"Bosonic Gaussian Channels"

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Convexity of State space $\Omega \subset \mathcal{B}(\mathcal{H})$

$$\Omega = \{ \rho \mid \rho^{\dagger} = \rho,\ \rho \geq 0,\ \mathrm{tr}\,\rho = 1 \}$$

$$\partial\Omega = \{ \rho \in \Omega \mid \det \rho = 0 \} = \text{boundary of } \Omega$$

$$\Omega^{(ext)} = \{ \rho = |\psi\rangle\langle\psi| \mid |\psi\rangle \in \mathcal{H} \} \subset \partial\Omega$$

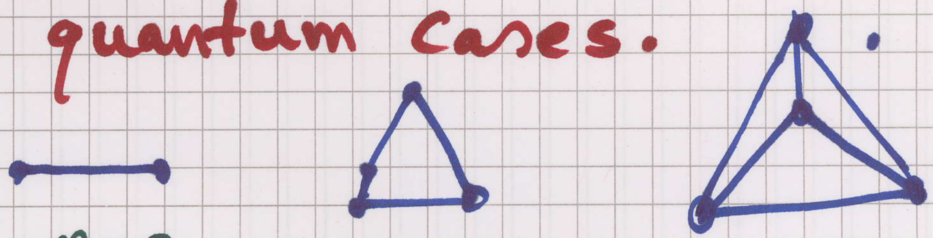
If $\dim \mathcal{H} = n$, then Ω is a bounded closed subset of $\mathbb{R}^{n^2-1}$:

$$\rho \in \Omega \implies \rho = \frac{1}{n}\left[\mathbb{1} + \underline{x} \cdot \underline{\lambda} \right]$$

$\underline{\lambda}$ are Gellman-like matrices, $\underline{x} \in \mathbb{R}^{n^2-1}$.

Ω is ~~classical~~ convex in the classical and in the quantum cases.

Classical case



$n=2$ $\quad$ $n=3$ $\quad$ $n=4$

The pure states of classical n-state system are exactly n in number.

The state space is a regular simplex, and

Every non-extremal or mixed state is a UNIQUE convex sum of extremal (pure) states.

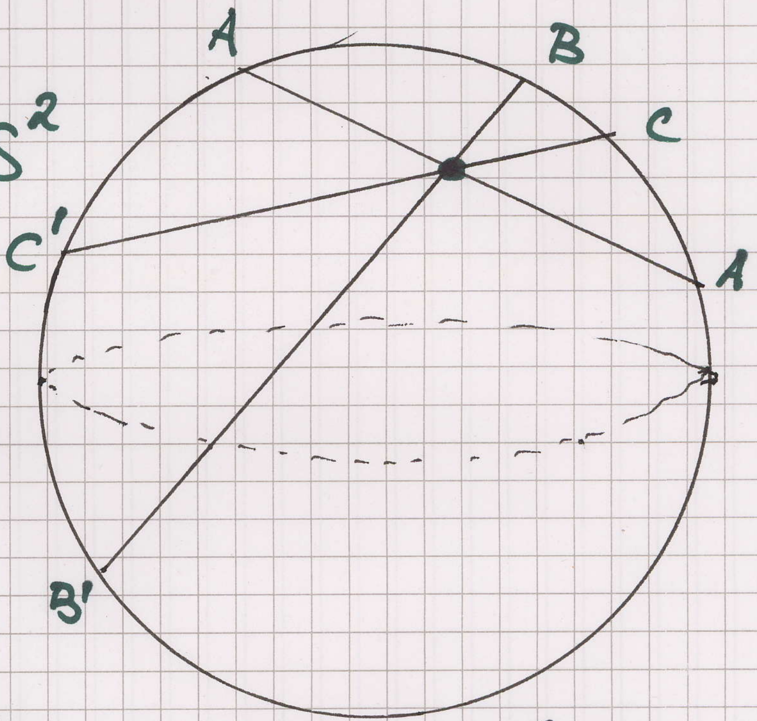
CARATHEODORY Theorem

For an n-level quantum system:

The state space has CP^{n-1} worth of pure states or extremal points. Distinct pure states are not orthogonal, and hence are not reliably distinguishable.

No Caratheodory: A non-extremal state can be written as convex sum of extremal or pure states — but in arbitrarily large number of ways.

$n=2: \partial\Omega = \Omega^{(ext)} = S^2$



Start with the spectral decomposition

$$\rho = \sum_{\alpha=1}^{k}{}' \lambda_\alpha |\varphi_\alpha\rangle\langle\varphi_\alpha|$$

$$= \sum_{\alpha}{}' |\chi_\alpha\rangle\langle\chi_\alpha|, \quad k = \operatorname{rank}\rho \le n.$$

Define $|\varphi_j\rangle = \sum_{\alpha}{}' U_{j\alpha} |\chi_\alpha\rangle, \quad j = 1, 2, \dots, m \geq k$

Verify: $\sum' |\varphi_j\rangle\langle\varphi_j| = \sum_{j,\alpha,\beta} U_{j\alpha} U^*_{j\beta} |\chi_\alpha\rangle\langle\chi_\beta|$

$$= \sum |\chi_\alpha\rangle\langle\chi_\alpha|$$

$$\text{if } U^\dagger U = \mathbb{1}$$

$$\begin{array}{cc} & m \\ k & \boxed{\quad U^\dagger \quad} \end{array}$$

$2mk - k^2$
$= k(2m-k)$ free parameters.

