sine-Gordon theory in noncommutative spacetime

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Outline

1. Commutative Warm-up: sine-Gordon model and a summary of well-known results.
2. What is meant by "noncommutative" in this talk: Moyal algebra $\mathcal{A}_\theta(\mathbb{R}^d)$ and $\star$-product.
3. Finding the Model: Dimensional reduction from self-dual Yang-Mills(SDYM) theory.
4. Properties of the Model: Classical and Quantum.
5. Conclusions and Outlook.

Consider the following theory for a real scalar field in $1+1$ dimensions.

$$S = \int dtdy \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + 4\alpha^2(\cos\phi - 1)\,.$$

- We use the metric $\eta_{\mu\nu} = \mathrm{diag}(1\,, -1)$, and α has the dimensions of mass.
- The equation of motion for ϕ is

$$\partial_\mu\partial^\mu\phi = -4\alpha^2 sin\phi\,.$$

- It has kink and anti-kink solutions, which are static and given by

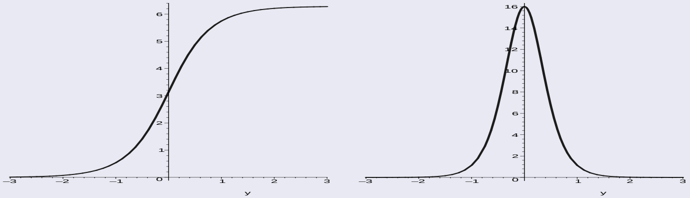
$$\phi(y) = \pm 4\arctan e^{2\alpha y}\,.$$

- It is energy density is localized. It is given by

$$\epsilon = \frac{1}{2}(\partial_y \phi)^2 + 4\alpha^2(1 - \cos\phi) = \frac{16\alpha^2}{\cosh^2 2\alpha y}$$

- The kink and its energy density have the profiles

Profiles



- Its classical mass is $M_{kink} = \int dy \epsilon = 16\alpha$.
- Kink has topological charge $Q = 1$. It is disconnected from the vacuum sector with $Q = 0$.

Some well-known properties of the sine-Gordon model are

1. It is super-renormalizable. It is in fact integrable at the quantum level: Its S-matrix completely factorizes into two-particle S-matrices and obey Yang-Baxter equation. No particle production occurs!!!

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3. It is equivalent to a fermionic theory, namely the massive Thirring model.

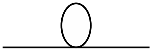
- To explore indications of the model at quantum level, a simple analysis is to compute the corrections to M_{kink} by semi-classical means.

Super-Renormalizable: It is sufficient to normal order the interactions to cancel all the divergences.

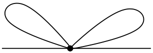
$$:4\alpha^2(\cos\phi - 1) := 4(\alpha^2 - \delta\alpha^2)(\cos\phi - 1)$$

We can observe this quickly from the Feynman graphs. We have $\cos\phi - 1 = -\frac{1}{2}\phi^2 + \frac{1}{4!}\phi^4 - \frac{1}{6!}\phi^6 + \cdots$.

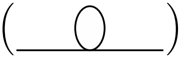
- All divergent contributions come from the self-contractions of the vertices.



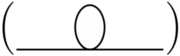
$$\approx \log\frac{4\alpha^2}{\Lambda^2}$$

- 

$$= \Big(\ \Big)^2 \approx \left(\log\frac{4\alpha^2}{\Lambda^2}\right)^2$$



(n-loop) $= \Big(\ \Big)^n \approx \left(\log\frac{4\alpha^2}{\Lambda^2}\right)^n$



Quantum Corrections to the Kink Mass

- This is done by finding the normal modes of the fluctuations around the kink solution. If ω_n are the frequencies of these modes, this implies

$$E_{kink-sector} = 16\alpha + \hbar \sum_n (k_n + \frac{1}{2})\omega_n + \nu_n + O(\alpha^2)$$

