Symmetry Preserving Discretization of Differential Equations with Applications to Numerical Solutions

Pavel Winternitz
Centre de recherches mathématiques
Université de Montréal
CP 6128, Succ. Centre Ville
Montréal, QC, Canada

References:

1. A. Bourlioux, C. Cyr-Gagnon, P.W., J. Phys. A. 39 (22), 6877–6896 (2006)
2. D. Levi, P.W., J. Phys A. 39 (2), R1–R63 (2006)
3. A. Bourlioux, R. Verge-Rebelo, P.W., Submitted, ArXives
4. R. Campoamor-Stursberg, M.A. Rodriguez, P.W. (in preparation).

Origin of Lie group and Lie algebra theory: groups of transformations of independent and dependent variables leaving the solution set of a system of ODEs, or PDEs, invariant.

The problem treated in this lecture: Given an ODE

$$E = y^{(n)} - F(x, y, y', \dots, y^{(n-1)}) = 0$$

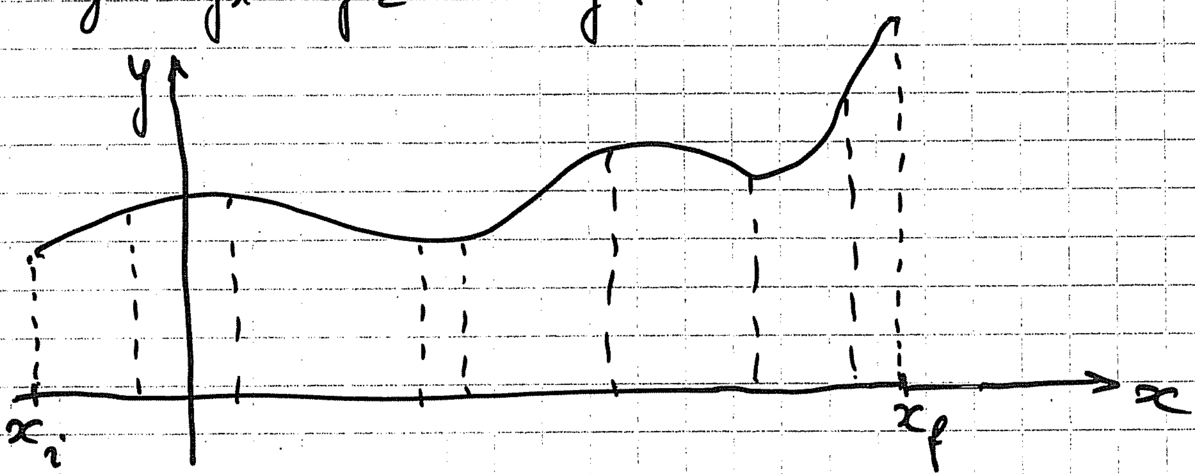
We wish to solve the ODE :

1. Analytically

$$y(x) = y(x, C_1, C_2, \ldots, C_n)$$

2. Numerically :

x	x_1	x_2	$\ldots$	x_n
y	y_1	y_2		y_n



E.g. : Solve an initial value problem
$y, y', \ldots, y^{(n-1)}$ given for $x = x_0$

How to obtain analytical solution of nonlinear ODE?

1. Group theory: S. Lie, ...
2. Singularity analysis, S. Kovalevskaya, P. Painlevé, ...

group theory: Find Lie point symmetry group of local point transformations

$$\tilde{x} = \Lambda_\lambda(x,y), \quad \tilde{y} = \Omega_\lambda(x,y)$$

such that

$$y(x) = \text{solution} \Rightarrow \tilde{y}(\tilde{x}) = \text{solution}$$

Basic tool: Lie algebra of vector fields:

$$X = \xi(x,y)\frac{\partial}{\partial x} + \phi(x,y)\frac{\partial}{\partial y}$$

X known $\Rightarrow$

$$\frac{d\tilde{x}}{d\lambda} = \xi(\tilde{x},\tilde{y}) \qquad \tilde{x}(\lambda=0) = x$$

$$\frac{d\tilde{y}}{d\lambda} = \phi(x,\tilde{y}) \qquad \tilde{y}(\lambda=0) = y$$

$$\Rightarrow \tilde{x} = \Lambda_\lambda(x,y) \qquad \tilde{y} = \Omega_\lambda(x,y)$$

