Napoli 2009

QFT ON STAR GRAPHS AND ANYONIC LUTTINGER JUNCTIONS

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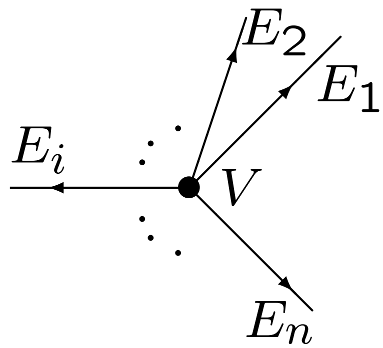


A star graph Γ with n edges.

Each point P in Γ is parametrized by:

$x \in \mathbb{R}_+$ - distance of P from V;

$i = 1, ..., n$ - index of the edge.

$B \equiv \Gamma \setminus V$ - bulk of Γ.

The Tomonaga-Luttinger model on Γ:

$$\mathcal{L} = \mathrm{i}\psi_1^*(\partial_t - v_F\partial_x)\psi_1 + \mathrm{i}\psi_2^*(\partial_t + v_F\partial_x)\psi_2$$

$$-g_+(\psi_1^*\psi_1 + \psi_2^*\psi_2)^2 - g_-(\psi_1^*\psi_1 - \psi_2^*\psi_2)^2\,,$$

where $\psi_{1,2}(t,x,i)$ are fermions.

The model is exactly solvable on $\mathbb{R}$ via bosonization (massless scalar field φ and its dual $\widetilde{\varphi}$).

On Γ one must fix in addition the boundary conditions in V:

$$\mathcal{L}_V = \psi_\alpha^*(t,0,i)\mathcal{B}_{ij}^{\alpha\beta}\psi_\beta(t,0,j)$$

No longer exactly solvable by bosonization – exponential boundary interactions of φ and $\widetilde{\varphi}$.

For Γ with $n = 3$ Fisher, Ludwig, Lin and Nayak (1999) discovered a non-trivial fixed point with enhanced conductance

$$G = \frac{4}{3}\,G_{\text{line}}\,.$$

Question: Are there BC which preserve the exact solvability of the TL model on Γ?

If yes:

(i) what are the corresponding critical points?

(ii) is the Fisher et al. point reproduced?

(iii) what is the behavior away of criticality?

(iv) stability of the critical points.

(v) what about $n > 3$?

The main question has affirmative answer – there exist BC, which are linear in φ and $\tilde{\varphi}$ and therefore quadratic in ψ, which preserve exact solvability.

Physical idea - treat the vertex V of Γ as a point-like defect and use QFT with defects (Delfino, Mussardo, Sorba, Ragoucy, M.M.)

Basic tools:

(a) analytic - simple elements of the spectral theory of linear operators on graphs - "quantum graphs": (Kuchment, Smilansky, Exner, Kostrykin, Schrader, Harmer, ...)

(b) algebraic - convenient basis in field space "reflection-transmission" (R-T) algebra, which translates the analytic boundary value problem at hand in algebraic terms: (Ragoucy, Sorba, M. M.)

Combine (a) and (b) with standard methods in QFT.

Plan

I. General features of QFT on star graphs.

1. Symmetries of QFT on Γ.
2. Boundary conditions in V.
3. The scalar field φ and its dual $\widetilde{\varphi}$ on Γ.
4. Scale invariance, critical points.
5. Vertex operators and boundary dimensions.

II. Anyon Luttinger liquid on Γ.

1. Anyon solution of the TL model on $\mathbb{R}$.
2. Extension of the solution to Γ.
3. Conductance of the anyon Luttinger liquid.
4. Critical points and their stability.

III. Further developments.

1. Boundary bound states.
2. From star graphs towards generic graphs.
3. Boundary conditions breaking time-reversal.

1. Symmetries on Γ and Kirchhoff's rules.

As usual, symmetries are associated with conserved currents

$$\partial_t j_t(t,x,i) - \partial_x j_x(t,x,i) = 0\,.$$

The conservation of the relative charge

$$Q = \sum_{i=1}^{n} \int_0^{\infty} \mathrm{d}x\, j_t(t,x,i)$$

needs however special attention on Γ.

Q is time independent iff the Kirchhoff's rule

$$\sum_{i=1}^{n} j_x(t,0,i) = 0$$

holds in the vertex V of Γ.

N.B. The Kirchhoff's rules corresponding to different conserved currents are in general not equivalent - obstructions are expected in lifting the symmetry content from $\mathbb{R}$ to Γ.

