One-Loop Amplitudes from Generalised Unitarity

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Outline

- Loop: Cutting Loops $\Leftrightarrow$ Sewing Trees
- Unitarity & Cut-Constructibility
- General Algorithm for Multiple-Cuts in D-dim
- Quadruple-Cuts
- Double-Cuts
- Triple-Cuts
- Analytic result for generic one-loop amplitudes

Need for Next-to-Leading-Order

Process ($V \in \{Z, W, \gamma\}$)	Relevance
$pp \to t\,\bar{t}\,b\,\bar{b}$	$t\,\bar{t}\,H$
$pp \to t\,\bar{t} + 2\text{jets}$	$t\,\bar{t}\,H$
$pp \to VV\,b\,\bar{b}$	$\text{VBF} \to H \to VV,\ t\,\bar{t}\,H$
$pp \to VV + 2\text{jets}$	$\text{VBF} \to H \to VV$
$pp \to V + 3\text{jets}$	New Physics
$pp \to b\,\bar{b}\,b\,\bar{b}$	Higgs + New Physics

▷ Accurate estimates for signal and background new physics processes

▷ Less sensitivity to unphysical input scales ($\to$ renormalization & factorization scale)

▷ More realistic process modelling: initial state radiation, jet clustering, richer virtuality

▷ NLO Building Blocks

- Born level n-point
- Real contribution $(n+1)$-point
- Virtual contribution n-point
- IR safety: $\mathrm{R} + \mathrm{V}$

)(Virtual contribution: *Tensor Reduction* $\to$ $I^{\mu\nu\rho\ldots} = \int d^D\ell \, \frac{\ell^\mu \ell^\nu \ell^\rho \ldots}{D_1 D_2 \ldots}$

Spinor Formalism

Xu, Zhang, Chang

Berends, Kleiss, De Causmaeker

Gastmans, Wu

Gunion, Kunzst

- on-shell massless (Weyl) spinors

$$|i\rangle \equiv |k_i^+\rangle \equiv u_+(k_i) = v_-(k_i) \ , \qquad [i| \equiv \langle k_i^+| \equiv \overline{u_+(k_i)} = \overline{v_-(k_i)} \ ,$$

- $k^2 = 0 :$ $\quad \not{k} = |k\rangle[k| + |k]\langle k|$

- Spinor Inner Products

$$\langle i\, j\rangle \equiv \langle i^-|j^+\rangle = \sqrt{|s_{ij}|}\, e^{i\Phi_{ij}} \ , \qquad [i\, j] \equiv \langle i^+|j^-\rangle = -\langle i\, j\rangle^* \ ,$$

with $s_{ij} = (k_i + k_j)^2 = 2k_i \cdot k_j = \langle i|k_j|i] = \langle j|k_i|j] = \langle i\, j\rangle[j\, i]$.

One Loop Amplitudes

$$A^{4-2\epsilon} = c_4(\epsilon)\, I_4 + c_3(\epsilon)\, I_3 + c_2(\epsilon)\, I_2 + c_1(\epsilon)\, I_1 \quad (1)$$

$$= c_4\, I_4^{\#} + c_3\, I_3^{\#} + c_2\, I_2^{\#} + c_1\, I_1^{\#} \quad (2)$$

$$\overset{\epsilon \to 0}{=} \text{PolyLogarithms} + \text{Rational} \quad (3)$$

where $I_i = I_i^{4-2\epsilon}$, while $I_i^{\#} = I_i^{4-2\epsilon+\#}$ are *D-shifted* Master Integrals

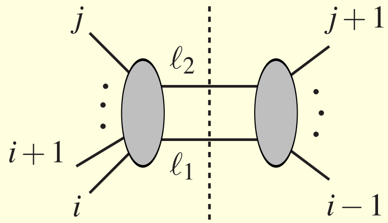
- $I_i^{\#}$ and $I_i^{4-2\epsilon}$ are linked by known recurrence relations;
- the scalar functions associated to **boxes** $(I_4^{(4m)}, I_4^{(3m)}, I_4^{(2m,\mathrm{e})}, I_4^{(2m,\mathrm{h})}, I_4^{(1m)}, I_4^{(0m)})$, **triangles** $(I_3^{(3m)}, I_3^{(2m)}, I_3^{(1m)})$, **bubbles** (I_2), and **tadpoles** (I_1) are analytically known in $(4-2\epsilon)$-dim ;

- A is known, once the coefficients c_4, c_3, c_2, c_1 appearing in (2) are known: they all are rational functions of spinor products $\langle i\, j \rangle$, $[i\, j]$

Unitarity & Cut-Constructibility

- Discontinuity accross the Cut

Cut Integral in the P_{ij}^2-channel



$$C_{i\dots j} = \Delta(A_n^{\text{1-loop}}) = \int d^4\Phi \, A^{\text{tree}}(\ell_1, i, \dots, j, \ell_2) \, \times \, A^{\text{tree}}(-\ell_2, j+1, \dots, i-1, -\ell_1)$$

with

$$d^4\Phi = d^4\ell_1 \, d^4\ell_2 \, \delta^{(4)}(\ell_1 + \ell_2 - P_{ij}) \, \delta^{(+)}(\ell_1^2) \, \delta^{(+)}(\ell_2^2)$$

- loop-Reconstruction

Bern, Dixon, Dunbar & Kosower
Anastasiou & Melnikov
Brandhuber, Mc Namara, Spence & Travaglini