- $k_n = 0$ for the quantum kink particle, $k_n \neq 0$ for the scattering states of mesons in the presence of the kink particle.
- To find M_{kink} at this approximation, one subtracts E_{vacuum} and regularizes the remaining divergences by renormalizing α^2. This gives

$$M_{kink} = 16\alpha - \frac{2}{\pi}\alpha + O(\alpha^2)$$

Noncommutative Spacetime: Moyal Algebra and $\star$-product

- Let $\mathcal{A}$ be the algebra of functions over $\mathbb{R}^d$. We multiply $f, g \in \mathcal{A}$ w.r.t. the pointwise multiplication map $\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$:

$$\mu(f(x) \otimes g(x)) \equiv (f \cdot g)(x).$$

- Flat noncommutative spacetime is the associative algebra $\mathcal{A}_\theta(\mathbb{R}^d)$(Moyal Algebra) obtained by replacing μ with $\mu_\star$:

$$\mu_\star(f(x) \otimes g(x)) \equiv (f \star g)(x) \, .$$

- $\star$-product is given by the formula

$$(f \star g)(x) = f(x) e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}} g(x)$$

- The coordinate functions x_μ generate $\mathcal{A}_\theta(\mathbb{R}^d)$ and they fulfil the commutation relation

$$x_\mu \star x_\nu - x_\nu \star x_\mu =: [x_\mu\,, x_\nu]_\star = i\theta_{\mu\nu}\,.$$

- $\theta_{\mu\nu}$ is a real antisymmetric tensor of rank 2, with constant components.
- In $\mathcal{A}_\theta(\mathbb{R}^{1+1})$ we will sometimes use the light-cone coordinates:

$$u = \frac{1}{2}(t+y)\,, \quad v = \frac{1}{2}(t-y)\,, \quad \partial_u = (\partial_t+\partial_y)\,, \quad \partial_v = (\partial_t-\partial_y)\,.$$

- They fulfil

$$[v\,, u]_\star = i\theta\,.$$

Now, A First Guess

We would like to have a NC sine-Gordon theory

Properties

- Classically Integrable: There is a linear system of equations, whose compatibility conditions implies a noncommutative version of sine-Gordon field equations.
- To have the correct commutative limit.
- To possess kink, anti-kink solutions.
- Causal S-matrix at tree-level.

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Further Properties?

- Semi-Classical behavior: Spectrum of quadratic fluctuations around the vacuum and kink solutions.
- Quantum corrections to the mass of the kink. Regularization of divergences.
- SUSY extensions and their properties.

1. A well-known result is that $SU(2)$ self-dual Yang-Mills (SDYM) theory on $R^{(2,2)}$ can be reduced to sine-Gordon model on $R^{(1,1)}$. Note that $R^{(2,2)}$ has signature $(-++-)$.
2. So consider the self-dual $U(2)$ SDYM on $\mathcal{A}_\theta(\mathbb{R}^{(2,2)})$. (We follow Lechtenfeld et. al. *Nucl.Phys.B* **705**(2005))

$$F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu\,, A_\nu]_\star$$

- First we will reduce to a theory in $\mathcal{A}_\theta(\mathbb{R}^{(2,1)})$.
- We take $A_4 = \Lambda$ and demand translational invariance of A_μ along x_4-direction: i.e. $\partial_4(A_a\,, A_4) = 0$. This gives:

$$\partial_a\Lambda + [A_a\,, \Lambda]_\star = \frac{1}{2}\varepsilon_{abc}F^{bc}\,.$$

Gauge Fixing

We fix the gauge by

- $$\begin{aligned} A_t - A_y &= 0\,, \quad A_t + A_y = \Phi^{-1} \star (\partial_t + \partial_y)\Phi\,, \\ A_x + \Lambda &= 0\,, \quad A_x + \Lambda = \Phi^{-1} \star \partial_x \Phi\,. \end{aligned}$$
- Here Φ is a $U(2)$ valued field.