2. Boundary conditions in V.

Select all boundary conditions providing time independent Hamiltonian.

In order to implement this requirement, one must impose on the energy-momentum tensor

$$\theta_{tt}(t,x,i)\,, \quad \theta_{tx}(t,x,i)\,,$$

the Kirchhoff's rule

$$\sum_{i=1}^{n} \theta_{tx}(t,0,i) = 0\,.$$

N.B. For $n=1$ (half-line) one has

$$\theta_{tx}(t,0) = 0 \quad \text{(complete reflection)}\,,$$

being the starting point of BCFT (Cardy).

Besides reflection, for $n>1$ one has transmission as well.

To be more precise, the Kirchhoff's rule

$$\sum_{i=1}^{n} \theta_{tx}(t, 0, i) = 0$$

implies t-independence of the bulk Hamiltonian H_{bulk} and parametrizes all possible self-adjoint extensions of H_{bulk} from $\Gamma \backslash V$ to Γ.

It does not ensure the existence of self-adjoint extensions!

Self-adjoint extensions exist iff H_{bulk} has equal deficiency indices:

$$n_+(H_{\text{bulk}}) = n_-(H_{\text{bulk}}) \, .$$

This condition is usually hard to be verified directly.

According to a theorem of von Neumann, the index condition is automatically satisfied for system which are invariant under time-reversal,

$$T \, H_{\text{bulk}} \, T^{-1} = H_{\text{bulk}} \, , \qquad T - \text{antilinear} \, .$$

3. Free scalar field on Γ.

- equation of motion:

$$\left(\partial_t^2 - \partial_x^2\right) \varphi(t,x,i) = 0\,, \qquad x > 0\,,$$

- initial condition (equal-time CCR):

$$[\varphi(t,x_1,i_1)\,,\,\varphi(t,x_2,i_2)] =$$

$$[(\partial_t\varphi)(t,x_1,i_1)\,,\,(\partial_t\varphi)(t,x_2,i_2)] = 0\,,$$

$$[(\partial_t\varphi)(t,x_1,i_1)\,,\,\varphi(t,x_2,i_2)] = -\mathrm{i}\delta_{i_1}^{i_2}\,\delta(x_1-x_2)\,.$$

- boundary condition: $\forall\, t \in \mathbb{R}$

$$\left[\lambda(\mathbb{I}-U)_i^j\,\varphi(t,0,j) - \mathrm{i}(\mathbb{I}+U)_i^j(\partial_x\varphi)(t,0,j)\right] = 0\,,$$

$\lambda > 0 \rightarrow$ parameter with dimension of mass;
$U \rightarrow n \times n$ complex matrix.

Kirchhoff's rule for θ_{tx} on Γ implies unitary time evolution of φ iff

$$U^* = U^{-1}\,.$$

(Kostrykin, Schrader, Harmer)

U parametrizes all selfadjoint extensions of H_{bulk}.

Supplementary conditions on U following from:

(i) Hermiticity: $\varphi^*(t,x,i) = \varphi(t,x,i)$

(ii) Time-reversal invariance:

$$T\varphi(t,x,i)T^{-1} = \varphi(-t,x,i)\,, \qquad T - \text{antilinear}\,.$$

These conditions imply that

$$U^t = U\,.$$

(iii) Invariance under scale transformations:

$$x \longmapsto \rho x\,, \qquad t \longmapsto \rho t\,, \qquad \rho > 0\,.$$

The corresponding Kirchhoff's law implies

$$U^* = U\,.$$

(iv) $U(1)$-Kirchhoff's rule:

$$j_t(t,x,i) = \partial_t \varphi(t,x,i)\,, \quad j_x(t,x,i) = \partial_x \varphi(t,x,i)\,,$$

$$U\mathbf{v} = \mathbf{v}\,, \quad \mathbf{v} \equiv (1,1,...,1)\,.$$

(entries along each line of U sum up to 1.)

(v) $\widetilde{U}(1)$-Kirchhoff's rule:

$$\widetilde{j}_t(t,x,i) = \partial_t \widetilde{\varphi}(t,x,i)\,, \quad \widetilde{j}_x(t,x,i) = \partial_x \widetilde{\varphi}(t,x,i)\,,$$

$$U\mathbf{v} = -\mathbf{v}\,.$$

(entries along each line of U sum up to -1.)