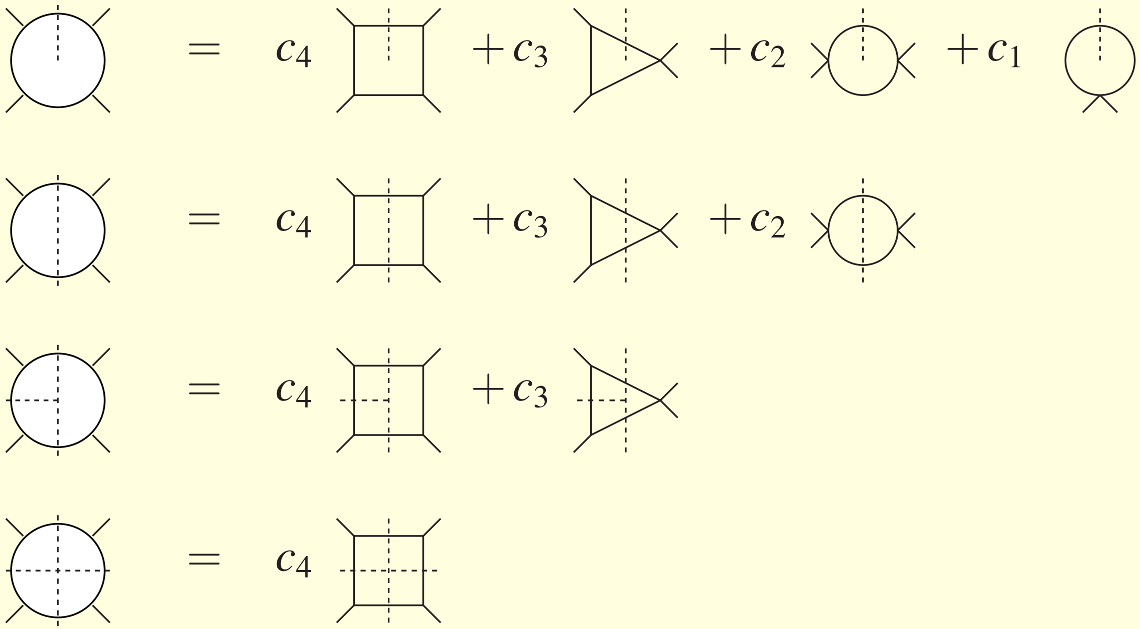
- channel-by-channel reconstruction of the loop-intgeral: $\delta^{(+)}(p^2) \leftrightarrow \frac{1}{(p^2 - i0)}$

- loop-tools integrations: PV-tensor reduction, or integration-by-parts identitities

Generalised Unitarity

- coefficients show up entangled in a given cut: how do we disentangle them?

The polylogarithmic structure of boxes, triangles, and bubbles is different. Therefore their multiple cuts have specific signature which enable us to distinguish unequivocally among them.



- Cuts in 4-dim carry informations about the *coefficients* of the polylogarithmic structure; but no informations about the *rational* term

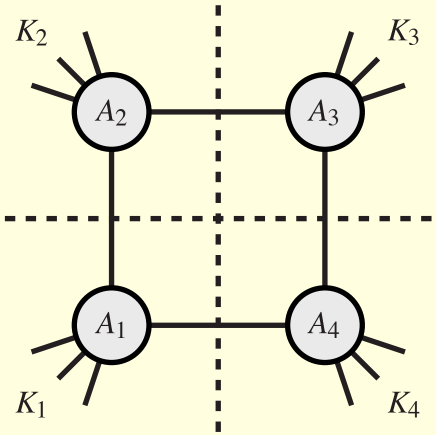
- Cuts in D-dim detect also *rational term*

Quadruple Cuts

Britto, Cachazo, Feng (2004)

Boxes

The discontinuity across the leading singularity, *via* quadruple cuts, is **unique**, and corresponds to the **coefficient** of the master **box**



- 4PLE-cut integrand: $I_4(\ell) = A_1^{\text{tree}} \times A_2^{\text{tree}} \times A_3^{\text{tree}} \times A_4^{\text{tree}}$
- Momentum-decomposition ansatz: $\ell^\mu = \alpha\, K_1^\mu + \beta\, K_2^\mu + \gamma\, K_3^\mu + \delta\, \varepsilon^\mu_{\nu\rho\sigma} K_1^\nu K_2^\rho K_3^\sigma$
- Cut-conditions: $D_1 = D_2 = D_3 = D_4 = 0 \qquad \Leftrightarrow$ coefficient constraints
- Solutions: $\ell^\mu_\pm = \alpha_0\, K_1^\mu + \beta_0\, K_2^\mu + \gamma_0\, K_3^\mu + \delta_\pm\, \varepsilon^\mu_{\nu\rho\sigma} K_1^\nu K_2^\rho K_3^\sigma$

$$c_4 = \frac{I_4(\ell_+) + I_4(\ell_-)}{2}$$

Double-Cut Phase Space Measure

- 4-dim LIPS Cacahazo, Svrček & Witten

$$\int d^4\Phi = \int d^4\ell_0 \, \delta^{(+)}(\ell_0^2)\, \delta^{(+)}((\ell_0 - K)^2) = \int \frac{\langle \ell \, d\ell\rangle [\ell \, d\ell]}{\langle \ell | K | \ell]} \int t \, dt \, \delta^{(+)}\left(t - \frac{K^2}{\langle \ell | K | \ell]}\right)$$

$$\Leftarrow \qquad \ell_0^2 = 0 \, , \quad \ell_0 = |\ell_0\rangle [\ell_0| \equiv t |\ell\rangle [\ell|$$

- D-dim LIPS Anastasiou, Britto, Feng, Kunszt, P.M.

$$\int d^{4-2\epsilon}\Phi \quad = \quad \chi(\epsilon) \int d\mu^{-2\epsilon} \int d^4\Phi_\mu \, ,$$

