The Worldline Formalism in Flat and Curved Space

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Outline

- Introduction

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- Worldline formalism in a simple example ($\lambda\phi^3$)

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- Conclusions

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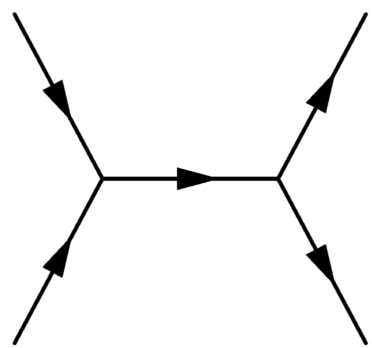
  II quantized approach: quantization of “wave fields”
- The worldline formalism is a first quantized approach: coordinates of each relativistic “particle” are quantized



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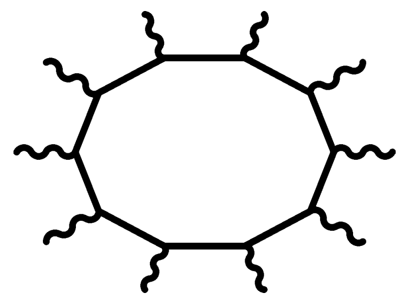

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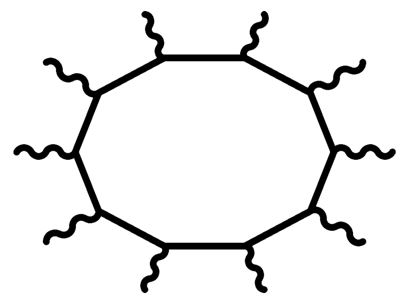
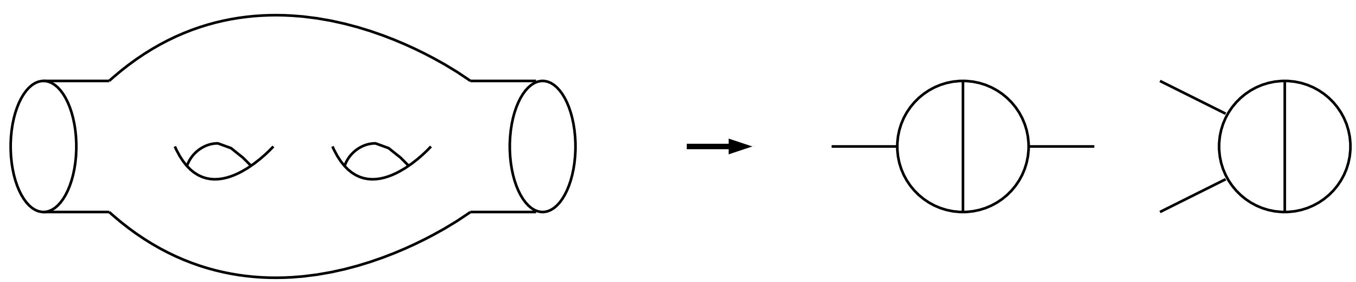


  - useful to compare with string theory (string inspired Feynman rules)



The worldline formalism

It is an old formalism

PHYSICAL REVIEW VOLUME 80, NUMBER 3 NOVEMBER 1, 1950

Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction

R. P. FEYNMAN*
Department of Physics, Cornell University, Ithaca, New York
(Received June 8, 1950)

The validity of the rules given in previous papers for the solution of problems in quantum electrodynamics is established. Starting with Fermi's formulation of the field as a set of harmonic oscillators, the effect of the oscillators is integrated out in the Lagrangian form of quantum mechanics. There results an expression for the effect of all virtual photons valid to all orders in $e^2/\hbar c$. It is shown that evaluation of this expression as a power series in $e^2/\hbar c$ gives just the terms expected by the aforementioned rules.

In addition, a relation is established between the amplitude for a given process in an arbitrary unquantized potential and in a quantum electrodynamical field. This relation permits a simple general statement of the laws of quantum electrodynamics.

A description, in Lagrangian quantum-mechanical form, of particles satisfying the Klein-Gordon equation is given in an Appendix. It involves the use of an extra parameter analogous to proper time to describe the trajectory of the particle in four dimensions.

A second Appendix discusses, in the special case of photons, the problem of finding what real processes are implied by the formula for virtual processes.

Problems of the divergences of electrodynamics are not discussed.