Summary:

$$\begin{aligned} U^* &= U^{-1}\,, && \text{(unitary time evolution)} \\ U^t &= U\,, && \text{(time}-\text{reversal inv.)} \\ U^* &= U\,, && \text{(scale inv.)} \\ U\mathbf{v} &= \mathbf{v}\,, && (U(1)-\text{inv.)} \\ U\mathbf{v} &= -\mathbf{v}\,, && (\widetilde{U}(1)-\text{inv.)} \end{aligned}$$

The solution in algebraic terms:

$$\varphi(t,x,i) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi\sqrt{2|k|}}$$
$$\left[a^{*i}(k)\mathrm{e}^{\mathrm{i}(|k|t-kx)} + a_i(k)\mathrm{e}^{-\mathrm{i}(|k|t-kx)}\right]\,,$$

$\{a_i(k),\, a^{*i}(k)\ :\ k \in \mathbb{R}\}$ generate an associative algebra $\mathcal{A}$ with identity element $\mathbf{1}$ and satisfy the commutation relations

$$\begin{aligned} a_{i_1}(k_1)\, a_{i_2}(k_2) - a_{i_2}(k_2)\, a_{i_1}(k_1) &= 0\,, \\ a^{*i_1}(k_1)\, a^{*i_2}(k_2) - a^{*i_2}(k_2)\, a^{*i_1}(k_1) &= 0\,, \\ a_{i_1}(k_1)\, a^{*i_2}(k_2) - a^{*i_2}(k_2)\, a_{i_1}(k_1) &= \end{aligned}$$
$$2\pi[\delta_{i_1}^{i_2}\delta(k_1-k_2) + S_{i_1}^{i_2}(k_1)\delta(k_1+k_2)]\mathbf{1}\,,$$

and the constraints

$$a_i(k) = S_i^j(k)a_j(-k)\,, \qquad a^{*i}(k) = a^{*j}(-k)S_j^i(-k)\,,$$

where $S(k)$ is the S-matrix characterizing the defect.

$\mathcal{A}$ is a special case of the R-T algebras, (Sorba, Ragoucy, Caudrelier, M. M) - a convenient choice of coordinates in the presence of point-like defects.

In our case (Kostrykin, Schrader, Harmer,...):

$S(k) = -[\lambda(\mathbb{I}-U)+k(\mathbb{I}+U)]^{-1}[\lambda(\mathbb{I}-U)-k(\mathbb{I}+U)]\,.$

The main properties of $S(k)$:

(i) Unitarity:

$$S(k)^* = S(k)^{-1}\,;$$

(ii) Hermitian analyticity

$$S(k)^* = S(-k)\,;$$

N.B. As a consequence, one has

$$S(k)\,S(-k) = \mathbb{I}\,,$$

which ensures the consistency of the constraints:

$a_i(k) = S_i^j(k)a_j(-k)\,, \quad a^{*i}(k) = a^{*j}(-k)S_j^i(-k)\,.$

(iii) Invariance under time reversal:

$$S(k)^t = S(k)\,.$$

(iv) Normalization:

$$S(\lambda) = U\,.$$

(v) analytic properties in the complex k-plane:
Let $\mathcal{U}$ be the unitary matrix diagonalizing U and let us parametrize

$$U_d = \mathcal{U}\, U\, \mathcal{U}^{-1}$$

as follows

$$U_d = \mathrm{diag}\left(\mathrm{e}^{2\mathrm{i}\alpha_1}, \mathrm{e}^{2\mathrm{i}\alpha_2}, ..., \mathrm{e}^{2\mathrm{i}\alpha_n}\right)\,, \qquad \alpha_i \in \mathbb{R}\,.$$

$\mathcal{U}$ diagonalizes $S(k)$ for any k as well and

$$S_d(k) = \mathcal{U} S(k) \mathcal{U}^{-1} =$$

$$\mathrm{diag}\left(\frac{k+\mathrm{i}\eta_1}{k-\mathrm{i}\eta_1}, \frac{k+\mathrm{i}\eta_2}{k-\mathrm{i}\eta_2}, ..., \frac{k+\mathrm{i}\eta_n}{k-\mathrm{i}\eta_n}\right)$$

where

$$\eta_i = \lambda \tan(\alpha_i)\,, \qquad -\frac{\pi}{2} \leq \alpha_i \leq \frac{\pi}{2}\,.$$

$S(k)$ is a meromorphic function with simple poles located on the imaginary axis and different from 0.

Boundary Bound States (BBS) $\rightarrow \eta_j > 0$

We focus below on the case without BBS and comment in the Conclusions about this case!