X acts on functions of x and y

$pr^{(n)}X$ acts on f-ns of $x, y, \dot{y}, \dots y^{(n)}$, e.g. $E=0$

$$\mathrm{pr}\, X = \xi(x,y)\partial_x + \phi(x,y)\partial_y + \phi^x(x,y,y')\partial_{y'} + \phi^{xx}(y,y,y',y'')\partial_{y''} + \cdots$$

$$\phi^x = D_x\phi - (D_x\xi)y'$$

$$\phi^{xx} = D_x\phi^x - (D_x\xi)y''$$

$$\vdots$$

$$\phi^{nx} = D_x\phi^{(n-1)x} - (D_x\xi)y^{(n)}$$

How to find symmetry algebra ?

$$\mathrm{pr}\, X\, E\Big|_{E=0} = 0$$

$\Rightarrow$ determining equations for $\xi(x,y)$ and $\phi(x,y)$: overdetermined system of linear PDEs

Once found : What to do with symmetry group ? Answer : lower the order of the ODE.

Possibly: to order zero $\Leftrightarrow$ "solve"

Possible problems

1. Equation of order n $\Rightarrow$ we need a symmetry algebra of order

$$N \geq n$$

($N = n$ if algebra is solvable).

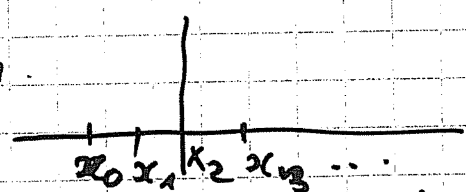
2. Even if algebra is large enough, the solution may be implicit

Symmetry group of an ODE: determines many properties of the solutions

How to obtain numerical solutions.

"Standard method" : discretize

$$y' \to \frac{y_{n+1}-y_n}{x_{n+1}-x_n} = \frac{y_{n+1}-y_n}{h}$$



$x_n = x_0 + n h$ .. lattice

$$y'' \to \frac{y_{n+1}-2y_n+y_{n-1}}{h^2} \qquad \Delta = \frac{T-1}{h} \qquad Tf(x) = f(x+h)$$

$$y''' \to \frac{y_{n+2}-3y_{n+1}+3y_n-y_{n-1}}{h^3} \qquad y^{(n)} \to \left(\frac{T-1}{h}\right)^n y$$

$$\frac{y_{n+2}-3y_{n+1}+3y_n-y_{n-1}}{h^3} = F\left(x_n, y_n, \frac{y_{n+1}-y_n}{h}, \frac{y_{n+2}-2y_{n+1}+y_n}{h^2}\right)$$

"Universal": for any equation. Convergence?

Give y_{n-1}, y_n, y_{n+1}, calculate $y_{n+2}, \dots$

This way: we loose ~~all~~ most symmetry properties, all consequences of Lorentz invariance, Galilei invariance, etc.

Idea: Discretize while preserving point symmetries: V. Dorodnitsyn, V. Dorodnitsyn R. Kozlov, P. Winternitz, M. A. Rodriguez + P. W., D. Levi, ...

Discrete derivatives: Invariant under $x \to x + x_0$

$y \to y + y_0$

ODE $\to$ OΔS (ordinary difference scheme)

$$\boxed{\begin{aligned} &E_1(n, x_{n+K}, x_{n+K+1}, \dots, x_{n+L}, y_{n+K}, \dots, y_{n+L}) = 0 \\ &E_2(n, x_{n+K}, x_{n+K+1}, \dots, x_{n+L}, y_{n+K}, \dots, y_{n+L}) = 0 \end{aligned}}$$

$$\frac{\partial(E_1, E_2)}{\partial(x_{n+L}, y_{n+L})} \neq 0$$

$L - K = N - 1$ = order of scheme

N = # of points

$N \geq n+1$

Continuous limit: $x_{n+K+1} - x_{n+K} \to 0$

$E_1 \to$ ODE

$E_2 \to$ identity $(0=0)$

Construct E_1 and E_2 out of group invariants.

ODE given $\Rightarrow$ symmetry algebra L and symmetry group G known.