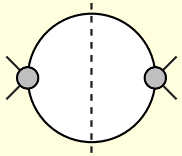
$$\begin{aligned} \int d^4\Phi_\mu \quad &= \quad \int d^4L \, \delta^{(+)}(L^2 - \mu^2)\, \delta^{(+)}((L-K)^2 - \mu^2) \\ &= \quad \int dz \, \delta(z - z_0) \int \frac{\langle \ell \, d\ell\rangle [\ell \, d\ell]}{\langle \ell | K | \ell]} \int t \, dt \, \delta^{(+)}\left(t - \frac{(1-2z)K^2}{\langle \ell | K | \ell]}\right) \end{aligned}$$

$$\Leftarrow \qquad L = \ell_0 + zK \, , \qquad \text{with} \;\; \ell_0^2 = 0 \, , \qquad \ell_0 \equiv t |\ell\rangle [\ell| \qquad z_0 = \frac{1 - \sqrt{1 - \frac{4\mu^2}{K^2}}}{2} \, ,$$

Double-Cut $\oplus$ Spinor-Integration

Britto, Buchbinder, Cachazo & Feng (2005); Britto, Feng & P.M. (2006)

Anastasiou, Britto, Feng, Kunszt & P.M. (2006)



$$M = \chi(\epsilon) \int d\mu^{-2\epsilon}\, \Delta\,, \qquad \Delta = \int d^4\Phi_\mu\, A_L^{\rm tree} \otimes A_R^{\rm tree}$$

- t-integration $\oplus$ z-integration

the 4D-discontinuity reads,

$$\Delta = \int \langle \ell\, d\ell\rangle [\ell\, d\ell]\; G\Big(|\ell\rangle\Big) \frac{[\eta\, \ell]^n}{\langle \ell|P_1|\ell]^{n+1} \langle \ell|P_2|\ell]}$$

- Feynman Parametrization:

$$\Delta = \int \langle \ell\, d\ell\rangle [\ell\, d\ell]\; (n+1) \int dx\, (1-x)^n\, G\Big(|\ell\rangle\Big) \frac{[\eta\, \ell]^n}{\langle \ell|R|\ell]^{n+2}}\,, \qquad \not{R} = x\not{P}_1 + (1-x)\not{P}_2$$

Log-term of 4D-Double Cut

- Integration-by-Parts in $|\ell]$

$$[\ell\, d\ell] \frac{[\eta\, \ell]^n}{\langle \ell|R|\ell]^{n+2}} = \frac{[d\ell\; \partial_{|\ell]}]}{(n+1)} \frac{[\eta\, \ell]^{n+1}}{\langle \ell|R|\ell]^{n+1} \langle \ell|R|\eta]} \, .$$

- Integration in $|\ell\rangle$: Holomorphic δ-function

Cachazo, Svrcek, Witten; Cachazo;
Britto, Cachazo, Feng

$$\Delta = \int dx\, (1-x)^n \int \langle \ell\, d\ell \rangle [d\ell\; \partial_{|\ell]}] \frac{G\Big(|\ell\rangle\Big)\, [\eta\, \ell]^{n+1}}{\langle \ell|R|\ell]^{n+1} \langle \ell|R|\eta]}$$

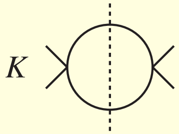
$$= \int dx\, (1-x)^n \left\{ \frac{G\Big(\not{R}|\eta]\Big)}{(R^2)^{n+1}} + \sum_j \lim_{\ell \to \ell_j} \langle \ell\, \ell_j \rangle \frac{G\Big(|\ell\rangle\Big)\, [\eta\, \ell]^{n+1}}{\langle \ell|R|\ell]^{n+1} \langle \ell|R|\eta]} \right\} = F^{(1)} + F^{(2)}$$

where $|\ell_j\rangle$ are the simple poles of G, and $R^2 = a(x-x_1)(x-x_2)$

- Double-Cut

$$M = \chi(\epsilon) \int d\mu^{-2\epsilon} \left(F^{(1)} + F^{(2)} \right)$$

- I_2

$$\Delta I_2 = $$ 

$$= \int d^4\ell_0\, \delta^{(+)}(\ell_0^2)\, \delta^{(+)}((\ell_0-K)^2) = \int_0^\infty t\, dt \int \langle \ell\, d\ell\rangle [\ell\, d\ell]\, \delta^{(+)}((\ell_0-K)^2)$$

$$\delta^{(+)}((\ell_0-K)^2) = \delta^{(+)}(K^2 - 2\ell_0\cdot K) = \delta^{(+)}(K^2 - \langle \ell_0|K|\ell_0]) = \frac{1}{\langle \ell|K|\ell]}\, \delta^{(+)}\left(t - \frac{K^2}{\langle \ell|K|\ell]}\right)$$

▷ t-integration $\Rightarrow$ $$\Delta I_2 = \int \langle \ell\, d\ell\rangle [\ell\, d\ell] \frac{K^2}{\langle \ell|K|\ell]^2}$$

▷ Integration-by-Parts in $|\ell]$

$$\frac{[\ell\, d\ell]}{\langle \ell|K|\ell]^2} = [d\ell\, \partial_{|\ell]}]\, \frac{[\eta\, \ell]}{\langle \ell|K|\eta]\langle \ell|K|\ell]}\,, \qquad \forall \eta : \eta^2 = 0$$

$$\Rightarrow \qquad \Delta I_2 = \int \langle \ell\, d\ell\rangle [d\ell\, \partial_{|\ell]}]\, \frac{K^2\, [\eta\, \ell]}{\langle \ell|K|\eta]\langle \ell|K|\ell]}$$