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On Gauge Invariance and Vacuum Polarization

JULIAN SCHWINGER
Harvard University, Cambridge, Massachusetts
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This paper is based on the elementary remark that the extraction of gauge invariant results from a formally gauge invariant theory is ensured if one employs methods of solution that involve only gauge covariant quantities. We illustrate this statement in connection with the problem of vacuum polarization by a prescribed electromagnetic field. The vacuum current of a charged Dirac field, which can be expressed in terms of the Green's function of that field, implies an addition to the action integral of the electromagnetic field. Now these quantities can be related to the dynamical properties of a "particle" with space-time coordinates that depend upon a proper-time parameter. The proper-time equations of motion involve only electromagnetic field strengths, and provide a suitable gauge invariant basis for treating problems. Rigorous solutions of the equations of motion can be obtained for a constant field, and for a plane wave field. A renormalization of field strength and charge, applied to the modified lagrange function for constant fields, yields a finite, gauge invariant result which implies nonlinear properties for the electromagnetic field in the vacuum. The contribution of a zero spin charged field is also stated. After the same field strength renormalization, the modified physical quantities describing a plane wave in the vacuum reduce to just those of the maxwell field; there are no nonlinear phenomena for a single plane wave, of arbitrary strength and spectral composition. The results obtained for constant (that is, slowly varying fields), are then applied to treat the two-photon disintegration of a spin zero neutral meson arising from the polarization of the proton vacuum. We obtain approximate, gauge invariant expressions for the effective interaction between the meson and the electromagnetic field, in which the nuclear coupling may be scalar, pseudoscalar, or pseudovector in nature. The direct verification of equivalence between the pseudoscalar and pseudovector interactions only requires a proper statement of the limiting processes involved. For arbitrarily varying fields, perturbation methods can be applied to the equations of motion, as discussed in Appendix A, or one can employ an expansion in powers of the potential vector. The latter automatically yields gauge invariant results, provided only that the proper-time integration is reserved to the last. This indicates that the significant aspect of the proper-time method is its isolation of divergences in integrals with respect to the proper-time parameter, which is independent of the coordinate system and of the gauge. The connection between the proper-time method and the technique of "invariant regularization" is discussed. Incidentally, the probability of actual pair creation is obtained from the imaginary part of the electromagnetic field action integral. Finally, as an application of the Green's function for a constant field, we construct the mass operator of an electron in a weak, homogeneous external field, and derive the additional spin magnetic moment of $\alpha/2\pi$ magnetons by means of a perturbation calculation in which proper-mass plays the customary role of energy.

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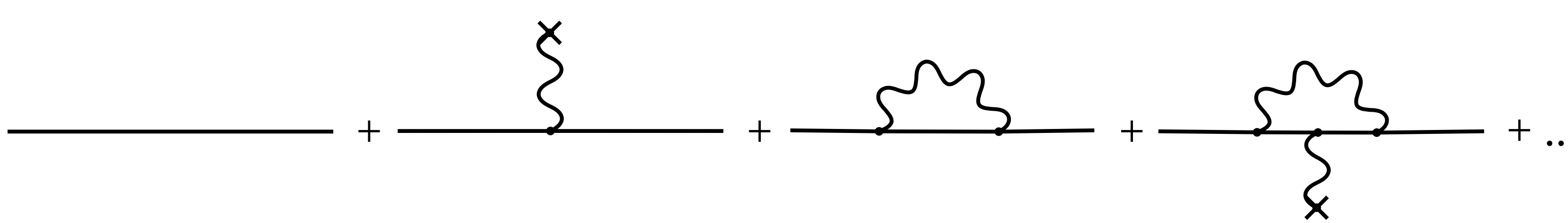
The worldline formalism

In app. A of Feynman paper: "The formulation given here ... is given only for its own interest as an alternative to the formulation of second quantization."

$$\int_0^\infty ds \int_{x(0)=x}^{x(s)=x'} \mathcal{D}x(\tau) \exp\Big(-\frac{i}{2}m^2 s\Big)$$

$$\exp\Big[-i\int_0^s d\tau\, \frac{1}{2}\Big(\frac{dx_\mu}{d\tau}\Big)^2 - i\int_0^s d\tau\, \frac{dx_\mu}{d\tau} A_\mu(x(\tau))$$

$$-i\frac{e^2}{2}\int_0^s d\tau \int_0^s d\tau'\, \frac{dx_\mu}{d\tau}\frac{dx_\nu}{d\tau'}\delta_+^{\mu\nu}(x(\tau)-x(\tau'))\Big]$$



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  - string inspired Feynman rules of Bern and Kosower (rederived by Strassler using only a particle picture)
  - and surely many more interesting examples ...