Calculate "difference invariants"

$$X = \xi(x,y)\frac{\partial}{\partial x} + \phi(x,y)\frac{\partial}{\partial y}$$

$$\mathrm{pr}\, X = \sum_{j=K}^{L} \left\{ \xi(x_{n+j}, y_{n+j})\frac{\partial}{\partial x_{n+j}} + \phi(x_{n+j}, y_{n+j})\frac{\partial}{\partial y_{n+j}} \right\}$$

$$\mathrm{pr}\, X_a\, I(x_{n+j}, y_{n+j}) = 0$$

$a = 1, \dots, M$

$M = \dim L = \dim G$

$L \leq j \leq M$

Invariant manifolds:

$$\mathrm{rank}\begin{pmatrix} \xi_{1,n+k}, \dots, \xi_{1,n+L}, \phi_{1,n+k}, \dots, \phi_{1,n+L} \\ \vdots \\ \xi_{M,n+k}, \dots, \xi_{M,n+L}, \phi_{1,n+k}, \dots, \phi_{1,n+L} \end{pmatrix} < \text{maximal rank}$$

<u>Advantages</u> of invariant discretization.

1) First order ODEs (M.A. Rodriguez, P.W. J. Phys. A. 2004)
"<u>Exact</u> discretization" (exactly the same solutions)

2) Second order ODEs with dim $L \geq 3$ (V. Dorodnitsyn, R. Kozlov, P.W., J. Math. Phys. 2000 and 2004). Discrete systems solved <u>analytically</u>; they converge to exact solutions like ε^2.

3. This talk: 1. Compare <u>accuracy</u> of <u>standard</u> and <u>symmetry preserving</u> schemes. 2nd and 3rd order ODEs
<u>Result</u>: accuracy improved by $10^1 - 10^3$ (1 to 3 orders of magnitude)

2. Compare qualitative behaviour for solutions with singularities at points close to a singularity

Algorithm for constructing invariant difference scheme.

1. Start from ODE

$$F(x, y, y', \ldots, y^{(N)}) = 0.$$

Find symmetry algebra L (and group G)

$$X = \xi(x,y)\partial_x + \phi(x,y)\partial_y$$

2. Construct n-th order differential prolongation

$$pr^N X = \xi\partial_x + \phi\partial_y + \phi^x\partial_{y'} + \phi^{xx}\partial_{y''} + \cdots + \phi^{Nx}\partial_{y^{(N)}}$$

and construct Nth order differential invariants : $I^c_\alpha(x, y, y', \ldots y^{(N)})$, $\alpha = 1, \ldots, J$

Rewrite ODE in terms of invariants

$$\tilde{F}(I^c_1, I^c_2, \ldots, I^c_J) = 0 \quad \ldots \text{ODE}$$

We have : $pr^N X \, I^c_\alpha = 0$

3. Construct N+1 point difference prolongation of X

$$pr_\Delta X = \sum_{i=n-K}^{n+L} [\xi(x_i, y_i)\frac{\partial}{\partial x_i} + \phi(x_i, y_i)\frac{\partial}{\partial y_i}]$$

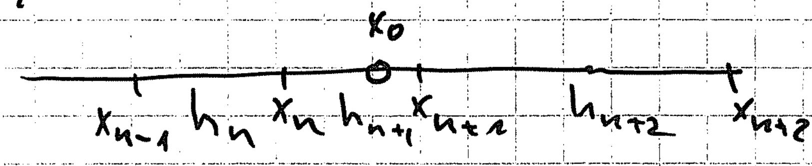
$$L + K = N$$

Find a basis of <u>difference invariants</u>

$$pr_{\Delta} X\, F(x_{n-k}, y_{n-k}, x_{n-k+1}, y_{n-k+1}, \dots, x_{n+L}, y_{n+L}) = 0$$

$$\{I_1, I_2, \dots, I_M\}$$

4. Expand the difference invariants in Taylor series about some point x_0



Choose such a basis that

$$I_j = I_j^c + O(\varepsilon)$$

5. Form <u>invariant difference scheme</u>

$$E_1(I_1, \dots, I_M) = 0 \xrightarrow{\varepsilon \to 0} \tilde{F}(I_1^c, \dots I_J^c) = 0$$

$$E_2(I_1, \dots, I_M) = 0 \xrightarrow{\varepsilon \to 0} 0 = 0$$

$$\det \left\{ \frac{\partial(E_1, E_2)}{\partial(x_{n+M}, y_{n+M})} \right\} \neq 0$$