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- Introducing the light-cone coordinates

$$u = \frac{1}{2}(t+y)\,, \quad v = \frac{1}{2}(t-y)\,, \quad \partial_u = (\partial_t + \partial_y)\,, \quad \partial_v = (\partial_t - \partial_y)\,.$$

- We find

$$\partial_x(\Phi^{-1} \star \partial_x \Phi) - \partial_v(\Phi^{-1} \star \partial_u \Phi) = 0\,.$$

Linear System

- Consider the system of equations

$$(\zeta\partial_x - \partial_u)\Psi = \Phi^{-1} \star \partial_u \Phi \star \Psi \,, \quad (\zeta\partial_v - \partial_x)\Psi = \Phi^{-1} \star \partial_x \Phi \star \Psi$$

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Compatibility Condition

- Compatibility condition for this linear system is

$$\partial_x(\Phi^{-1} \star \partial_x\Phi) - \partial_v(\Phi^{-1} \star \partial_u\Phi) = 0\,.$$

- This is in fact the NC version of integrable Ward model.
- It is reminiscent to NC version of nonlinear sigma model and it does have multi-soliton solutions.

In two steps: First we factorize x-dependence, then restrict the form of the $U(2)$ matrices.

- We take $[t\,,x]_\star = 0\,, \quad [x\,,y]_\star = 0\,, \quad [t\,,y]_\star = i\theta$.
- Take the ansatz

$$\Phi(t,x,y) = V(x)\,g(t,y)\,V^\dagger(x)\,,\,V(x) = e^{i\alpha x\sigma_1}\,,g(t\,,y) \in U(2)\,.$$

- Linear system becomes

$$\begin{aligned} \partial_u\Psi - i\alpha\zeta[\sigma_1\,,\Psi] &= -V^{-1}g^{-1}\star\partial_u gV\star\Psi\,, \\ \zeta\partial_v\Psi - i\alpha[\sigma_1\,,\Psi] &= V^{-1}g^{-1}\star\partial_x gV\star\Psi\,. \end{aligned}$$

- We have some freedom to pick $g(t,y) \in U(2)$. We choose

$$g = \begin{pmatrix} g_+ & 0 \\ 0 & g_- \end{pmatrix} \quad \in U(1)\otimes U(1) \subset U(2)$$

Compatibility condition implies the equations

NC sine-Gordon equations

$$\begin{aligned}
\partial_v(g_+^{-1} \star \partial_u g_+) + \alpha^2(g_-^{-1} \star g_+ - g_+^{-1} \star g_-) &= 0 \\
\partial_v(g_-^{-1} \star \partial_u g_-) + \alpha^2(g_+^{-1} \star g_- - g_-^{-1} \star g_+) &= 0
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\end{aligned}$$

- It is possible to parameterize $g_\pm$ by

$$g_+ = e_\star^{-i\phi_+}\,, \quad g_- = e_\star^{i\phi_-}$$

- Taking $\theta \to 0$ and using $\varphi := \phi_+ + \phi_-$ and $\rho := \phi_+ - \phi_-$, leads to

$$\partial_u \partial_v \varphi = -4\alpha^2 sin\varphi\,, \quad \partial_u \partial_v \rho = 0\,.$$

- Thus we propose the equations above as the field equations for the NC sine-Gordon model.

If α was 0, we would have had

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- The action should be consisting of WZW action for g_+ and g_-, plus an interaction term:

Action

$$S[g_+, g_-] = S_{WZW}[g_+] + S_{WZW}[g_-] + \alpha^2 \int dtdy\,(g_+^\dagger \star g_- + g_-^\dagger \star g_+ - 2)\,.$$

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$$S_{WZW}[f] = -\frac{1}{2}\int dtdy\,\partial_\mu f^{-1} \star \partial^\mu f - \frac{1}{3}\int dtdy \int_0^1 d\lambda\,\varepsilon^{\mu\nu\sigma}\,\hat{f}^{-1}\partial_\mu \hat{f} \star \hat{f}^{-1}\partial_\nu \hat{f} \star \hat{f}^{-1}\partial_\sigma \hat{f}\,.$$

The model has the standard static kink, anti-kink solutions.