4. Scale invariance and critical points.

The BC fixed by:

$$U^* = U^{-1}, \quad U^t = U\,, \quad U^* = U\,,$$

imply that $S(k)$ is k-independent,

$$S(k) = U \quad \forall\, k\,.$$

Classification of the critical points $\equiv$
all orthogonal symmetric real matrices.

The eigenvalues of $S = U$ are ± 1.
Let p be the number of eigenvalues -1.

Classification

$p = 0 \Rightarrow S_N = \mathbb{I}$ - (Neumann);

$0 < p < n \Rightarrow p(n-p)$-parameter family;

$p = n \Rightarrow S_D = -\mathbb{I}$ - (Dirichlet);

$$0 < p < n$$

Example $n = 2$: (Bachas, Dijkgraaf, Ooguri,...)

$$p = 1 \longrightarrow S_1(\alpha) = \frac{1}{1+\alpha^2} \begin{pmatrix} \alpha^2 - 1 & -2\alpha \\ -2\alpha & 1 - \alpha^2 \end{pmatrix}$$

Example $n = 3$:

$$p = 1 \longrightarrow S_1(\alpha_1, \alpha_2) = -\frac{1}{1+\alpha_1^2+\alpha_2^2} \times$$

$$\begin{pmatrix} \alpha_1^2 - \alpha_2^2 - 1 & 2\alpha_1\alpha_2 & 2\alpha_1 \\ 2\alpha_1\alpha_2 & -\alpha_1^2 + \alpha_2^2 - 1 & 2\alpha_2 \\ 2\alpha_1 & 2\alpha_2 & 1 - \alpha_1^2 - \alpha_2^2 \end{pmatrix}$$

$$p = 2 \longrightarrow S_2(\alpha_1, \alpha_2) = -S_1(\alpha_1, \alpha_2)$$

$\alpha_{1,2} \in \mathbb{R}$ are not fixed by scale invariance and describe therefore marginal couplings.

Imposing in addition the $U(1)$-Kirchhoff rule:

$$U\mathbf{v} = \mathbf{v} \Rightarrow (U(1) - \text{inv.})$$

$p = n \Rightarrow S_D = -\mathbb{I}$ - (Dirichlet) - impossible;

One is left with:

$p = 0 \Rightarrow S_N = \mathbb{I}$ - (Neumann);

$0 < p < n \Rightarrow p(n - p - 1)$-parameter family;

Example: $0 < p < n = 3$:

$$p = 1: \quad S_1(\alpha_1, \alpha_2) \to S_1(\alpha) =$$

$$\frac{1}{1+\alpha+\alpha^2}\begin{pmatrix} \alpha+1 & -\alpha & \alpha(\alpha+1) \\ -\alpha & \alpha(\alpha+1) & \alpha+1 \\ \alpha(\alpha+1) & \alpha+1 & -\alpha \end{pmatrix}$$

New - not symmetric under edge permutations

$$p = 2: \quad S_2(\alpha_1, \alpha_2) \to \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

The Fisher et al. point.

We show later that all these are critical points of the T-L model for $n = 3$.

5. Vertex operators and their dimensions

- The dual field $\widetilde{\varphi}$.

$$\partial_t \widetilde{\varphi}(t,x,i) = -\partial_x \varphi(t,x,i)\,,$$

$$\partial_x \widetilde{\varphi}(t,x,i) = -\partial_t \varphi(t,x,i)\,,$$

  $\widetilde{\varphi}$ is nonlocal with respect to φ.

- Right and left chiral fields:

$$\varphi_{i,R}(t-x) = \varphi(t,x,i) + \widetilde{\varphi}(t,x,i)\,,$$

$$\varphi_{i,L}(t+x) = \varphi(t,x,i) - \widetilde{\varphi}(t,x,i)\,.$$

- Vertex operators: $\zeta = (\sigma,\tau) \in \mathbb{R}^2$

$$V(t,x,i;\zeta) \sim$$

$$\sim \eta_i : \mathrm{exp}\left\{\mathrm{i}\sqrt{\pi}\left[\sigma\varphi_{i,R}(t-x) + \tau\varphi_{i,L}(t+x)\right]\right\} :\,,$$

  $:\cdots:$ - normal product in $\mathcal{A}$
  η_i - Klein factors - in general anyonic.