The basic example

- The propagator

$$
= \frac{1}{p^2+m^2} = \int_0^\infty dT \ \exp[-T \underbrace{(p^2+m^2)}_{H}]
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where the worldline action is

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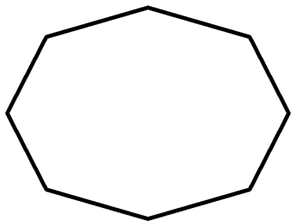
S[x(\tau)] = \int_0^T d\tau \left(\frac{1}{4}(\dot{x}^\mu)^2 + m^2\right)
$$

The basic example

- The one-loop induced action



$$
\begin{aligned}
&= -\log \mathrm{Det}^{-\frac{1}{2}}(p^2+m^2) = \frac{1}{2}\mathrm{Tr}\,\log \underbrace{(p^2+m^2)}_{H} \\
&= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\,\mathrm{Tr}\,e^{-TH} \\
&= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int_{S^1}\mathcal{D}x(\tau)\;e^{-S[x(\tau)]}
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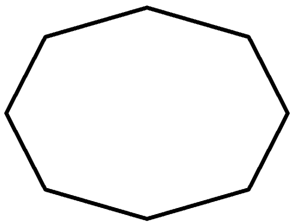

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Also with background fields

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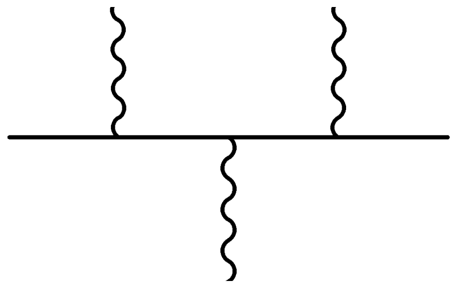

$$= \frac{1}{p^2 + m^2 + \lambda\phi(x)} \equiv \frac{1}{H}$$

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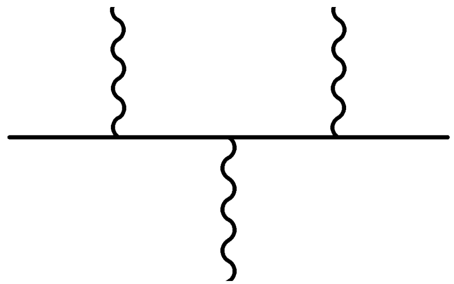

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where

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Quantization of bosonic particle

- Obtain this from the quantization of the bosonic particle

$$S[x,p,e] = \int_0^1 d\tau \left[p_\mu \dot{x}^\mu - \frac{1}{2} e(p^2 + m^2) \right]$$

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- Canonical quantization gives a constraint

$$[\hat{x}^\mu, \hat{p}_\nu] = i\hbar \delta^\mu_\nu$$

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- Using wave functions $\phi(x) = \langle x|\phi\rangle$ get Klein-Gordon eq.

$$(\hat{p}^2 + m^2)|\phi\rangle = 0 \quad \Rightarrow \quad (-\partial_\mu \partial^\mu + m^2)\phi(x) = 0$$

Path integral quantization

- Configuration space action (Wick rotated)

$$S[x,e] = \int_0^1 d\tau \; \frac{1}{2}\Big(e^{-1}(\dot{x}^\mu)^2 + e m^2\Big)$$

with reparametrization invariance (gauge field e)

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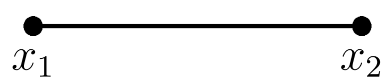
- Propagator

$$\langle \phi(x_1)\phi(x_2)\rangle_{QFT} =$$



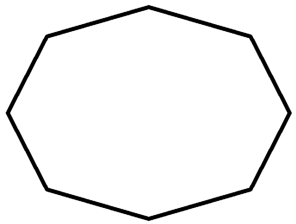
$$= \int_I \frac{\mathcal{D}x\mathcal{D}e}{\text{Vol(Gauge)}} \, \mathrm{e}^{-S[x,e]}$$

$$= \int_0^\infty dT \, \underbrace{\int_I \mathcal{D}x \, e^{-S[x,e=2T]}}_{A(x_1,x_2;T)=\langle x_2|e^{-HT}|x_1\rangle} = \int_0^\infty \frac{dT}{(4\pi T)^{\frac{D}{2}}} \, e^{-\frac{(x_2-x_1)^2}{4T} - m^2 T}$$

Path integral quantization

- One-loop effective action

$$\ln Z[g] = $$



$$= \int_{S^1} \frac{\mathcal{D}x\mathcal{D}e}{\mathrm{Vol(Gauge)}}\, \mathrm{e}^{-S[x,e]}$$

$$= -\frac{1}{2}\int_0^\infty \frac{dT}{T} \underbrace{\int_{S^1} \mathcal{D}x\; e^{-S[x,e=2T]}}_{\mathrm{Tr}\; e^{-TH}} = -\frac{1}{2}\int_0^\infty dT\, \frac{e^{-m^2 T}}{T^{\frac{D}{2}+1}} \int \frac{d^D x}{(4\pi)^{\frac{D}{2}}}$$

$$= \int d^D x \underbrace{\int \frac{d^{D-1}p}{(2\pi)^{D-1}}\, \frac{1}{2}\sqrt{\vec{p}^{\,2} + m^2}}_{\text{vacuum energy density}}$$