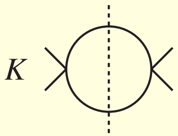
▷ Cauchy Residue Theorem in $|\ell\rangle$ would imply the contribution at the pole

$$|\ell\rangle = K|\eta] \quad (\Leftrightarrow |\ell] = K|\eta\rangle)$$

With the final result

$$\Rightarrow \qquad \Delta I_2 = \lim_{|\ell\rangle \to K|\eta]} \langle \ell|K|\eta] \left(\frac{K^2[\eta\, \ell]}{\langle \ell|K|\eta]\langle \ell|K|\ell]} \right) = \left(\frac{K^2[\eta|K|\eta\rangle}{[\eta|KKK|\eta\rangle} \right) = 1.$$

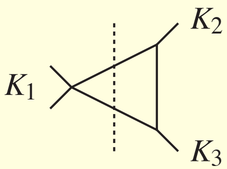
- I_2



$$= \int d^4\ell\, \delta^{(+)}(\ell^2)\, \delta^{(+)}((\ell-K)^2) = K^2 \int \frac{\langle \ell\, d\ell\rangle [\ell\, d\ell]}{\langle \ell|K|\ell]^2} = 1\ ;$$

The discontinuity of a bubble is **rational** !!!

- I_3^{3m}



$$= \int d^4\ell\, \delta^{(+)}(\ell^2)\, \frac{\delta^{(+)}((\ell-K_1)^2)}{(\ell+K_2)^2} = \int \frac{\langle \ell\, d\ell\rangle [\ell\, d\ell]}{\langle \ell|K_1|\ell]\langle \ell|Q|\ell]} = \int_0^1 dx \int \frac{\langle \ell\, d\ell\rangle [\ell\, d\ell]}{\langle \ell|R|\ell]^2} = \int_0^1 dx \frac{1}{R^2}$$

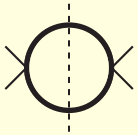
$$\not{Q} = (K_2^2/K_1^2)\not{K}_1 + \not{K}_2\ , \quad \not{R} = (1-x)\not{K}_1 + x\not{Q} \ \Rightarrow R^2 \text{ quadratic in } x$$

The discontinuity of a 3m-Triangle is a $\ln$**(irrational argument)** !!!

- I_4

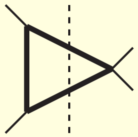
The double cut detect box-coefficient as well. One can show that the discontinuity of a 1m-,2m-,3m-box is a $\ln$**(rational argument)** – but boxes are known from 4-ple cuts.

- I_2



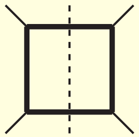
$$= \sqrt{1 - \frac{4\mu^2}{K^2}}$$

- I_3^{1m}



$$= \frac{1}{K^2} \ln \left(\frac{1 - \sqrt{1 - \frac{4\mu^2}{K^2}}}{1 + \sqrt{1 - \frac{4\mu^2}{K^2}}} \right)$$

- I_4^{0m}



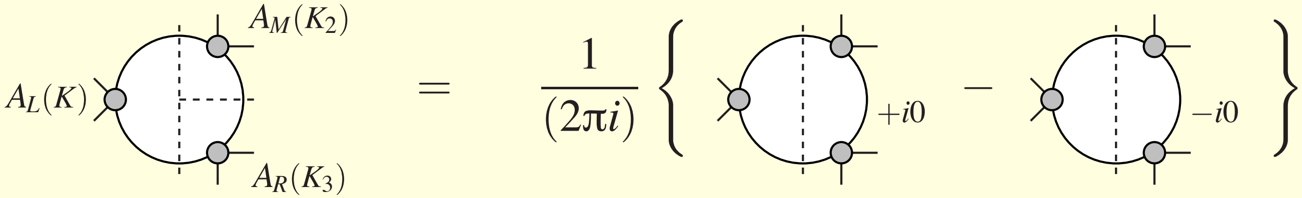
$$= \frac{2}{st\sqrt{1 - \frac{4\mu^2(s+t)}{s}}} \ln \left(\frac{1 - \sqrt{1 - \frac{4\mu^2(s+t)}{s}}}{1 + \sqrt{1 - \frac{4\mu^2(s+t)}{s}}} \right)$$

- μ-integration $\equiv$ Dimension-Shift

$$\begin{aligned} \int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} (\mu^2)^r f(\mu^2) &= \int d\Omega_{-1-2\epsilon} \int \frac{d\mu^2}{2(2\pi)^{-2\epsilon}} (\mu^2)^{-1-\epsilon+r} f(\mu^2) = \frac{(2\pi)^{2r} \int d\Omega_{-1-2\epsilon}}{\int d\Omega_{2r-1-2\epsilon}} \int \frac{d^{2r-2\epsilon}\mu}{(2\pi)^{2r-2\epsilon}} f(\mu^2) \\ &= -\epsilon(1-\epsilon)(2-\epsilon)\cdots(r-1-\epsilon)(4\pi)^r \int \frac{d^{2r-2\epsilon}\mu}{(2\pi)^{2r-2\epsilon}} f(\mu^2) \end{aligned}$$

Triple-Cut ⊕ Spinor-Integration

P.M., *Phys. Lett.* B644 (2007) 272



The integration over the Feynman parameter is frozen.

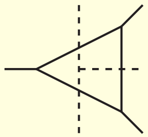
▷ *Cuts* in Feynman Parameters

$$\frac{1}{(ax+b)+i0} \quad \rightarrow \quad K_1(x) = \frac{1}{a}\delta(x-x_0)$$

$$\frac{1}{(ax^2+bx+c)+i0} \quad \rightarrow \quad K_2(x) = \frac{1}{a\,|x_1-x_2|}\Big(\delta(x-x_1)+\delta(x-x_2)\Big)$$

where $x_{0,1,2}$ are the *zeroes* of the corresponding denominators.