Kink, Anti-Kink

- $$\varphi_0 = \pm 4 \arctan e^{2\alpha y}\,, \quad \rho_0 = 0\,, \quad g_0 = e^{-\frac{i}{2}\varphi_0}\,, \quad .$$
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- Multi-soliton configurations can be constructed using the linear system via the "dressing" method.
- We will study the quadratic fluctuations around this solution. Invoking the semi-classical reasoning, the energy spectrum for the kink particle should be given by

$$E_{kink-sector} = 16\alpha + \frac{1}{2}\sum_n (\omega_n + \nu_n) + O(\alpha^2)$$

where ω_n and ν_n are the frequencies for the normal modes.

We split the fields g_+ , g_- by setting

$$g_+ = g_0 e^{-i(\eta+\xi)}\,, \qquad g_- = e^{i(\eta-\xi)} g_0^{-1}\,,$$

- $\eta\,,\xi$ are fluctuations in the static background g_0.
- We expand $S[g_+, g_-]$ up to cubic order in η and ξ.

$$S[g_+, g_-] = S[g_0] - \int dtdy\, (\partial_\mu \eta)^2 + (\partial_\mu \xi)^2 + \text{interaction terms}$$

1. First, we find the field equations for η and ξ and expand them to second order in θ.
2. Next, we expand the fluctuations in modes by assuming

$$\eta(t, y) = \sum_n e^{i\omega_n t} \psi_n(y)\,, \quad \xi(t, y) = \sum_n e^{i\nu_n t} \chi_n(y)\,.$$

Eigenmodes fulfill the Schrödinger-type equations:

Equations

$$\big[-\partial_z^2 + V_0(z) + \theta V_1(z) + \theta^2 V_2(z)\big]\tilde{\psi}_n(z) = \frac{\omega_n^2}{4\alpha^2}\tilde{\psi}_n(z)\,,$$

$$\big[-\partial_z^2 + \theta W_1(z) + \theta^2 W_2(z)\big]\tilde{\chi}_n(z) = \frac{\nu_n^2}{4\alpha^2}\tilde{\chi}_n(z)\,.$$

With $z := 2\alpha y$, $\tilde{\psi}_n := e^{\frac{i}{4}\omega_n\theta\partial_y\varphi_0}\psi_n\,,\quad \tilde{\chi}_n := e^{\frac{i}{4}\nu_n\theta\partial_y\varphi_0}\chi_n$ and,

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Potentials

$$V_0 = (2\tanh^2 z - 1)\,,\; V_1 = -\omega_n^2\frac{\sinh z}{\cosh^2 z}$$

$$V_2 = -\omega_n^2\alpha^2\Big(\frac{2}{\cosh^4 z} - \frac{\sinh^2 z}{\cosh^4 z}\Big)$$

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Potentials

$$W_1(z) = -\nu_n^2\frac{\sinh z}{\cosh^2 z}$$

$$W_2(z) = \nu_n^2\alpha^2\frac{\sinh^2 z}{\cosh^4 z}$$

Consider first θ=0

- We have the equation

$$\left[-\partial_z^2 + 2\tanh^2 z - 1\right]\psi_n(z) = \frac{\omega_n^2}{4\alpha^2}\psi_n(z)\,,$$

- The solution consists of the discrete zero-mode

$$\psi_0(z) = \partial_z\varphi_0 = -\frac{2}{\cosh z}\,, \quad \omega_0 = 0\,,$$

followed by the continuum states

$$\psi_q(z) = e^{iqz}(\tanh z - iq)\,, \quad {}_0\omega_q^2 = 4\alpha^2(q^2+1)\,, \quad q \geq 0\,.$$

- $\psi_q(z)$ can be normalized, by putting the system in a box of length L.