Correlation functions - scale-invariant case:

$$\langle V(t_1, x_1, i_1; \zeta) V^*(t_2, x_2, i_2; \zeta)\rangle \sim$$

$$\left[\frac{1}{\mathrm{i}(t_{12} - x_{12}) + \epsilon}\right]^{\sigma^2 \delta_{i_1}^{i_2}} \left[\frac{1}{\mathrm{i}(t_{12} + x_{12}) + \epsilon}\right]^{\tau^2 \delta_{i_1}^{i_2}}$$

$$\left[\frac{1}{\mathrm{i}(t_{12} - \widetilde{x}_{12}) + \epsilon}\right]^{\sigma\tau S_{i_1}^{i_2}} \left[\frac{1}{\mathrm{i}(t_{12} + \widetilde{x}_{12}) + \epsilon}\right]^{\sigma\tau S_{i_1}^{i_2}}$$

with $t_{12} = t_1 - t_2,\ x_{12} = x_1 - x_2,\ \widetilde{x}_{12} = x_1 + x_2$.
Scaling matrix: $(x \longmapsto \rho x,\ t \longmapsto \rho t)$

$$D = (\sigma^2 + \tau^2)\mathbb{I}_n + 2\sigma\tau S\,.$$

Scaling dimensions - eigenvalues of D:

$$d_i = \frac{1}{2}(\sigma + s_i \tau)^2 \geq 0\,,$$

$s_i = \pm 1$ being the eigenvalues of S.

$$d_i = \frac{1}{2}(\sigma^2 + \tau^2) + s_i \sigma\tau \equiv d_{\mathrm{line}} + d_i^b$$

d_i^b - boundary dimension.

II. The Tomonaga-Luttinger model on Γ

1. Anyon solution of the TL model on $\mathbb{R}$.

$$\mathcal{H} = \int dx \left[v_F(\psi_1^* \mathrm{i}\partial_x \psi_1 - \psi_2^* \mathrm{i}\partial_x \psi_2) + g_+ \rho_+^2 + g_- \rho_-^2 \right]$$

v_F – Fermi velocity,
$g_\pm$ – coupling constants,
$\rho_\pm$ – charge densities, ($U(1) \otimes \widetilde{U}(1)$-symmetry):

$$\rho_\pm(t,x) = \left[\psi_1^*(t,x)\psi_1(t,x) \pm \psi_2^*(t,x)\psi_2(t,x)\right] .$$

Equations of motion:

$$\mathrm{i}(\partial_t - v_F \partial_x)\psi_1(t,x) = 2g_+\, \rho_+(t,x)\psi_1(t,x) + 2g_-\, \rho_-(t,x)\psi_1(t,x)\,,$$
$$\mathrm{i}(\partial_t + v_F \partial_x)\psi_2(t,x) = 2g_+\, \rho_+(t,x)\psi_2(t,x) + 2g_-\, \rho_-(t,x)\psi_2(t,x)\,.$$

Solution:

$$\psi_1(t,x) \;\propto\; :\mathrm{e}^{\mathrm{i}\sqrt{\pi}\left[\sigma\varphi_R(vt-x)+\tau\varphi_L(vt+x)\right]}:\,,$$
$$\psi_2(t,x) \;\propto\; :\mathrm{e}^{\mathrm{i}\sqrt{\pi}\left[\tau\varphi_R(vt-x)+\sigma\varphi_L(vt+x)\right]}:\,,$$

v – velocity of the anyon excitations;
σ, $\tau, v \in \mathbb{R}$ to be determined.

$$\psi_1^*(t,x_1)\psi_1(t,x_2) = \mathrm{e}^{-\mathrm{i}\pi\kappa\varepsilon(x_{12})}\psi_1(t,x_2)\psi_1^*(t,x_1)\,,$$
$$\psi_1^*(t,x_1)\psi_1^*(t,x_2) = \mathrm{e}^{\mathrm{i}\pi\kappa\varepsilon(x_{12})}\psi_1^*(t,x_2)\psi_1^*(t,x_2)\,,$$

ε being the sign function and

$$\kappa = \tau^2 - \sigma^2 \to \text{statistic parameter}$$

In terms of the variables $\zeta_\pm = \tau \pm \sigma$, one has

$$\begin{aligned} \zeta_+\zeta_- &= \kappa\,, && \text{(definition of } \kappa\text{)} \\ v\zeta_+^2 &= v_F\kappa + \frac{2}{\pi}g_+\,, && \text{(eq. of motion for } \psi_1\text{)} \\ v\zeta_-^2 &= v_F\kappa + \frac{2}{\pi}g_-\,, && \text{(eq. of motion for } \psi_2\text{)} \end{aligned}$$

implying

$$\begin{aligned} \zeta_\pm^2 &= |\kappa|\left(\frac{\pi\kappa v_F + 2g_+}{\pi\kappa v_F + 2g_-}\right)^{\pm 1/2}\,, \\ v &= \frac{\sqrt{(\pi\kappa v_F + 2g_-)(\pi\kappa v_F + 2g_+)}}{\pi|\kappa|}\,. \end{aligned}$$

$\kappa = 1$ - conventional fermion solution;
$\kappa \neq 1$ - anyon solution;
$2g_\pm > -\pi\kappa v_F$ - stability conditions.