Example 1.

$$x^2 y'''' + 4xy' + 2y = (2xy + x^2 y')^{\frac{k-2}{k-1}}$$

$$k \neq 0, \tfrac{1}{2}, 1, 2$$

Symmetry algebra

$$X_1 = \frac{\partial}{\partial x} - \frac{2y}{x}\frac{\partial}{\partial y}, \quad X_2 = \frac{1}{x^2}\frac{\partial}{\partial y}$$

$$X_3 = x\frac{\partial}{\partial x} + (k-2)y\frac{\partial}{\partial y}$$

$$[X_1, X_2] = 0, \quad \begin{pmatrix} [X_1, X_3] \\ [X_2, X_3] \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

Difference invariants:

$$X_a\, I(x_{n-1}, x_n, x_{n+1}, y_{n-1}, y_n, y_{n+1}) = 0 \quad a = 1,2,3$$

$$\left.\begin{array}{l} \text{6 points} \\ \dim L = 3 \end{array}\right\} \Rightarrow \text{3 invariants}$$

$$I_1 = \frac{x_{n+1} - x_n}{x_n - x_{n-1}} \qquad I_2 = \frac{x_{n+1}^2 y_{n+1} - x_n^2 y_n}{(x_{n+1} - x_n)^k}$$

$$I_3 = \frac{x_n^2 y_n - x_{n-1}^2 y_{n-1}}{(x_n - x_{n-1})^k}$$

Put: $h_{n+1} = x_{n+1} - x_n = \alpha\varepsilon \qquad \alpha \sim \beta \sim 1$

$\qquad h_n = x_n - x_{n-1} = \beta\varepsilon$

Discrete derivatives: not invariant

(No ∂_x, nor ∂_y in algebra)

Expand $y_{n\pm 1} \equiv y(x_{n\pm 1})$ into Taylor series about x_n ; $y(x_n) \equiv y$; $x_{n\pm 1} = x_n \pm h$

$$y_{n\pm 1} = y \pm h y' + \frac{h^2}{2} y'' \pm \frac{h^3}{3!} y''' + \frac{h^4}{4!} y^{IV} \pm \dots$$

Form new invariants out of I_1, I_2, I_3 to approximate the ODE

$$\frac{2 I_1}{I_1+1}\left(I_2 - \frac{1}{(I_1)^{k-1}} I_3\right) = (h_{n+1})^{2-k} \left\{ (x^2 y'' + 4 x y' + 2y) + \frac{1}{3}(h_{n+1} - h_n)(x^2 y''' + 6 x y'' + 6 y') + O(\varepsilon^2) \right\}$$

$$\frac{1}{2}\left[I_2 + \frac{1}{(I_1)^{k-1}} I_3\right]^{\frac{k-2}{k-1}} = (h_{n+1})^{2-k} (x^2 y' + 2xy)^{\frac{k-2}{k-1}} \cdot \left\{ 1 + \frac{k-2}{k-1}(h_{n+1} - h_n) \frac{x^2 y'' + 4 x y' + 2y}{x^2 y' + 2xy} + O(\varepsilon^2) \right\}$$

$\Rightarrow$ Invariant $O\Delta S$

$$\boxed{\begin{gathered} \frac{2 I_1}{I_1+1}\left(I_2 - \frac{1}{(I_1)^{k-1}} I_3\right) = \frac{1}{2}\left[I_2 + \frac{1}{(I_1)^{k-1}} I_3\right]^{\frac{k-2}{k-1}} \\ I_1 = k \ (= \text{const}) \end{gathered}}$$

For $k=1$: second order approximation on a uniform lattice

$$\frac{x_{n+1} - x_n}{x_n - x_{n-1}} = 1 \Rightarrow x_n = x_0 + hn$$

$k \neq 1 \Rightarrow$ exponential lattice

Comment: In this case the ODE can be solved exactly:

$$y(x) = \left(\frac{1}{k-1}\right)\frac{1}{kx^2}(x-x_0) + \frac{y_0}{x} \qquad k \neq 0,1$$