We consider θ-dependent potentials as perturbations

1. $\psi_0(z) = -\frac{2}{\cosh z}$ is static, and remains a zero-mode to all order in θ.
2. For corrections to the spectrum of ω_n^2 we can write

$$\omega_n^2 - {}_0\omega_n^2 =: \Sigma_k \theta^k \left(\Delta_n^k(V_1) + \Delta_n^k(V_2) \right)$$

3. At order θ:
   - We observe that

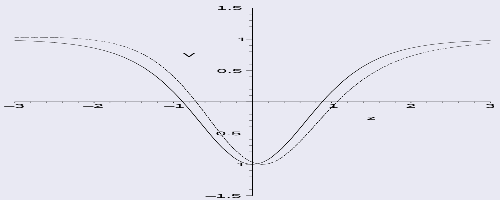
$$V_1 = -\omega_n^2 \frac{\sinh z}{\cosh^2 z}, \quad W_1 = -\nu_n^2 \frac{\sinh z}{\cosh^2 z}$$

odd under parity. So $\Delta_n^1(V_1)$ and $\Delta_n^1(W_1)$ both vanish.

At order θ^2:

- We find that $\Delta_n^1(V_2) \approx \frac{1}{L}$ and $\Delta_n^1(W_2) \approx \frac{1}{L}$, thus they too vanish as $L \to \infty$.
- It seems not possible to compute $\Delta_n^2(V_1)$ and $\Delta_n^2(W_1)$ analytically, but it is unlikely that they change the spectrum considerably.

Potential Profiles



In the vacuum sector we have

- $$g_0 = e^{-\frac{i}{2}\varphi_0} = 1\,, \quad \varphi_0 = 0\,, \quad \rho_0 = 0\,,$$
- Fluctuation equations are
$$-\partial_\mu \partial^\mu \eta - 4\alpha^2 \eta = 0\,, \quad -\partial_\mu \partial^\mu \xi = 0\,.$$
- Thus η and ξ are plane waves:
$$\eta(t, y) = e^{\pm iky + i\omega t}\,, \quad \xi(t, y) = e^{\pm iry + i\nu t}\,,$$
- The dispersion relations
$$\omega^2 = k^2 + 4\alpha^2\,, \quad \nu^2 = r^2.$$
- These are in agreement with the ordinary sine-Gordon theory results.

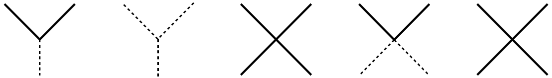
- Our perturbative treatment of noncommutativity for the spectrum of fluctuations around the kink implies no changes in the spectrum of fluctuations.
- Thus $E_{kink} - E_{vacuum}$ is in agreement with the results of the ordinary sine-Gordon model.

We move on to discuss the properties of the two-point functions of the model.

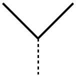
- The propagators are

$$\equiv \langle \varphi\varphi \rangle = \frac{2}{k^2 + 4\alpha^2}\,, \qquad \equiv \langle \rho\rho \rangle = \frac{2}{k^2}$$

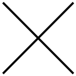
- For our purposes we only need the interactions to quadratic order in the fields ψ and ρ. The vertices are then



Feynman rules for these vertices read

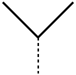
- 

$$= -\frac{1}{2^2}(k_1 \wedge k_2)\sin\left(\theta\frac{k_1 \wedge k_2}{2}\right)e^{-\frac{i}{2}\theta(k_1\wedge k_2 + k_2\wedge k_3)}$$

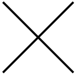
- 

$$= \frac{1}{12}\alpha^2 e^{\left(-\frac{i}{2}\theta\sum_{i<j}^{n}k_i\wedge k_j\right)} - \frac{i}{2^2\cdot 4!}k_1\cdot(k_3 - k_2)$$
$$\times \sin\left(\theta\frac{k_2\wedge k_3}{2}\right)e^{-\frac{i}{2}\theta(k_1\wedge k_2 + k_1\wedge k_3 + k_1\wedge k_4 + k_2\wedge k_4 + k_3\wedge k_4)}$$

Feynman rules for these vertices read

- 

$$= -\frac{1}{2^2}(k_1 \wedge k_2) \sin\left(\theta \frac{k_1 \wedge k_2}{2}\right) e^{-\frac{i}{2}\theta(k_1 \wedge k_2 + k_2 \wedge k_3)}$$