Symmetries on $\mathbb{R}$: $U(1) \otimes \widetilde{U}(1)$.

Charge densities:

$$\rho_\pm(t,x) = [\psi_1^*(t,x)\psi_1(t,x) \pm \psi_2^*(t,x)\psi_2(t,x)] \to$$

$$-\frac{1}{2\sqrt{\pi}\zeta_\pm}\left[(\partial\varphi_R)(vt-x) \pm (\partial\varphi_L)(vt+x)\right]$$

Normalization – fixed by the Word identities:

$$[\rho_+(t,x_1)\,,\,\psi_\alpha(t,x_2)] = -\delta(x_{12})\,\psi_\alpha(t,x_2)\,,$$
$$[\rho_-(t,x_1)\,,\,\psi_\alpha(t,x_2)] = -(-1)^\alpha\delta(x_{12})\,\psi_\alpha(t,x_2)\,.$$

Currents:

$$j_\pm(t,x) = \frac{v}{2\sqrt{\pi}\zeta_\pm v_F}\left[(\partial\varphi_R)(vt-x) \mp (\partial\varphi_L)(vt+x)\right]$$

satisfy

$$\partial_t\rho_\pm(t,x) - v_F\partial_x j_\pm(t,x) = 0\,.$$

N. B. Imposing Kirchhoff's rule for θ_{tx}, on Γ one can save only one of the factors $U(1) \otimes \widetilde{U}(1)$.

2. Extension of the solution to Γ.

Keeping the values of σ, τ, v, perform in the solution on $\mathbb{R}$ the substitution

$$\varphi_R(vt - x) \longmapsto \varphi_{i,R}(vt - x)\,,$$

$$\varphi_L(vt + x) \longmapsto \varphi_{i,L}(vt + x)\,,$$

$\varphi_{i,Z}$ satisfying the U-boundary condition.

Statement: One gets the solution of the T-L model on Γ, satisfying the following boundary conditions at criticality:

$$j_+(t, 0, i) = -\sum_{k=1}^{n} S_{ik}\, j_+(t, 0, k)\,,$$

where $j_+(t, x, i)$ is the $U(1)$-current.

This BC is quadratic in ψ and describes the splitting of the current in the junction.

Symmetries on Γ: $U(1)$ or $\widetilde{U}(1)$

$U(1)$-Kirchhoff's rule:

$$U\mathbf{v} = \mathbf{v}\,, \quad \mathbf{v} \equiv (1, 1, ..., 1)\,.$$

$U(1)$ (electric) charge is conserved.

$\widetilde{U}(1)$-Kirchhoff's rule:

$$U\mathbf{v} = -\mathbf{v}\,.$$

Electric charge is no longer conserved.

By duality, in this case the electric charge density (and not current) satisfies

$$\sum_{i=1}^{n} \rho_+(t, 0, i) = 0$$

Characteristic feature (Das, Rao) of superconducting junctions (Sodano, Giuliani,...).

3. Conductance of the anyon Luttinger liquid:

Consider the $U(1)$-symmetric model and couple the system to an external time dependent classical field $A_\nu(t,x,i)$ according to

$$\partial_\nu \longmapsto \partial_\nu + \mathrm{i} A_\nu \,, \qquad \nu = t,\, x \,.$$

The next step is to compute the expectation value

$$\langle j_+(t,x,i)\rangle_{A_\nu} \,.$$

At a critical point one finds:

$$\langle j_+(t,0,i)\rangle_{A_x} \sim \sum_{j=1}^{n} \left(\delta_i^j - S_{ij}\right) A_x(t,j) + O(A_x^2)\,,$$

which gives the conductance tensor

$$G_{ij} = G_{\text{line}} \left(\delta_{ij} - S_{ij}\right)\,,$$

where

$$G_{\text{line}} = \frac{1}{2\pi\zeta_+^2}$$

is the conductance of a single wire.