Example: $\lambda\phi^3$ theory

- Example of Bern-Kosower master formula in $\lambda\phi^3$

$$S[\phi] = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{3!}\phi^3 \right]$$

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- Quantum-background splitting $\phi \to \varphi + \phi$

$$S_2[\varphi] = \int d^D x \, \frac{1}{2} \varphi \Big(-\partial^2 + m^2 + \lambda \phi \Big) \varphi$$

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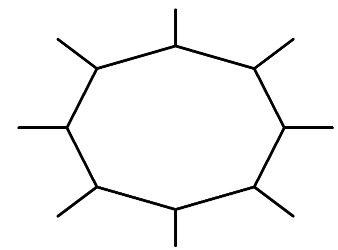
- 1-loop effective action is

$$\mathrm{e}^{-\Gamma[\phi]} = \int D\varphi \; \mathrm{e}^{-S_2[\varphi]} = \mathrm{Det}^{-\frac{1}{2}}(-\partial^2 + m^2 + \lambda\phi(x))$$

Example: $\lambda\phi^3$ theory

- Thus

$$\Gamma[\phi] = \frac{1}{2}\mathrm{Tr}\,\log\left(-\partial^2 + m^2 + \lambda\phi\right) = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\;\mathrm{Tr}\,\mathrm{e}^{-(-\partial^2+m^2+\lambda\phi)T}$$

$$= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\;\int_{PBC} Dx\;\mathrm{e}^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2+m^2+\lambda\phi(x)\right)} =$$ 

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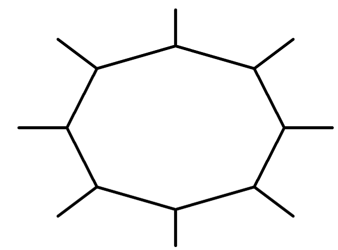

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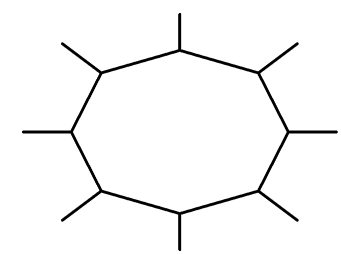
$$= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\ \int_{PBC} Dx\ \mathrm{e}^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2+m^2+\lambda\phi(x)\right)} =$$ 

- Can now use perturbation theory for quantum mechanical path integrals
- Get effective action in the derivative expansion (standard calculation in the heat kernel approach)

Example: $\lambda\phi^3$ theory

- Alternatively, expand in powers of ϕ taken as sum of N plane waves, and get averages of vertex operators

$$\phi(x) = \sum_{i=1}^{N} e^{ip_i \cdot x} \quad \rightarrow \quad \left\langle e^{ip_1 \cdot x(\tau_1)}\; e^{ip_2 \cdot x(\tau_2)} \dots e^{ip_N \cdot x(\tau_N)} \right\rangle$$

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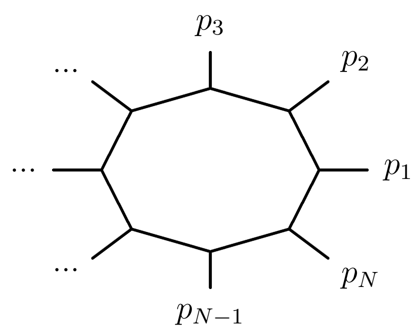
- obtain Bern-Kosower type of master formula

$$\Gamma[p_1, \ldots, p_N] =$$



$$= -\frac{(-\lambda)^N}{2} (2\pi)^D \delta^D \left(\sum_{i=1}^{N} p_i \right)$$

$$\int_0^\infty \frac{dT}{T} \, \frac{e^{-m^2 T}}{(4\pi T)^{\frac{D}{2}}} \Big(\prod_{i=1}^{N} \int_0^T d\tau_i \Big) \exp \sum_{i,j=1}^{N} \Big[\frac{1}{2} \Delta_{ij} \, p_i \cdot p_j \Big]$$

Example: $\lambda\phi^3$ theory

- Note that $\Delta_{ij} = \Delta(\tau_i, \tau_j)$ is the worldline propagator with PBC (zero mode excluded)

$$\langle x^\mu(\tau) x^\nu(\sigma) \rangle = -\eta^{\mu\nu} \Delta(\tau, \sigma)$$

with

$$\Delta(\tau, \sigma) = |\tau - \sigma| - \frac{1}{T}(\tau - \sigma)^2 - \frac{T}{6}$$

and satisfies

$$\begin{aligned} \partial_\tau \Delta(\tau, \sigma) &= {}^\bullet\Delta(\tau, \sigma) = \epsilon(\tau - \sigma) - \frac{2}{T}(\tau - \sigma) \\ \partial_\tau^2 \Delta(\tau, \sigma) &= {}^{\bullet\bullet}\Delta(\tau, \sigma) = 2\delta(\tau - \sigma) - \frac{2}{T} \end{aligned}$$