- 

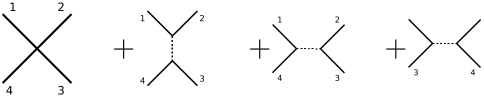
$$= \frac{1}{12}\alpha^2 e^{(-\frac{i}{2}\theta \sum_{i<j}^{n} k_i \wedge k_j)} - \frac{i}{2^2 \cdot 4!} k_1 \cdot (k_3 - k_2)$$
$$\times \sin\left(\theta \frac{k_2 \wedge k_3}{2}\right) e^{-\frac{i}{2}\theta(k_1 \wedge k_2 + k_1 \wedge k_3 + k_1 \wedge k_4 + k_2 \wedge k_4 + k_3 \wedge k_4)}$$

- $a \wedge b = a_t b_y - a_y b_t$

Scattering Amplitudes

It was shown by Lechtenfeld et. al.*Nucl.Phys.B*705(2005)) that this model do not exhibit any acausal behaviour at tree level.

- $A_{\varphi\varphi\to\varphi\varphi} =$



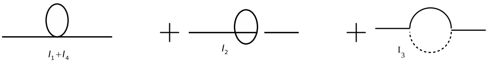
$= 2i\alpha^2$

- All other amplitudes, $A_{\rho\rho\to\rho\rho}$, $A_{\varphi\rho\to\varphi\rho}$, $A_{\varphi\varphi\to\rho\rho}$ and $A_{\rho\rho\to\varphi\varphi}$ vanish.
- Thus the model has no acausal effects.
- Amplitudes for $\varphi\varphi \to \varphi\varphi\varphi\varphi$ and $\varphi\varphi\varphi \to \varphi\varphi\varphi$ also vanish. This is in agreement with the commutative sine-Gordon model.

One-loop two-point functions in vacuum sector...

- Two-point function for φ is $I_\varphi(P^2)$

$I_\varphi(P^2) =$



- Non-planar diagram I_2 leads to UV/IR mixing. We observe this from

$$I_2 = \frac{-\alpha^2}{6\pi} \log\left[\alpha^2\theta^2 P^2 + \frac{4\alpha^2}{\Lambda^2}\right] + \text{subleading terms}\,,$$

- I_3 and I_4 vanish as $\theta \to 0$. There is no UV/IR mixing due to I_1, I_3 and I_4.

Renormalization; mass and field strength counter terms

1. For $P \neq 0$, $\theta \neq 0$, the leading terms for $I_\varphi(P^2)$ reads

$$I_\varphi(P) \approx \left[\frac{-\alpha^2}{3\pi} + \frac{P^2}{2^6\pi}\right] \log \frac{4\alpha^2}{\Lambda^2} + \text{finite terms} + \text{subleading terms}$$

2. For $P = 0$, $I_\varphi(P^2)$ is the same as that of the ordinary sine-Gordon model, thus only mass renormalization is sufficient to render the theory finite.

Renormalization in the Euclidean Signature.

- We can write renormalized self-energy as:

$$\Sigma_R(P^2) = (1+\delta Z_\varphi)^{-1} I_\varphi(P^2) + \delta m_\varphi^2 + \delta Z_\varphi P^2 \,,$$

and assume the renormalization conditions

$$\Sigma_R(P^2)\Big|_{P^2=P_0^2} = 0\,, \qquad \frac{d}{dP^2}\Sigma_R(P^2)\Big|_{P^2=P_0^2} = 0$$

- From these considerations we find for δm_φ^2 and δZ_φ:

$$\delta m_\varphi^2 = \frac{1}{1+\delta Z_\varphi}\left[\frac{\alpha^2}{3\pi}\log\frac{4\alpha^2}{\Lambda^2}\right]\,, \quad \delta Z_\varphi = \frac{-1+\sqrt{1-\frac{1}{2^4\pi}\log\frac{4\alpha^2}{\Lambda^2}}}{2}\,.$$