Numerical comparison:

$$y_{ref}(x) \equiv y_{Exact}(x)$$

Compare: $|y_{stand}(x_n) - y_{ref}(x_n)|$

$|y_{inv}(x_n) - y_{ref}(x_n)|$

In this case: both on uniform lattice

15

8. Figures

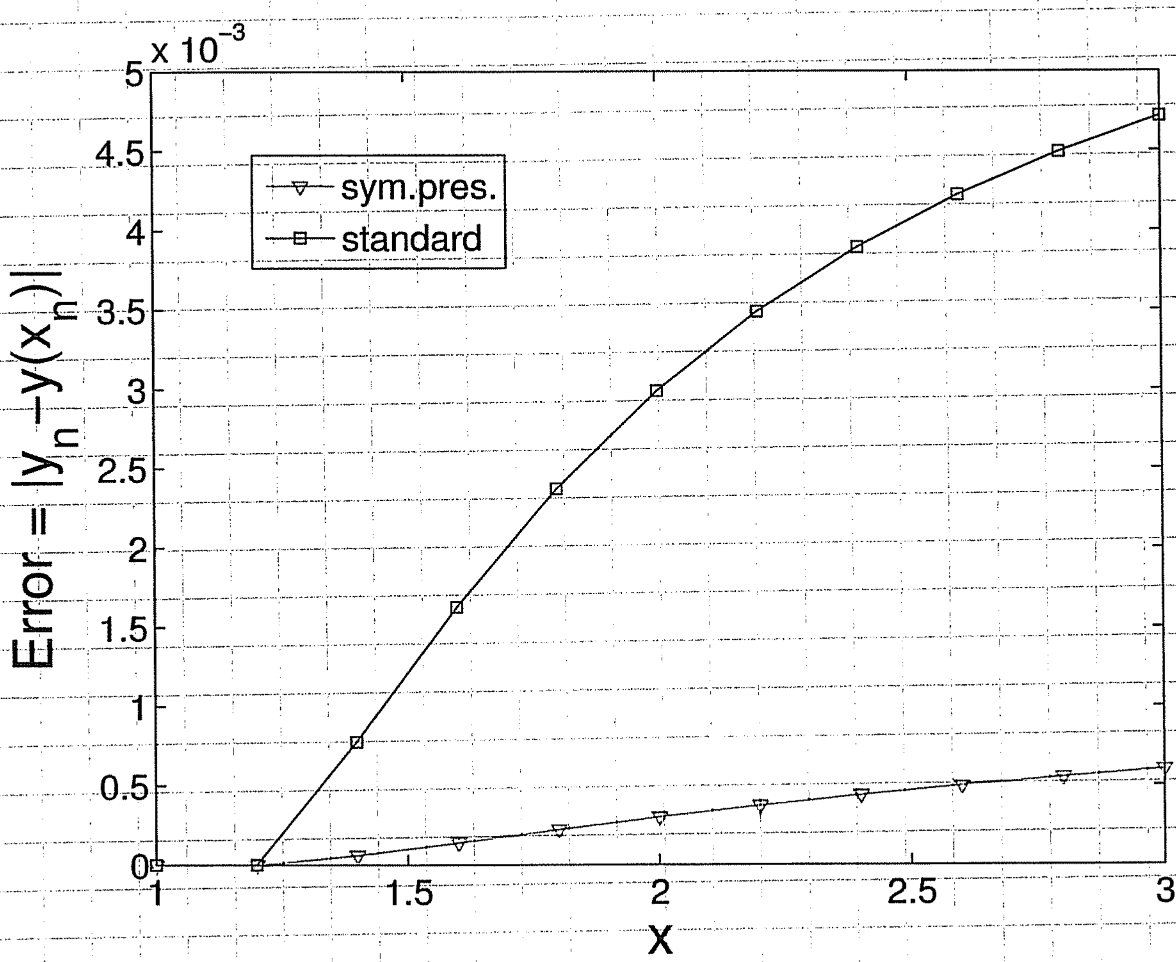


Figure 1. Discretization errors for the symmetry preserving scheme and the standard scheme, Example 1.

