(R) \to -2\Lambda_0\,, \quad R \to +\infty\,,$ current acceleration
c. local stability of the matter.
Typically, these models have curvature singularities.

## ♣ Hu-Sawicki viable model

The HS model is represented by:

$$f(R) = -A\frac{(R/m^2)^n}{1+(R/m^2)^n}, \qquad n \geq 1\,.$$

$A > 0$, and $m^2$ are arbitrary constants.
When $R \to 0$,

$$f(R) \to -(A-1)\left(\frac{R}{m^2}\right)^n$$

$f(0) = 0$: pure Einstein gravity without cosmological constant.
For $R \to +\infty$,

$$f(R) \to -A\,,$$

an effective cosmological constant.

♣ Starobinsky viable model

It is similar to the HS one, with slightly different algebraic dependence

$$f(R) = A\left(\frac{1}{(1+b^2R^2)^n} - 1\right)\,, \qquad\qquad n \geq 1\,.$$

$A$, $b$ are constants. When $R \to 0$, the behaviour is :

$$f(R) \to -nA\,(b^2R^2)$$

$f(0) = 0$: pure Einstein gravity without cosmological constant.
For $R \to +\infty$,

$$f(R) \to -A\,,$$

again an effective cosmological constant.

## ♣ A further example of viable model

As a last example (Cognola et al. PRD 08)

$$f(R) = -\alpha \left( \tanh \left( \frac{b\,(R - R_1)}{2} \right) + \tanh \left( \frac{bR_1}{2} \right) \right)$$

When $R \to 0$,

$$f(R) \to -\frac{\alpha b R}{2 \cosh^2 \left( \frac{bR_1}{2} \right)}$$

$f(0) = 0$: pure Einstein gravity without cosmological constant.
For $R \to +\infty$,

$$f(R) \to -2\Lambda_0 \equiv -\alpha \left( 1 + \tanh \left( \frac{bR_1}{2} \right) \right) \; .$$

$R \gg R_1$, $R_1$ small enough, $\Lambda_0$ effective cosmological constant. Its advantages are a better formulation in the Einstein frame and a generalization that also includes the inflation era.

Futhermore, besides $F(R)$ models, like $F(R) = R - \frac{\mu^4}{R}$,

or $F(R) = R + aR^2 - b$, for viable models, the determination of the existence of de Sitter may present technical difficulties, since one has to solve:

$$R_0 = K(R_0)\,, \quad K(R_0) = R_0 f'(R_0) - 2f(R_0)\,.$$

and this, in general, may be a difficult task, since is an higher order algebraic Eq. or a trascendent Eq. As an example, for the simplest Starobinsky viable model ($n = 1$, realistic models $n > 2$)

$$f(R) = -c_1 \frac{R^2}{1 + c_2 R^2}\,,$$

one has

$$\left(1 + c_2 R_0^2\right)^2 R_0 - 2c_1 c_2 R_0^2 - 2c_1 R_0^2 \left(1 + c_2 R_0^2\right) = 0$$

which is an algebraic equation of fifth order in $R_0$!

As further example, consider the simple model

$$f(R) = \alpha \left( e^{-bR} - 1 \right) .$$

Note $f(0) = 0$ and $f(R) \to -\alpha$ for large $R$, thus it is a viable model. The existence condition for de Sitter solution is the trascendental equation:

$$R_0 = K(R_0) = 2\alpha + \alpha \, (bR_0 - 1) \, e^{-bR_0} .$$

Here, since $K(0) = 0$ and $K'(0) = \alpha b$, it follows that for $\alpha b > 1$ there exists a non vanishing solution $R_0 \simeq 2\alpha$, while for $\alpha b < 1$ there is no solution, since the growth of $K(R_0)$ is slower than $R_0$. This is an example of $F(R)$ model which may not have dS critical points. For $\alpha b > 1$, the scalaron mass is positive and the dS solution is stable. Note, however, that this model may have antigravity effects in the future, since $1 + f'(R)$ may be negative.

# Generalized Local Models

(Cognola, Gastaldi and S.Z. 08, Cognola and S.Z. 08). DSA is very powerful here. Start parametrizing the FRW space-time as

$$ds^2 = -e^{2n(t)}dt^2 + e^{2\alpha(t)}d\vec{x}^{\,2}\,, \quad N(t) = e^{n(t)}\,, \quad a(t) = e^{\alpha(t)}\,.$$

and consider

$$\mathcal{L} = -\frac{1}{2\chi}F(R, P, Q, ...) + \mathcal{L}_m\,,$$

where $P = R^{\mu\nu}R_{\mu\nu}$ and $Q = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ are the second order quadratic invariants and the dots means other independent algebraic invariants of higher order, and $L_m$ depending on $\rho$ and $p = p(\rho)$. Introduce $X = (\dot{H}/H^2 - \dot{n}/H)$, thus all curvature invariants may be expressed as functions of $H$, $n$, $X$