Coupling to spin 1

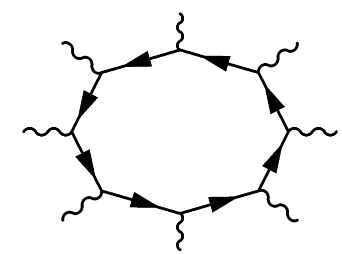
- Scalar contribution to scalar QED at 1-loop

$$S[\phi,\phi^*,A] = \int d^Dx\,(\,|(\partial_\mu + ieA_\mu)\phi|^2 + m^2|\phi|^2\,)$$

$$e^{-\Gamma[A]} = \int D\phi D\phi^*\, e^{-S[\phi,\phi^*,A]} = \mathrm{Det}^{-1}(-\nabla_A^2 + m^2)$$

$$\Gamma[A] = \mathrm{Tr}\,\log\,(-\nabla_A^2 + m^2) = -\int_0^\infty \frac{dT}{T}\ \mathrm{Tr}\ e^{-(-\nabla_A^2+m^2)T}$$

$$= -\int_0^\infty \frac{dT}{T}\ \int_{PBC} Dx\ e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2+ieA_\mu(x)\dot{x}^\mu+m^2\right)} =$$



Coupling to spin 1

- Take A_μ as sum of plane waves

$$A_\mu(x) = \sum_{i=1}^{N} \varepsilon_\mu^{(i)} \, e^{ip_i \cdot x}$$

and get averages of “N photon vertex operators”

$$\left\langle \varepsilon_{\mu_1}^{(1)} \, \dot{x}^{\mu_1}(\tau_1) e^{ip_1 \cdot x(\tau_1)} \ldots \varepsilon_{\mu_N}^{(N)} \, \dot{x}^{\mu_N}(\tau_N) e^{ip_N \cdot x(\tau_N)} \right\rangle$$

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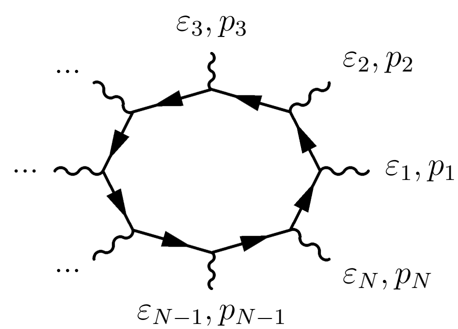
- can compute by Wick contractions (or by Gaussian integration after exponentiation of the $\epsilon_\mu \dot{x}^\mu$’s)

Coupling to spin 1

- get the Bern-Kosower master formula

$$\Gamma[p_i, \varepsilon_i] =$$



$$=$$

$$-(-ie)^N (2\pi)^D \delta^D\left(\sum_{i=1}^N p_i\right) \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{(4\pi T)^{\frac{D}{2}}} \prod_{i=1}^N \int_0^T d\tau_i$$

$$\exp \sum_{i,j=1}^N \left[\frac{1}{2} \Delta_{ij}\, p_i \cdot p_j - i\, {}^\bullet\!\Delta_{ij}\, \varepsilon_i \cdot p_j + \frac{1}{2}\, {}^{\bullet\bullet}\!\Delta_{ij}\, \varepsilon_i \cdot \varepsilon_j\right]\Bigg|_{\mathrm{lin}\,\varepsilon_{\mathrm{i}}}$$

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- Attempts to study QFT with boundaries (FB, O. Corradini, P. Pisani, JHEP 0702 (2007), [hep-th/0612236])

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- Consider the chiral current $J_5^\mu = \bar\Psi \gamma^\mu \gamma_5 \Psi$

$$\nabla_\mu \left\langle J_5^\mu \right\rangle \sim \mathrm{Tr}\left[\gamma^5 e^{-\beta \hat{H}}\right] = \int_{PBC} \mathcal{D}x \mathcal{D}\psi \; e^{-S_{\mathrm{SQM}}} = \hat{\mathrm{A}}(R)\mathrm{ch}(E)$$

$$\hat{H} = -\frac{1}{2} \nabla\!\!\!/ \nabla\!\!\!/ = -\frac{1}{2}\nabla^2 + \frac{1}{8}R = \hat{Q}^2$$

$$\hat{Q} = \frac{i}{\sqrt{2}} \nabla\!\!\!/$$

$$S_{\mathrm{SQM}} = \int_0^\beta d\tau \, \frac{1}{2} g_{\mu\nu}(x)\Big(\dot{x}^\mu \dot{x}^\nu + \psi^\mu(\dot{\psi}^\nu + \dot{x}^\lambda \Gamma^\nu_{\lambda\rho}(x)\psi^\rho)\Big)$$