- I_3^{3m}

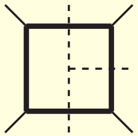


$$= \quad \cdots = \frac{1}{(2\pi i)} \int dx \left\{ \frac{1}{R^2 + i0} - \frac{1}{R^2 - i0} \right\} = \int dx\, \delta(R^2) = \int dx\, K_2(x) = \frac{(-2)}{\sqrt{\Lambda}}$$

with

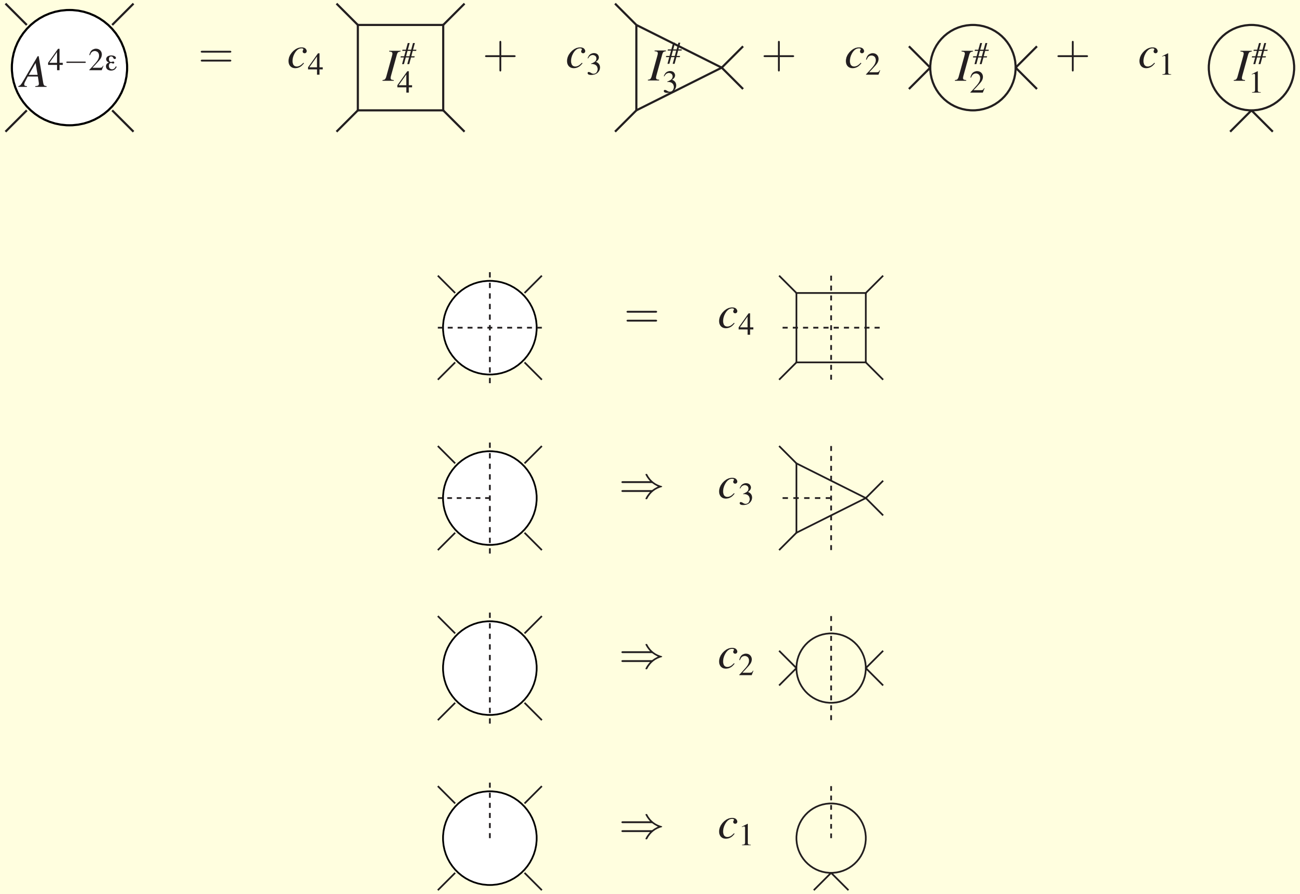
$$R^2 \quad = \quad ax^2 + 2bx + c\,, \qquad x_{1,2} = \frac{-b \pm \sqrt{\Lambda}}{a}\,, \qquad \Lambda = \text{Källen func'n}$$

- massive-I_4^{0m}



$$= \frac{(-2)}{st\sqrt{1 - 4\frac{(s+t)}{s}\mu^2}}$$

Cut-Construction of One-Loop Amplitudes



On-Shell Complex Momenta enable the *fulfillment* of the cut-constraints!

Master Formulae

Schouten identity to reduce $|\ell]$

$$\frac{[\ell\, a]}{[\ell\, b]\,[\ell\, c]} = \frac{[b\, a]}{[b\, c]}\frac{1}{[\ell\, b]} + \frac{[c\, b]}{[c\, b]}\frac{1}{[\ell\, c]} \tag{4}$$

Integration-by-Parts in $|\ell]$

$$[\ell\; d\ell]\frac{[\eta\,\ell]^n}{\langle\ell|P|\ell]^{n+2}} = \frac{[d\ell\;\partial_{|\ell]}]}{(n+1)}\frac{[\eta\,\ell]^{n+1}}{\langle\ell|P|\ell]^{n+1}\langle\ell|P|\eta]}\;. \tag{5}$$

Cauchy's Residue Theorem in $|\ell\rangle$,

$$[d\ell\;\partial_{|\ell]}]\frac{1}{\langle\ell\, x\rangle} = 2\pi\delta\Big(\langle\ell\, x\rangle\Big)\;, \qquad \int\langle\ell\; d\ell\rangle\;\delta\Big(\langle\ell\, x\rangle\Big)\; f(|\ell\rangle,|\ell]) = f(|x\rangle,|x]) \tag{6}$$

Residues in Feynman parameters, at the *zeroes* of the Standard Quadratic Function.