$$R = 6H^2e^{-2n}(2+X)\,, \quad P = 12H^4e^{-4n}(3+3X+X^2)\,,$$

$$Q = 12H^4e^{-4n}(2+2X+X^2)\,, ...$$

The generalized Friedman equations are, in the gauge $n = 0$, $t$ cosmological time

$$H\dot{F}_{\dot{H}} - HF_H + F - \dot{H}F_{\dot{H}} + 3H^2F_{\dot{H}} = 2\rho\,,$$

$$\ddot{F}_{\dot{H}} - \dot{F}_H + 6H\dot{F}_{\dot{H}} - 3HF_H + 3F + 3\dot{H}F_{\dot{H}} + 9H^2F_{\dot{H}} = -6p\,.$$

Introduce the new variables

$$X = \frac{\dot{H}}{H^2}\,, \quad Y = \frac{F - HF_H}{H^2F_{\dot{H}}} = \frac{F}{F_X} - X\,, \quad Z = \frac{\dot{F}_{\dot{H}} - F_H}{HF_{\dot{H}}} = \frac{F'_X}{F_X} - 2X - \xi\,,$$

prime means derivative with respect to $\alpha$ and the quantity

$$\xi = \xi(X, Y) = \frac{F_H}{HF_{\dot{H}}} = \frac{HF_H}{F_X}\,,$$

is a function of the variables $X$ and $Y$.

The autonomous system is

$$X' = -2X^2 - \gamma X + \beta(Z + \xi)$$

$$Y' = -(2X + Z + \xi)Y - XZ$$

$$Z' = -3(1 + w)(Z + Y + 3) - (Z + \xi)(Z + 3) - X(Z + 6)$$

where $X' \equiv \frac{dX}{d\alpha} = \frac{1}{H}\frac{dX}{dt}$ and $p = w\rho$,and

$$\beta = \beta(X, Y) = \frac{F_{\dot{H}}}{H^2 F_{\dot{H}\dot{H}}} = \frac{F_X}{F_{XX}}, \quad \gamma = \gamma(X, Y) = \frac{F_{H\dot{H}}}{HF_{\dot{H}\dot{H}}} = \frac{HF_{HX}}{F_{XX}}.$$

and at this point, one may apply the general DSA analysis.

Using

$$\Omega_p = w\Omega_\rho \,, \qquad\qquad \Omega_\rho = Z + Y + 3 \,,$$

the critical points can be chosen as

$$0 = X' = -2X^2 - \gamma X + \beta(Z + \xi)$$

$$0 = Y' = -(2X + Z + \xi)Y - XZ$$

$$0 = Z' = -3(1 + w)(Z + Y + 3) - (Z + \xi)(Z + 3) - X(Z + 6)$$

The number and the position of such points depends on the Lagrangian throughout the functions $\beta, \gamma$ and $\xi$. Again, thre is the inversion problem. In general, only numerical analysis is possible. In the following, some solvable examples.

- First example: $F = R + aR^2 + bP + cQ$ with $3a + b + c \neq 0$ – (Generalized Starobinsky-like model), here $P = R_{ij}R^{ij}$ Ricci square invariant, $G = G = R^2 - 4R_{ij}R^{ij} + R_{ijkr}R^{ijlm}$ Gauss-Bonnet.

  For $0 \leq w \leq 1/3$ there is only one critical point, that is

  – Minkowskian solution with $R_0 = 0$. Stable if $3a + b + c > 0$.

- Second example: $F = R + aR^2 + bP + cQ - d^2 Q_3$ – Generalisation of the previous one, motivated by two-loop corrections in quantum gravity
  ($Q_3 = R_{ijkr}R^{ijlm}R^{kr}_{lm}$ is a cubic invariant related the Goroff-Sagnotti two-loop term). There are at least two critical points, that is

  – Minkowskian solution with $R_0 = 0$. Stable if $3a + b + c > 0$.

  – de Sitter solution with $R_0 = 6/d$. Stable if $3a + b + c + 3d > 0$.

Note that only pure quadratic corrections do not lead to a dS solution, but the inclusion of a cubic correction succedes in obtaining it.