General features of the conductance tensor:

$$G_{ij} = G_{\text{line}} \left(\delta_{ij} - S_{ij} \right) \ ,$$

- Kirchhoff's rule: $U(1)$ symmetry implies

$$\sum_{j=1}^{n} G_{ij} = 0 \,, \qquad \forall \, i = 1, ..., n \,.$$

- Enhanced conductance:

$$G_{ii} > G_{\text{line}} \qquad \text{for} \qquad S_{ii} < 0 \,.$$

- Unitarity bound: $|S_{ii}| \leq 1$ leading to

$$0 \leq G_{ii} \leq 2 G_{\text{line}} \,.$$

- Sum rule:

$$\text{Tr}\, G = 2p\, G_{\text{line}} \,, \qquad (p = \text{number of } s_i = -1) \,.$$

Critical conductance on Γ for $n = 3$:

The Neumann point:

$$S_N = \mathbb{I} \Longrightarrow G = 0\,.$$

The vertex is an ideal isolator.

The Fisher et al. critical point $p = 2$:

$$S_2 = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

Enhanced conductance:

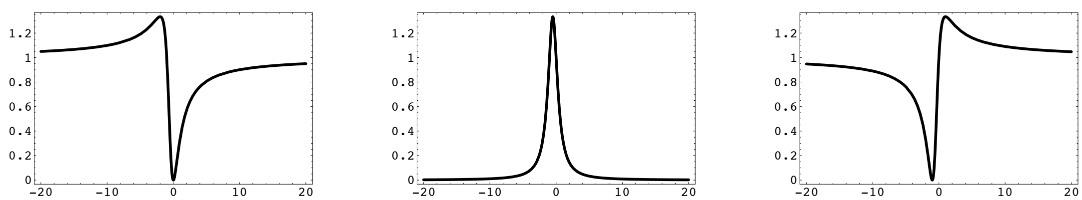
$$G_{11} = G_{22} = G_{33} = \frac{4}{3} G_{\text{line}}\,.$$

Physical explanation (Fisher et al.) - Andreev reflection from the vertex.

The $p = 1$ family of critical points

$$S_1(\alpha) = \frac{1}{1+\alpha+\alpha^2}\begin{pmatrix} \alpha+1 & -\alpha & \alpha(\alpha+1) \\ -\alpha & \alpha(\alpha+1) & \alpha+1 \\ \alpha(\alpha+1) & \alpha+1 & -\alpha \end{pmatrix}$$

Dependence on the marginal coupling α for $G_{\text{line}} = 1$:



$G_{11}(\alpha)$ $\qquad$ $G_{22}(\alpha)$ $\qquad$ $G_{33}(\alpha)$

Domains of enhancement (maxima $= 4G_{\text{line}}/3$);
Domains of reduction (minima $= 0$).
The sum rule gives ($p = 1$):

$$G_{11}(\alpha) + G_{22}(\alpha) + G_{33}(\alpha) = 2G_{\text{line}}$$

Conductance away of criticality without BBS.

$S(k)$ has no poles in the upper half-plane.

One gets

$$G_{ij}(\omega) = G_{\text{line}} \left(\delta_{ij} - S_{ij}(\omega)\right) \ ,$$

ω – frequency of the external field $A_\nu(t,i)$.

$G_{ij}(\omega)$ is in general complex $\Rightarrow$ the junction has non-trivial inductance and/or capacity.

4. Stability of the critical points - follows from the boundary dimensions

$$d_i^b = s_i \sigma \tau = \frac{s_i}{4}(\zeta_+^2 - \zeta_-^2)$$

where

$$\zeta_\pm^2 = |\kappa| \left(\frac{\pi \kappa v_F + 2g_+}{\pi \kappa v_F + 2g_-} \right)^{\pm 1/2} .$$

(i) $g_+ < g_-$ - attractive anyonic interactions:

$$d_i^b = \begin{cases} > 0 & \text{if } s_i = -1 \text{ - Dirichlet,} \\ < 0 & \text{if } s_i = 1 \quad \text{ - Neuman.} \end{cases}$$

Dirichlet ($s_i = -1$) is stable.