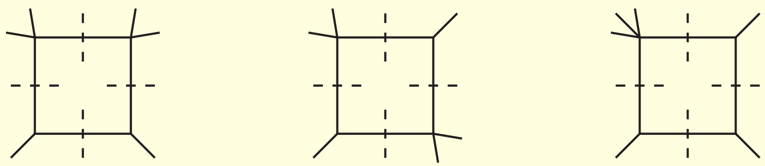
These *zeroes* are the signature of the Master Integrals: they correspond to branch points, therefore determining the polylogarithmic structure.

NLO 6-gluon Amplitude

- Numerical Result: Ellis, Giele, Zanderighi (2006)
- Analytical Result:

Amplitude	$\mathcal{N}=4$	$\mathcal{N}=1$	$\mathcal{N}=0\vert_{\rm CC}$	$\mathcal{N}=0\vert_{\rm rat}$
$(--++++)$	BDDK'94	BDDK'94	BDDK'94	BDK'05, KF'05
$(-+-+++)$	BDDK'94	BDDK'94	BBST'04	BBDFK'06, XYZ'06
$(-++-++)$	BDDK'94	BDDK'94	BBST'04	BBDFK'06, XYZ'06
$(---+++)$	BDDK'94	BBDD'04	BBDI'05, BFM'06	BBDFK'06
$(--+-++)$	BDDK'94	BBCF'05, BBDP'05	BFM'06	XYZ'06
$(-+-+-+)$	BDDK'94	BBCF'05, BBDP'05	BFM'06	XYZ'06

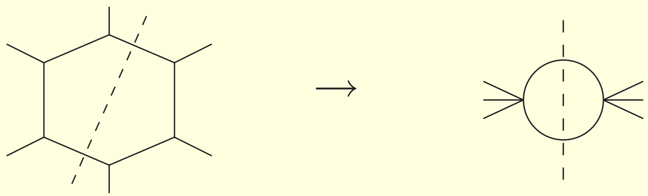
Quadruple Cuts

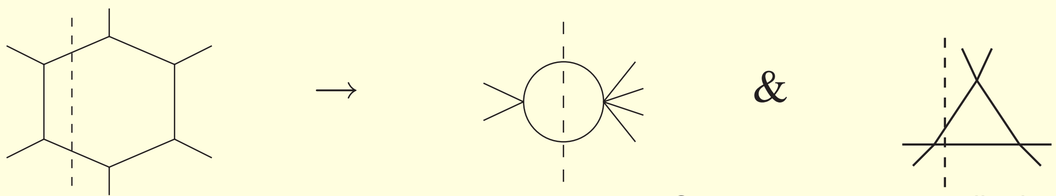


Bidder, Bjerrum-Bohr, Dunbar & Perkins (2005)

Double Cuts

Britto, Feng & P.M. (2006)





6-photon Amplitude

Mahlon (1996)

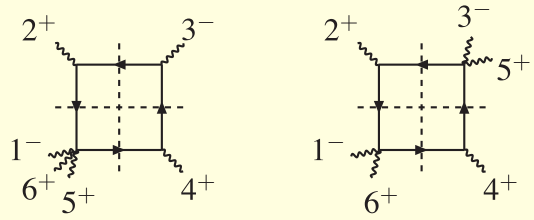
Nagy & Soper (2006)

Binoth, Guillet & Heinrich (2006)

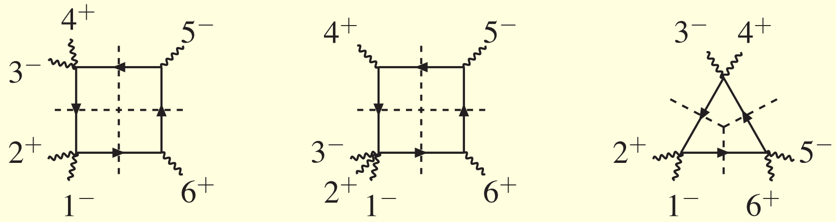
Binoth, Gehrmann, Heinrich & P.M. [hep-ph/0703311]

Ossola, Papadopoulous & Pittau (2007); Forde (2007)

- $(1^-, 2^+, 3^-, 4^+, 5^+, 6^+)$



- $(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$



NLO n-gluon⊕Higgs Amplitudes

- Heavy-top limit

- H + 4 partons Ellis, Giele, Zanderighi (2005); Campbell, Ellis, Zanderighi (2006)

- H + n-gluons

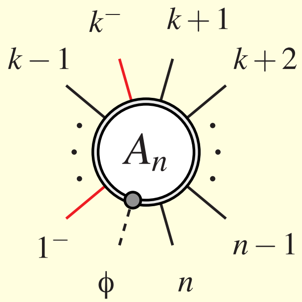
$$\phi = \frac{1}{2}(H + iA)$$

$$G^{\mu\nu}_{SD} = \frac{1}{2}(G^{\mu\nu} + \tilde{G}^{\mu\nu}) \ , \quad G^{\mu\nu}_{ASD} = \frac{1}{2}(G^{\mu\nu} - \tilde{G}^{\mu\nu}) \ , \quad \tilde{G}^{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$$

$$L_{\rm int} \propto H \ {\rm tr} G_{\mu\nu}G^{\mu\nu} + iA \ {\rm tr}\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu} \ = \ \phi \ {\rm tr} G_{SD,\mu\nu}G^{\mu\nu}_{SD} + \ \phi^\dagger \ {\rm tr}\tilde{G}_{ASD,\mu\nu}\tilde{G}^{\mu\nu}_{ASD} \ ,$$

- A(ϕ + n-gluons) $\longrightarrow$ A(n-gluons) w/out momentum conservation Dixon, Glover & Kohze