For ρ, the one-loop two-point function is

$$I_\rho(P^2) = \left(\frac{1}{2} I_3 + I_4\right)\Big|_{4\alpha^2 \to \mu^2}$$

- μ is a small mass for ρ introduced to regularize the IR limit.
- $I_\rho(P^2)$ is present purely due to the noncommutativity: $I_\rho(P^2) \to 0$ as $\theta \to 0$. However, it does not lead to any UV/IR mixing.
- There is no mass renormalization and the field-strength renormalization is given by

$$\delta Z_\rho = \frac{-1 + \sqrt{1 - \frac{3}{2^5 \pi} \log \frac{\mu^2}{\Lambda^2}}}{2}$$

Remarks

- **Remark1:** We stress that, these results are valid for $\theta \neq 0$. When $\theta \to 0$, in $I_\varphi(P^2)$ and $I_\rho(P^2)$, the divergent terms in Λ cancel with those in θ. In this case, the standard answer for the commutative sine-Gordon model is recovered, and a mass counter term for the field φ is sufficient to renormalize the theory.
- **Remark2:** When $I(P^2)$ are analytically continued to the Minkowski space, the logarithms develop branch cuts. This leads to imaginary parts in the total one-loop amplitudes, and for space-like external momenta to the violation of unitarity, as the optical theorem is no longer satisfied.

1. We have studied the quantum aspects of sine-Gordon model in noncommutative spacetime. Our aim has been to infer to what extent the classical integrability is useful in this respect.
   - We have presented a perturbative treatment of noncommutativity to study the spectrum of fluctuations around the kink. This implied the latter is in good agreement with the ordinary sine-Gordon model.
2. We have found that two-point functions at one-loop level show some interesting features.
   - There is UV/IR mixing due to interactions coupled with α^2, but it appears that there are non-planar diagrams which do not lead to UV/IR mixing effects.
   - We have exhibited the mass and field strength renormalizations in Euclidean signature. However, in Minkowski signature time-space noncommutatvity still causes unitarity violation.
   - Although, the usual vacuum subtraction can be performed it is not clear, how to regularize the divergences of the theory in Minkowski space.

3. It maybe be helpful to study the quantum effects in the $2+1$-dimensional Ward-model to gain more insights to the structure of the present class of models.
4. It will certainly be useful to study the SUSY generalizations of this model and see if it helps in regularizing the divergences of the bosonic theory. Investigations in this direction are already underway.

A First Guess.
A NC sine-Gordon action and why it is not useful.

Consider the action obtained by deforming all products to $\star$-products in the commutative theory.

$$S = \int dtdy \frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + 2\alpha^2(\cos_\star \phi - 1)$$

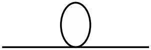
$\cos_\star \phi = 1 - \frac{1}{2}\phi \star \phi + \cdots$. The field equation becomes

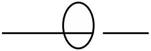
$$\partial_\mu \partial^\mu \phi = -4\alpha^2 sin_\star \phi \, .$$

This action is no good, for

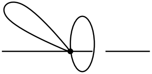
- The associated currents are deformed in the same way, but they are no longer conserved!.

- Normal ordering of the interaction is not sufficient to cancel the divergences any more.
- Consider, for example, the two-point functions. They come in three different kinds: planar, non-planar and mixed.

planar :  $$\approx \log \frac{4\alpha^2}{\Lambda^2}$$

non-planar :  $$\approx \log \left[\alpha^2 \theta^2 P^2 + \frac{4\alpha^2}{\Lambda^2} \right]$$

- But a mixed diagram has sub-diagram(s) which are planar

 $$\approx \log \left[\frac{4\alpha^2}{\Lambda^2} \right] \log \left[\alpha^2 \theta^2 P^2 + \frac{4\alpha^2}{\Lambda^2} \right]$$

- Thus, some diagrams get coefficients depending on the external momenta P, and it is not possible to sum the counter terms to get a $\cos_\star \phi$ interaction.

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