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- Susy Quantum Mechanics corresponds to I quantization of the spin 1/2 particle –> N=1 spinning particle

$N = 1$ spinning particle

- In fact, consider the $N = 1$ susy model

$$S[x,p,\psi] = \int d\tau \left[p_\mu \dot{x}^\mu + \frac{i}{2}\eta_{\mu\nu}\psi^\mu\dot{\psi}^\nu - \frac{1}{2}\eta_{\mu\nu}p^\mu p^\nu \right]$$

ψ^μ real Grassmann variables (1D Majorana fermions)

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- Gauging $N = 1$ susy gives the $N = 1$ spinning particle

$$S[x,p,\psi,e,\chi] = \int d\tau \left[p_\mu \dot{x}^\mu + \frac{i}{2}\eta_{\mu\nu}\psi^\nu\dot{\psi}^\mu - eH - i\chi Q \right]$$

$N=1$ spinning particle

- Canonical quantization

$$[\hat{x}^\mu, \hat{p}_\nu] = i\hbar\delta^\mu_\nu \ , \qquad \{\psi^\mu, \psi^\nu\} = \hbar\eta^{\mu\nu}$$

$$\hat{H}|\Psi\rangle = 0 \ , \quad \hat{Q}|\Psi\rangle = 0 \ , \quad |\Psi\rangle \in \text{physical Hilbert space}$$

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- ψ^μ are realized by the gamma matrices $\gamma^\mu \sim \sqrt{\frac{2}{\hbar}}\,\psi^\mu$
- Using wave functions $\Psi_\alpha(x) = \langle x, \alpha|\Psi\rangle$

$$\hat{Q}|\Psi\rangle = \hat{\psi}^\mu \hat{p}_\mu|\Psi\rangle = 0 \quad \Rightarrow \quad (\gamma^\mu)_\alpha{}^\beta\, \partial_\mu\, \Psi_\beta(x) = 0$$

i.e. massless Dirac equation $\gamma^\mu \partial_\mu \Psi = 0$

Chiral and trace anomalies

- Chiral anomalies computed with worldline path integral.

$$\nabla_\mu \left\langle J_5^\mu \right\rangle \sim \mathrm{Tr}\left[\gamma^5 e^{-\beta \hat{H}}\right] = \int_{PBC} \mathcal{D}x\mathcal{D}\psi \; e^{-S_{\rm SQM}} = \hat{\mathrm{A}}(R)\mathrm{ch}(E)$$

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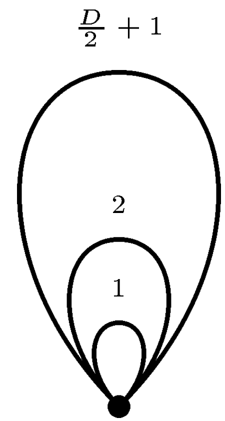
Trace anomalies

- A simple power counting shows that to get the one-loop trace anomalies in D dimensions need worldline calculations at $\frac{D}{2}+1$ loops on the worldline

$$\frac{1}{(4\pi T)^{\frac{D}{2}}} = \frac{1}{(2\pi\beta)^{\frac{D}{2}}} \quad \Leftrightarrow$$



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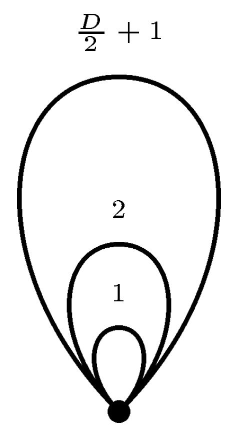

- Need to construct the QM path integral in curved space

Path integral in curved space

- The 1D nonlinear sigma model has derivative interactions and is super-renormalizable

$$S[x] = \frac{1}{\beta}\int_0^1 d\tau\ \frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu \sim \underbrace{\dot{x}^2}_{prop} + \underbrace{g_3 x \dot{x}^2 + g_4 x^2 \dot{x}^2 + \ldots}_{vertices}$$

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- To reproduce the quantum hamiltonian $H = -\frac{1}{2}\nabla^2$ must in general add a local counterterm V_{CT}

$$\Delta S[x] = \int_0^1 d\tau\ \beta^2\, V_{CT}$$

Path integral in curved space

Three established regularization schemes

- **Mode regularization (MR)**

  $$V_{MR} = -\frac{1}{8}R - \frac{1}{24}\, g^{\mu\nu} g^{\alpha\beta} g_{\gamma\delta}\, \Gamma^{\gamma}_{\mu\alpha}\, \Gamma^{\delta}_{\nu\beta}$$