$x^2 y'' + 4x y' + 2y = (2xy + x^2 y')^{(k-2)/(k-1)}$ $\quad k = \underline{\underline{3}}$

$h = 0.1$

Uniform lattice also in invariant case

Both schemes second order : $h \to 10^{-1} h$

$|y - y_{ref}| \to 10^{-2} \; |y - y_{ref}|$

Comment: The ODE can be simplified and solved exactly

$$y(x) = \left(\frac{1}{k-1}\right)^{k-1} \frac{1}{kx^2}(x-x_0) + \frac{y_0}{x}$$

We want to test the OΔS.

Other examples: <u>third order ODE's</u>

Classification: O. Gat, J. Math. Phys. 1992

– " – of third order OΔS:

C. Cyr-Gagnon (M.Sc)

(Both incomplete)

<u>Example 2</u>

$$X_1 = \partial_x \qquad X_2 = \partial_y , \quad X_3 = y\partial_x - x\partial_y , \quad X_4 = x\partial_x + y\partial_y$$

Differential invariants:

$$\mathrm{pr}X_a \, F(x, y, y', y'', y''') = 0 \qquad a = 1, \dots 4$$

$\{X_1, X_2, X_3\} \Rightarrow$

$$I_1 = \frac{y''}{(1+y'^2)^{3/2}} \qquad I_2 = \frac{(1+y'^2)y''' - 3y'y''^2}{(1+y'^2)^3}$$

$$I_2 = F(I_1)$$

$\{X_1, X_2, X_3, X_4\} \Rightarrow \quad F(I_1) = I_1^2$

$$\boxed{(1+y'^2)\,y''' - 3y'y''^2 = K y''^2}$$

Analytic solution

implicit.

$$y(x) = \int_0^x u(t)\,dt + C_3$$

$$\frac{dt}{dx} = C_1 \int_0^u \frac{e^{-\kappa \arctan s}}{(1+s^2)^{3/2}}\, ds + C_2$$

Not too useful for getting numbers and graphs.

We need: $u = u(t)$

Difference invariants:

$$X_a F(x_{n-1}, x_n, x_{n+1}, x_{n+2}, y_{n-1}, y_n, y_{n+1}, y_{n+2}) = 0$$

$$a = 1, \dots 4$$

8 variables
$\dim G = \dim L = 4$ } $\Rightarrow$ 4 invariants

$X_1, X_2, X_3, 4 \Rightarrow$

$$\xi_1 = h_{n+2}\left[1 + \left(\frac{y_{n+2} - y_{n+1}}{h_{n+2}}\right)^2\right]^{1/2}$$

$$\xi_2 = h_{n+1}\left[1 + \left(\frac{y_{n+1} - y_n}{h_{n+1}}\right)^2\right]^{1/2}$$

$$\xi_3 = h_n\left[1 + \left(\frac{y_n - y_{n-1}}{h_n}\right)^2\right]^{1/2}$$

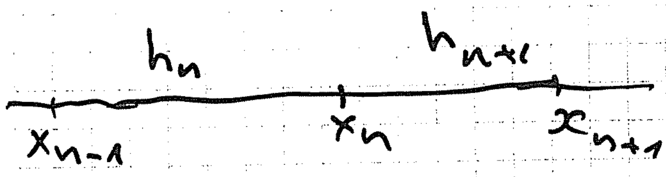
$$\xi_4 = (y_{n+2} - y_{n+1})h_{n+1} - (y_{n+1} - y_n)h_{n+2}$$

$$\xi_5 = (y_{n+1} - y_n)h_n - (y_n - y_{n-1})h_{n+1}$$

Add $X_4 \Rightarrow$ 4 ratios

Expand in Taylor series and form new invariants:

3 points

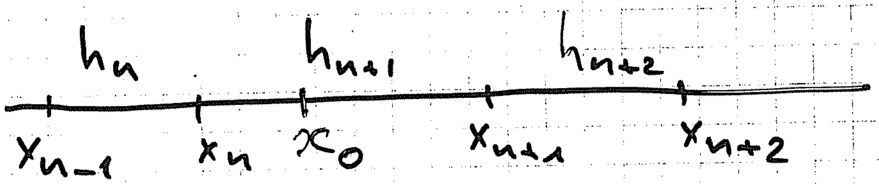


We expanded about $x \equiv x_n$

$x_{n+1} = x_n + h_{n+1}$
$x_{n-1} = x_n - h_n$

Uniform lattice $\Rightarrow$ x_n in the middle

4 points



We expanded about x_n
Maybe some x_0 would be better
In this case: uniform lattice not among invariant ones

$$\boxed{J_2 = \frac{6}{\xi_1+\xi_2+\xi_3}\left(\frac{\xi_4}{\xi_1\xi_2(\xi_1+\xi_2)} - \frac{\xi_5}{\xi_2\xi_3(\xi_2+\xi_2)}\right)}$$

$$= \frac{1}{(1+y'^2)^3}\Big\{\left[(1+y'^2)\,y''' - 3y'y''^2\right]$$

$$+ \frac{h_{n+2}+2h_{n+1}-h_n}{4}\left[(1+y'^2)\,y^{IV} - 10y'y''y''' + 15y'^2y''^3\right]$$

$$+ O(h_n^2)\Big\}$$

$$\boxed{J_1 = \frac{2\alpha\,\xi_4}{\xi_1\xi_2(\xi_1+\xi_2)} + \frac{2\beta\,\xi_5}{\xi_2\xi_3(\xi_2+\xi_3)}} \qquad \alpha+\beta=1$$

$$= \frac{1}{(1+y'^2)^{3/2}}\Big\{y'' + \frac{1}{3(1+y'^2)}\left[(1+y'^2)\,y''' - 3y'y''^2\right]$$

$$\times\left[\alpha(h_{n+2}+2h_{n+1}) + \beta(h_{n+1}-h_n)\right]\Big\}$$

$E(2) \Rightarrow J_2 = F(J_1)$

$Sim(2) \Rightarrow \boxed{J_2 = K\,J_1^2}$, $\boxed{A\xi_1 + B\xi_2 + C\xi_3 = 0}$ $\Big\}$ Invariant OΔS