# First generalization: A non local $F(R)$ model

They have been recently introduced motivated by non perturbative quantum corrections and string models.
(Deser-Woodward, Nojiri-Odintsov 07). The simplest one reads:

$$R \to R\left(1 + f(\Box^{-1}R)\right)$$

By introducing two auxiliary scalar fields $\phi$ and $\xi$, one has a local equivalent form

$$S = \int d^4x\sqrt{-g}\frac{1}{2\kappa^2}\left(R[1+f(\phi)] + \xi(\Box\phi - R)\right) + S_M\,.$$

Choosing $f(\phi) = f_0 e^{a\phi}$, the Eqs. of motion can be rewritten in an first oder diff. form (Jhingan, Nojiri,Odintsov,Sami,Thongkool,Z 08). Critical points are stable when $1/3 < a < 2/3$ which corresponds to $-\infty < w_{\rm eff} < -1/3$, dark energy regime with phantom non-phantom transition and the stable de Sitter fixed point $w_{\rm eff} = -1$ occurs when $a = 1/2$.

## ♣ A non local Gauss-Bonnet model

Other non local models, based on Gauss-Bonnet invariant have also been proposed (Capozziello et al PLB 671, 424 (2009). A sligthly generalized action (the original one has $m^2 = 0$) reads

$$S = -\int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} - \frac{\kappa^2}{2a}\mathcal{G}(\Box - m^2)^{-1}\mathcal{G}\right)$$

$R$ the scalar curvature, $\mathcal{G}$, the Gauss-Bonnet invariant and $\Box$ the Dalembertian operator, $\kappa^2 = 8\pi G/c^3$, with $m^2$ mass term, which may be thought of as a non-perturbative string correction, $a$ adimensional parameter. By introducing the scalar field $\phi$ (Note that $\phi$ is adimensional) one may rewrite the above action in a local form:

$$S = -\int d^4x\left(\frac{R}{2\kappa^2} - \frac{a}{2\kappa^2}g^{ij}\partial_i\phi\partial_j\phi - V(\phi) + \phi\mathcal{G}\right) + \ .$$

where the potential is simply quadratic

$$V(\phi) = \frac{a}{2k^2}m^2\phi^2,$$

The one-loop effective action of a more general model has been investigated in Cognola et al.:Eur.Phys.J.C64:483 (2009).

The model is described by the (Euclidean) action

$$S = \int d^4x \sqrt{g} \left[ R + f(\phi)\mathcal{G} - a\, g^{ij}\, \partial_i\phi\partial_j\phi - V(\phi) \right] .$$

When $f = \phi$ and $V(\phi) = \frac{a}{2k^2} m^2\phi^2$ one gets the original model. In this case a de Sitter solution exists with constant curvature $R_0$ as soon as

$$R_0^3 = \frac{36am^2}{k^4}, \quad \phi_0^3 = \frac{3}{2a\, m^2 k^2}$$

The on shell one-loop effective action can be computed in this model along the same lines of local $F(R)$ models and reads

$$\Gamma_{on-shell} = \frac{24\pi F_0}{GR_0^2} + \frac{1}{2}\,\text{log det}\,\left[\ell^2\left(-\Delta_2 + \frac{R_0}{6}\right)\right] - \frac{1}{2}\,\text{log det}\left[\ell^2\left(-\Delta_1 - \frac{R_0}{4}\right)\right] + \frac{1}{2}\,\text{log det}\left[\ell^2\left(-\Delta_0 + V''(\phi_0) - \frac{R_0}{6}f_0''\right)\right].$$

As a result, the stability condition for the local model is simply

$$V''(\phi_0) - \frac{R_0}{6}f_0'' > 0$$

which is satisfied by local model with $f(\phi) = \phi$ and with non negative quadratic potential, namely $a > 0$. Compatibility with Faraoni and Faraoni-Nadeau results for scalar tensor models (we have Gauss-Bonnet term).

# Conclusions

Modified gravity models have been proposed as phenomenological description of a fundamental unknow gravitational theory. From this point of view, corrections to Einstein-Hilbert action depending on higher order curvature invariants are likely to be expected (Lovelock gravity is an example).

A general feature of these models is to possess a further dynamical degree of freedom in addition to the ones of GR: Viable models issue

Different methods have been illustrated in order to study the stability of these models around de Sitter critical points. With regards to dark energy issue, the de Sitter critical point is important, and it has been considered in some details and the dS stability condition has been derived in all the methods.

One may make a comparison of the methods:

- Inhomogeneous perturbation in gauge invariant formalism : FWR metric is required, it has been developed only for $F(R)$ models, but it has a wider validity

- Perturbation of Eqs. of motion in the Jordan frame: Manifestly covariant. It deals only with dS points, but it covers generalized models as $F(R, G, P, ..)$

- One-loop gravity calculation around de Sitter background: Manifestly covariant. Up to now it has been developed only for dS and $F(R)$ models. Work in progress for $F(R, G, P, ..)$

- Dynamical system approach in FRW space-time: It is not manifestly covariant, FWR metric is required, but it is a general approach, which covers all critical points and generalized models.

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