- **Time slicing (TS)**

  $$V_{TS} = -\frac{1}{8}R + \frac{1}{8}\, g^{\mu\nu}\, \Gamma^{\beta}_{\mu\alpha}\, \Gamma^{\alpha}_{\nu\beta}$$

- **Dimensional regularization (DR)**

  $$V_{DR} = -\frac{1}{8}R$$

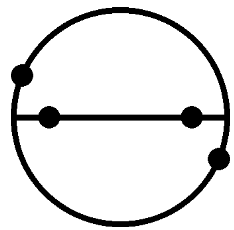
MR and TS schemes break manifestly general covariance and noncovariant counterterms restore it.

DR does not need noncovariant counterterms.

Path integral in curved space

- Example of superficially logarithmic divergent graph I.
  Use propagator Δ with Dirichlet boundary conditions:

$$I =$$



$$= \int_0^1 d\tau \int_0^1 d\sigma \; {}^{\bullet}\Delta \; {}^{\bullet}\Delta^{\bullet} \; \Delta^{\bullet} \sim \int \frac{dk}{k}$$

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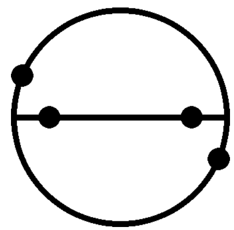

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- For details: F.B. and P. van Nieuwenhuizen,
  “Path Integrals and Anomalies in Curved Space”
  (Cambridge University Press, 2006)

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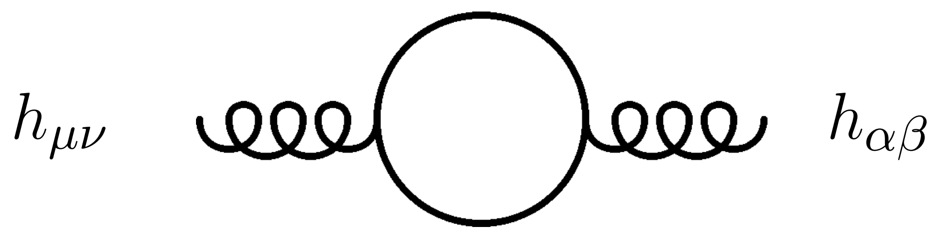

(with A. Zirotti, O. Corradini, P. Benincasa and S. Giombi)

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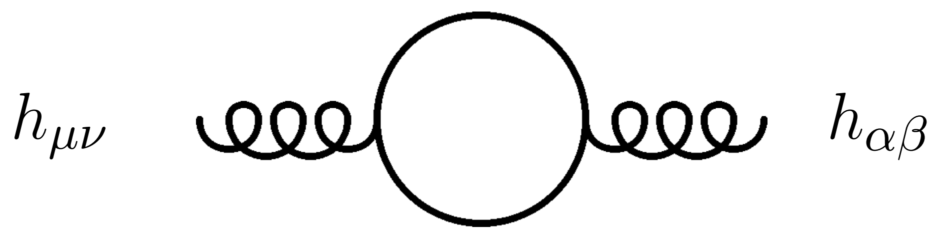
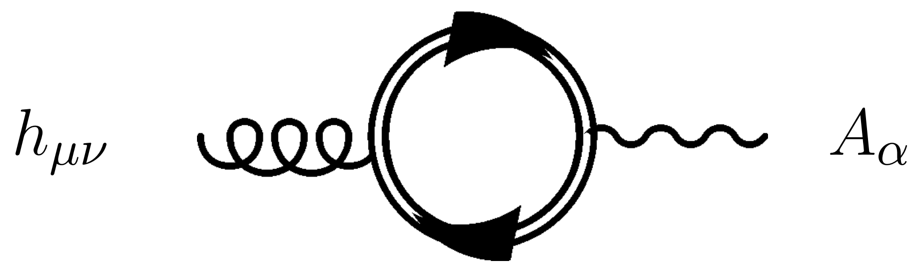


- Not easy to derive a Bern-Kosower type of formula: consider the various vertex operators

$$e^{ipx}\,, \qquad \epsilon\dot{x}\, e^{ipx} \sim e^{ipx+\epsilon\dot{x}} \qquad \Leftrightarrow \qquad \dot{x}\dot{x}\, e^{ipx}$$