  - ϕ-nite Berger, Del Duca, Dixon (2006)
  - ϕ-MHV amplitudes (nearest neighbour minuses) Badger, Glover, Risager (2007)
  - ϕ-MHV amplitudes (generic configuration) Glover, Williams, P.M. (wip)



Coefficients in Closed Form

Britto, Feng (2007)

Britto, Feng & P.M., arXiv: 0803.1989 [hep-ph]

- Generic Massive Double-cut in the K^2-channel:

$$\int d^4L\, \delta^{(+)}\Big(L^2 - M_1^2 - \mu^2\Big)\, \delta^{(+)}\Big((L-K)^2 - M_2^2 - \mu^2\Big)\, \frac{\prod_j \langle a_j|L|b_j]}{\prod_i \Big((L-K_i)^2 - m_i^2 - \mu^2\Big)}$$

$$= \int \langle \ell\, d\ell\rangle[\ell\, d\ell] \left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right) \frac{(K^2)^{n+1}}{\langle \ell|K|\ell]^{n+2}} \frac{\prod_{j=1}^{n+k} \langle \ell|R_j|\ell]}{\prod_{i=1}^{k} \langle \ell|Q_i|\ell]} \tag{7}$$

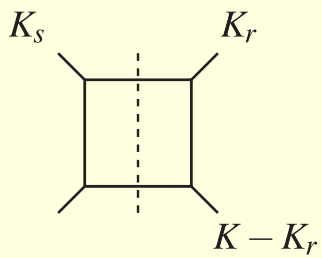
where

$$P_j = |a_j\rangle[b_j|$$

$$R_j = -\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right) P_j + \frac{-z(2P_j\cdot K)}{K^2} K,$$

$$Q_j = -\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right) K_i + \frac{K_i^2 + M_1^2 - m_i^2 - 2zK\cdot K_i}{K^2} K$$

• I_4-coefficient



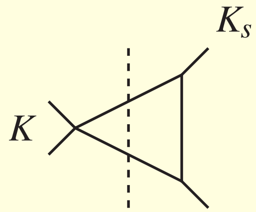
$$C[Q_r, Q_s, K] = \frac{(K^2)^{2+n}}{2} \left(\frac{\prod_{j=1}^{k+n} \langle P_{sr,1} | R_j | P_{sr,2}]}{\langle P_{sr,1} | K | P_{sr,2}]^{n+2} \prod_{t=1, t \neq i,j}^{k} \langle P_{sr,1} | Q_t | P_{sr,2}]} + \{ P_{sr,1} \leftrightarrow P_{sr,2} \} \right).$$

$$\Delta_{sr} = (2Q_s \cdot Q_r)^2 - 4Q_s^2 Q_r^2$$

$$P_{sr,1} = Q_s + \left(\frac{-2Q_s \cdot Q_r + \sqrt{\Delta_{sr}}}{2Q_r^2} \right) Q_r$$

$$P_{sr,2} = Q_s + \left(\frac{-2Q_s \cdot Q_r - \sqrt{\Delta_{sr}}}{2Q_r^2} \right) Q_r$$

• I_3-coefficient



$$C[Q_s,K] = \frac{(K^2)^{1+n}}{2}\frac{1}{(\sqrt{\Delta_s})^{n+1}}\frac{1}{(n+1)!\langle P_{s,1}P_{s,2}\rangle^{n+1}}\frac{d^{n+1}}{d\tau^{n+1}}\left(\frac{\prod_{j=1}^{k+n}\langle\gamma_{s,1}|R_j\,Q_s|\gamma_{s,1}\rangle}{\prod_{t=1,t\neq s}^{k}\langle\gamma_{s,1}|Q_t\,Q_s|\gamma_{s,1}\rangle}+\{P_{s,1}\leftrightarrow P_{s,2}\}\right)\Bigg|_{\tau=0}.$$

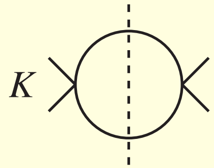
$$|\gamma_{s,1}\rangle = |P_{s,1}\rangle - \tau|P_{s,2}\rangle$$

$$\Delta_s = (2Q_s\cdot K)^2 - 4Q_s^2K^2$$

$$P_{s,1} = Q_s + \left(\frac{-2Q_s\cdot K+\sqrt{\Delta_s}}{2K^2}\right)K$$

$$P_{s,2} = Q_s + \left(\frac{-2Q_s\cdot K-\sqrt{\Delta_s}}{2K^2}\right)K$$

• I_2-coefficient



$$C[K] = (K^2)^{1+n} \sum_{q=0}^{n} \frac{1}{q!} \frac{d^q}{ds^q} \left(B^{(0)}_{n,n-q}(s) + \sum_{r=1}^{k} \sum_{a=q}^{n} \left(B^{(r;a-q;1)}_{n,n-a}(s) - B^{(r;a-q;2)}_{n,n-a}(s) \right) \right) \Bigg|_{s=0} ,$$

$$B^{(0)}_{n,t}(s) \equiv \frac{d^n}{d\tau^n} \left(\frac{1}{n! [\eta|\phi\, K|\eta]^n} \frac{(2\eta \cdot K)^{t+1}}{(t+1)(K^2)^{t+1}} \frac{\prod_{j=1}^{n+k} \langle \gamma_0 | R_j\ (K - s\eta) | \gamma_0 \rangle}{\langle \gamma_0\, \eta \rangle^{n+1} \prod_{p=1}^{k} \langle \gamma_0 | Q_p\ (K - s\eta) | \gamma_0 \rangle} \right) \Bigg|_{\tau=0} ,$$