Two Illustrative Consequences:

(1) Separability: A state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable iff there exists an ensemble realization of the product form

$$\rho = \sum_j p_j \, \rho_{Aj} \otimes \rho_{Bj} \sim \cancel{\sum_j p_j |\psi_{Aj}\rangle\langle\psi_{Aj}|}$$

$$\sim \sum_j p_j |\psi_{Aj}\rangle\langle\psi_{Aj}| \otimes |\psi_{Bj}\rangle\langle\psi_{Bj}|$$

$\rho = \sum p_k |\psi_k^{AB}\rangle\langle\psi_k^{AB}|$ can be realized in arbitrarily large number of ways. But we want every $|\psi_k^{AB}\rangle$ to be a product state — for separability. Not practical to run through all ensembles.

We do not know how to test, efficiently, if a given mixed state ρ_{AB} is separable or entangled, even in the simple case $\dim \mathcal{H}_A = 3 = \dim \mathcal{H}_B$.

(2) Entanglement of formation EoF.

For pure state $|\psi^{AB}\rangle$, we get reduced states $\rho_A = tr_B |\psi^{AB}\rangle\langle\psi^{AB}|$; $\rho_B = tr_A |\psi^{AB}\rangle\langle\psi^{AB}|$.

entanglement $E(|\psi^{AB}\rangle) = S(\rho_A) = S(\rho_B)$.

$S(\rho) = -tr\, \rho \log_2 \rho$: von Neumann entropy.

For mixed state

$$EoF = E(\rho_{AB}) = \inf_{\{p_j, |\psi_j^{AB}\rangle\}} \sum_j p_j E(|\psi_j^{AB}\rangle).$$

Positive & Completely Positive Maps

A linear map $M : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ is positive if $M(\rho) \geq 0$ for all $\rho \geq 0$.

It may appear that a positive map takes states to states, and hence is a physical process. This is false — in view of entanglement.

Given a map M on $\mathcal{B}(\mathcal{H}_B)$, we can extend it 'trivially' to the map $\mathbb{1}_A \otimes M$ on the composite system $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

M is positive $\not\Longrightarrow$ $\mathbb{1}_A \otimes M$ is positive

Transpose is a positive map, but partial transpose is not positive!

A positive map is said to be completely positive if all its 'trivial' extensions are positive.

Only completely positive maps correspond to physical processes or channels.

Maps which are ~~comp~~ positive but not CP will take product states into states, but will map some inseparable states into 'nonstates'.

$\Longrightarrow$ they are useful as entanglement witnesses.

Positive maps form a Convex Set.
$\Rightarrow$ Complete characterization of positive maps is equivalent to enumeration of all extremals of this Set. — Open problem.

CP maps form a Convex (sub) Set.
$\Rightarrow$ characterization of CP maps $\Longleftrightarrow$ enumeration of all the extremal CP maps

Obviously, for any operator $A \in \mathcal{B}(\mathcal{H})$, the map $\rho \longrightarrow A\rho A^{\dagger}$ is CP on $\mathcal{B}(\mathcal{H})$.
$\Rightarrow$ given a set of operators $\{A_k\}$ in $\mathcal{B}(\mathcal{H})$, the map $\rho \longrightarrow \sum_k A_k \rho A_k^{\dagger}$ is CP.

These are obvious examples of CP maps.
There are no CP maps beyond these!

Every CP map is of the form
$$\rho \longrightarrow \sum_k A_k \rho A_k^{\dagger} \quad \text{and Conversely.}$$

Such a simple characterization does not exist for maps which are P but not CP.

Every hermiticity-preserving map is of the form
$$\rho \longrightarrow \sum_k{}' \epsilon_k A_k \rho A_k^{\dagger}, \text{ and vice Versa}$$
Here, ϵ_k are signatures.
There exists no 'natural' intermediate stop between CP and hermiticity-preserving.

$$\rho \to \rho' = \sum_k A_k \rho A_k^\dagger$$ Kraus decomposition for CP maps

The map is identity preserving (or unital) iff $\sum_k A_k A_k^\dagger = \mathbb{1}$

The map is trace-preserving (or physical) iff $\sum A_k^\dagger A_k = \mathbb{1}$

Duality: If $\{A_k\}$ represents a trace-preserving CP map, then the dual set $\{A_k^\dagger\}$ represents an unital CP map and vice versa.

A CP map which is both trace-preserving and unital is said to be doubly stochastic.

In the classical case we have Birkhoff theorem which says that permutation maps are the only extremals of the convex set of all doubly ~~sto~~ stochastic maps.

Birkhoff is false in the quantum case.

The three Kraus operators $A_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ e^{i\varphi_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$A_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ e^{i\varphi_2} & 0 & 0 \end{pmatrix}$, $A_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & e^{i\varphi_3} & 0 \end{pmatrix}$ represent a doubly stochastic map, which is 'not' random unitary.