Applications

- Graviton-photon conversion in a constant em background



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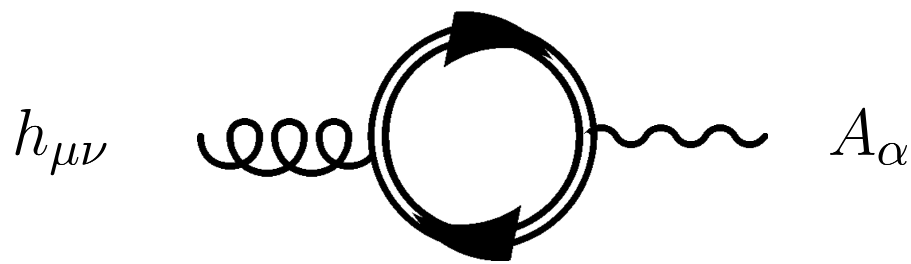
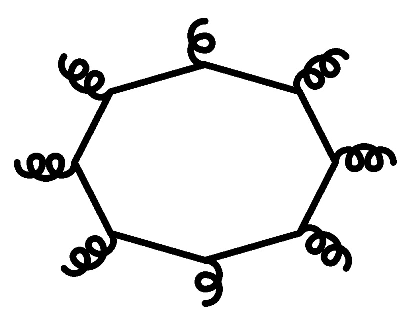


- One-loop effective action (first few Seeley DeWitt coeff.) for arbitrary differential form (massless or massive) using the N=2 spinning particle



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- $SO(N)$ spinning particle ($X = (x, \psi)$, $G = (e, \chi, a)$)

$$S[x,G] = \int d\tau \frac{1}{2}\Big(e^{-1}(\dot{x}^\mu - \chi_i \psi_i^\mu)^2 + \psi_i^\mu(\delta_{ij}\partial_\tau - a_{ij})\psi_j^\mu\Big)$$

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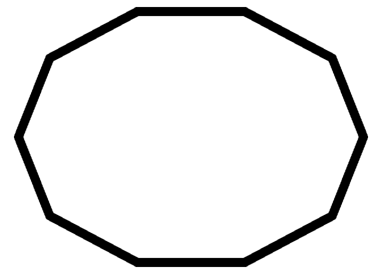
- gauge fixing on the circle $S^1 \rightarrow$ modular parameters

$$\begin{aligned} e &\rightarrow \beta && \text{(proper time)} \\ \chi_i &\rightarrow 0 \\ a_{ij} &\rightarrow \hat{a}_{ij}(\theta_k) && k = 1, .., s; \quad s = \text{rank } SO(N) \end{aligned}$$

Applications

One-loop partition function (PBC for bosons, ABC for fermions)

$$Z = \int_{S^1} \frac{\mathcal{D}X\,\mathcal{D}G}{\mathrm{Vol}\,(\mathrm{Gauge})}\, e^{-S[X,G]} =$$



$$= -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta} \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}}$$

$$\underbrace{\frac{2}{2^s s!}\prod_{k=1}^{s}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\left(\mathrm{Det}_{ABC}(\partial_\tau - \hat{a}_{vec})\right)^{\frac{D}{2}-1}\mathrm{Det}'_{PBC}(\partial_\tau - \hat{a}_{adj})}_{Dof(D,N)=Dof(2n,2s)=2^{s-1}\frac{(2n-2)!}{[(n-1)!]^2}\prod_{k=1}^{s-1}\frac{k\;(2k-1)!\;(2k+2n-3)!}{(2k+n-2)!\;(2k+n-1)!}}$$

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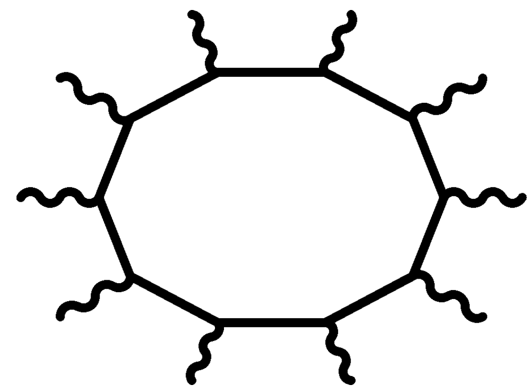

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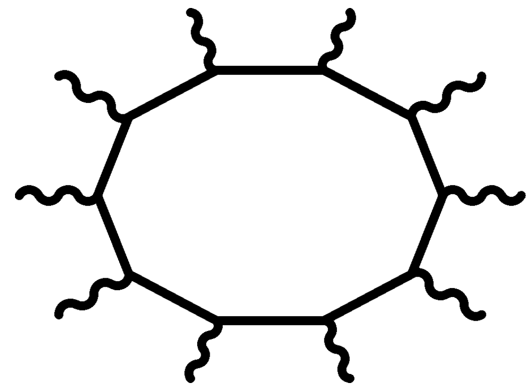

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- It does not seem possible to couple to a general curved background (higher spin coupling problem!)

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