$$B^{(r;b;1)}_{n,t}(s) \equiv \frac{(-1)^{b+1}}{b! \sqrt{\Delta_r}^{\,b+1} \langle P_{r,1}\, P_{r,2} \rangle^b} \frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\langle \gamma_{r,1} | \eta | P_{r,1}]^{t+1}}{\langle \gamma_{r,1} | K | P_{r,1}]^{t+1}} \frac{\langle \gamma_{r,1} | Q_r\ \eta | \gamma_{r,1} \rangle^b \prod_{j=1}^{n+k} \langle \gamma_{r,1} | R_j\ (K - s\eta) | \gamma_{r,1} \rangle}{\langle \gamma_{r,1} | \eta\ K | \gamma_{r,1} \rangle^{n+1} \prod_{p=1, p\neq r}^{k} \langle \gamma_{r,1} | Q_p\ (K - s\eta) | \gamma_{r,1} \rangle} \right) \Bigg|_{\tau=0} ,$$

$$B^{(r;b;2)}_{n,t}(s) \equiv (-1)^{2b+1} B^{(r;b;1)}_{n,t}(s) \ \Big|_{P_{r,1} \leftrightarrow P_{r,2}}$$

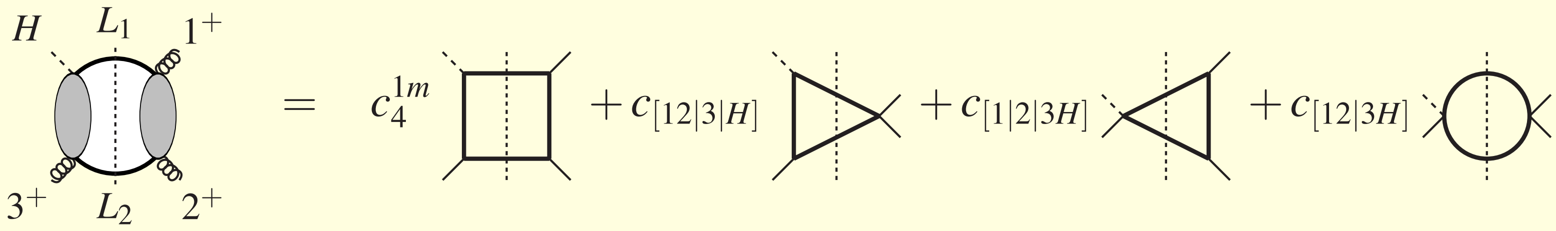
η and ϕ, massless reference momenta

$$|\gamma_0\rangle = (\not{K} - \tau\phi) |\eta] \ , \qquad |\gamma_{r,1}\rangle = |P_{r,1}\rangle - \tau |P_{r,2}\rangle \ , \qquad |\gamma_{r,2}\rangle = |P_{r,2}\rangle - \tau |P_{r,1}\rangle ,$$

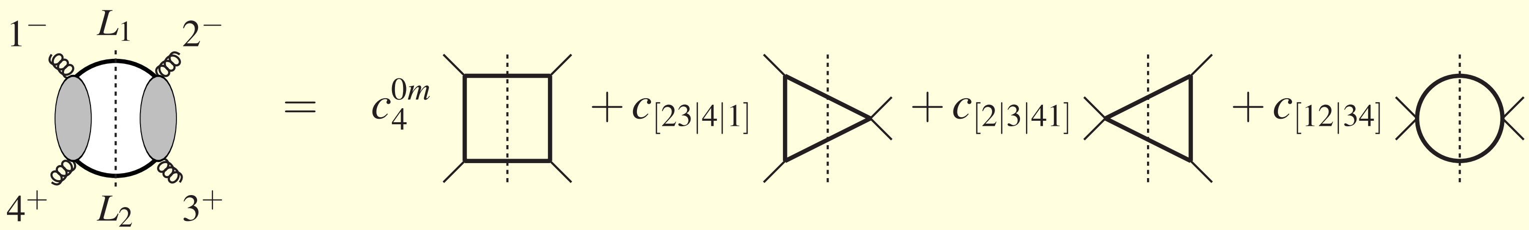
Integrals' Reduction by Pattern-matching

- Successfull Tests

Rozowsky (1997)



Bern-Morgan (1994)



- Implemented in S@M (Spinors @ MATHEMATICA) Mâitre & P.M. (2007)

Summary & Outlook

- Dramatic progress in evaluating on-shell scattering amplitudes: the exploitation of *factorization* and *unitarity*, turned into tools for computing them.

- *On-shell methods* restrict the propagating states to the physical ones and the *spinor-helicity formalism* suitable for avoiding the (intermediate) treatment of unphysical degrees of freedom.

- The *singularity* information can be extracted by defining amplitudes for suitable *complex, yet on-shell momenta*

- Generalised Unitarity realized through on-shell complex momenta parametrized by complex spinors

- Phase-Space Integration over spinorial variables

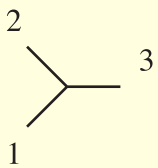
- Result: closed analytic expression for the coefficients of the reduction of *any* one-loop amplitude in terms of Master Integrals Britto, Feng & P.M.

- :: LHC-phenomenology

- :: UV-behavoiur of SuGra $N = 8$

- :: Numerical implementation of Generalised Unitarity

- on-shell 3-point amplitude: $k_i^2 = 0$



$$0 = k_1^2 = (k_2 + k_3)^2 = 2k_2 \cdot k_3 = \langle 2\,3 \rangle [3\,2] \begin{cases} \langle 2\,3 \rangle \neq 0 \\ \\ |3]\ //\ |2] \end{cases} \quad (k_3 \text{ on} - \text{shell \& complex})$$

The imaginary number is a fine and wonderful recourse of the divine spirit, almost an amphibian between being and non-being. [...] there is something fishy about [...] imaginaries, but one can calculate with them because their form is correct. Leibniz