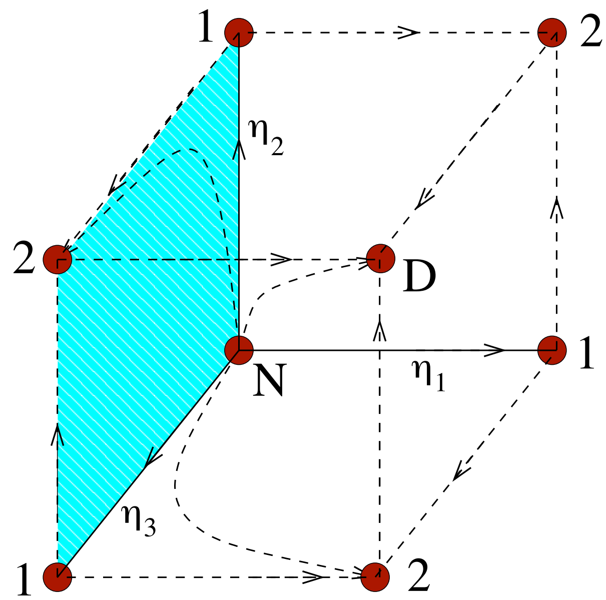
Neuman ($s_i = 1$) is unstable.

(ii) $g_+ > g_-$ - repulsive anyonic interactions:

Dirichlet ($s_i = -1$) is unstable.

Neuman ($s_i = 1$) is stable.

The phase diagram for n=3 and $g_+ < g_-$:



N - $p = 0$ (Neumann): $\eta_1 = \eta_2 = \eta_3 = 0$

D - $p = 3$ (Dirichlet): $\eta_1 = \eta_2 = \eta_3 = \infty$

1 - $p =$ 1-family: $(\alpha_{1,2} \longrightarrow_{U(1)} \alpha)$

2 - $p =$ 2-family: $(\alpha_{1,2} \longrightarrow_{U(1)}$ no prameters)

Cyan-shaded area - $U(1)$-Kirchhoff.

$g_+ > g_-$ - invert the arrows.

III. Further developments.

1. Boundary bound states on Γ (Bellazzini, Sorba, M. M.) arXiv: 0810.3101
BBS – poles of $S(k)$ in the upper half plane.
The system is necessarily non-ctritical.

Each BBS gives raise to a damped harmonic oscillator, whose contribution is completely fixed by causality (local commutativity).
The friction is positive in the right sector and negative in the left on.
Transfer of energy between left and right movers.

Effects:
(i) BBS drive the system out of equilibrium;
(ii) the time evolution is not unitary;
(iii) time-translation invariance is broken;
(iv) impact on the conductance

$$G_{ij}(\omega, t-t_0) =$$

$$G_{\mathsf{line}}\left[\delta_{ij} - S_{ij}(\omega) - \sum_{\eta\in\mathcal{P}} R_{ij}^{(\eta)} \frac{\eta}{\eta + \mathsf{i}\omega} \mathsf{e}^{(t-t_0)(\eta+\mathsf{i}\omega)}\right]$$

where:

– the external field is switched on at t_0;

– $\mathcal{P}$ - the set of poles of the S-matrix;

– "residue" matrix at the pole iη

$$R_{ij}^{(\eta)} = \frac{1}{\mathsf{i}\eta} \lim_{k\to \mathsf{i}\eta} (k - \mathsf{i}\eta) S_{ij}(k)\,, \qquad \mathsf{i}\eta \in \mathcal{P}$$

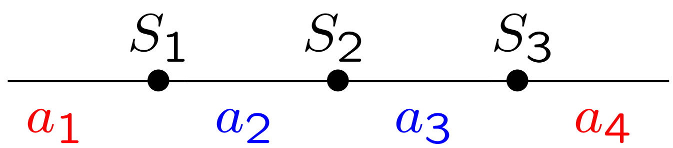
$\eta < 0 \to$ exponentially damped oscillations;

$\eta > 0 \to$ exponentially growing oscillations;

$U(1)$-Kirchhoff's rule is OK

2. From star graphs towards generic graphs

Basic idea – gluing star graphs by means of the R-T algebra (Ragoucy, M. M.).



$S_i(k)$ - local S-matrices;

Using the local R-T constraints

$$\begin{pmatrix} a_i(k) \\ a_{i+1}(k) \end{pmatrix} = S_i(k) \begin{pmatrix} a_i(-k) \\ a_{i+1}(-k) \end{pmatrix}$$

one can express $\{a_2, a_3\}$ in terms of $\{a_1, a_4\}$.

$\{a_1, a_4\}$ generate an RT algebra with a global scattering matrix $\mathbf{S}$ completely determined in terms S_i.

$\mathbf{S}$ has a complicated structure, taking into account all possible multiple reflections and transmissions occurring in the system.

3. Boundary conditions breaking time-reversal.

Important for applications – junctions with magnetic field.

$$U \neq U^t$$

Not conceptual but technical problem which is currently under investigation.