In general: First order

$K = \frac{\sqrt{3}}{2}$ $\quad \alpha=\beta=\frac{1}{2}$, $C=-A$, $B=2A$

$\Rightarrow$ Order 2 $\quad (\varepsilon^2)$

Alternative : $\frac{\xi_1}{\xi_2} = \frac{\xi_2}{\xi_3}$ ... lattice

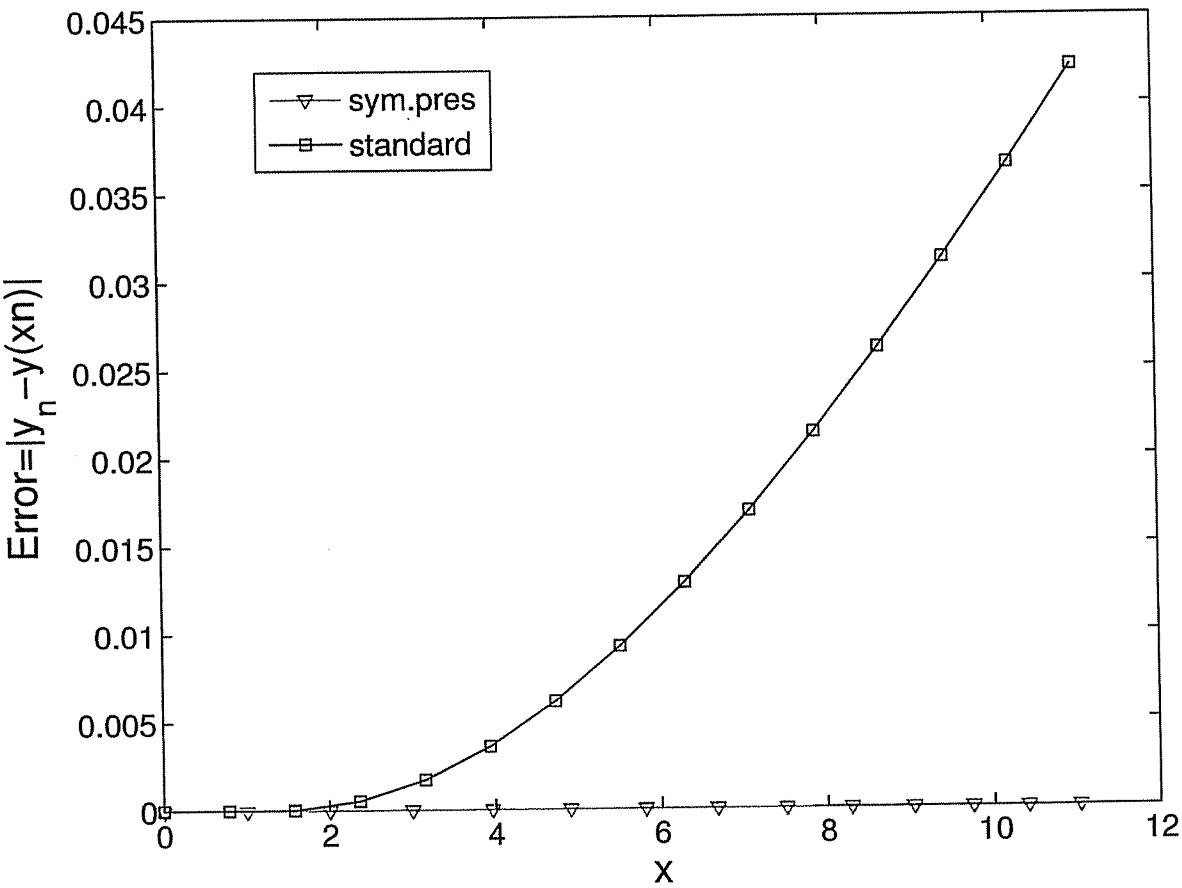


Figure 2. Discretization errors for the symmetry preserving scheme and the standard scheme, Example 2, $h = 1$.

$$(1+y'^2)y''' - 3y'y''^2 = K|y''|^2 \qquad K = 1$$

Mesh: $\dfrac{\xi_1}{\xi_2} = \dfrac{\xi_2}{\xi_3}$

Example 3

$X_1 = \partial_y \; , \; X_2 = x\partial_y \; , \; X_3 = (1+x^2)\partial_x + xy\partial_y \; , \; X_4 = y\partial_y$

Sim(2) group: Isomorphic to previous one; not conjugate

Differential invariants:

$X_1, X_2, X_3 \Rightarrow$
$$\boxed{\begin{aligned} I_1 &= (1+x^2)^{3/2} y'' \\ I_2 &= \left[(1+x^2)y''' + 3xy''\right](1+x^2)^{3/2} \end{aligned}}$$

Invariant ODE:
$$\boxed{I_2 = F(I_1)}$$

Add X_4: $\quad I_2 = k I_1 \Rightarrow$ linear

Difference invariants:

$$\xi_1 = (1+x_n^2)^{1/2}\left[\frac{y_{n+1}-y_n}{x_{n+1}-x_n} - \frac{y_n - y_{n-1}}{x_n - x_{n-1}}\right]$$

$$\xi_2 = (1+x_{n+1}^2)^{1/2}\left[\frac{y_{n+2}-y_{n+1}}{x_{n+2}-x_{n+1}} - \frac{y_{n+1}-y_n}{x_{n+1}-x_n}\right]$$

$$\xi_3 = \frac{x_n - x_{n-1}}{1 + x_n x_{n-1}}$$

$$\xi_4 = \frac{x_{n+1} - x_n}{1 + x_n x_{n+1}}$$

$$\xi_5 = \frac{x_{n+2} - x_{n+1}}{1 + x_{n+1} x_{n+2}}$$

$$J_1 = \frac{2\alpha\,\xi_1}{\xi_3+\xi_4} + \frac{2\beta\,\xi_2}{\xi_4+\xi_5} = (1+x^2)^{3/2}\,y'' + \mathcal{O}(\varepsilon)$$

$$J_2 = \frac{6}{\xi_3+\xi_4+\xi_5}\left(\frac{\xi_2}{\xi_4+\xi_5} - \frac{\xi_1}{\xi_3+\xi_4}\right)$$

$$= (1+x^2)^{3/2}\left[(1+x^2)\,y''' + 3x\,y''\right] + \mathcal{O}(\varepsilon)$$

OΔS

$$\boxed{J_2 = F(J_1) \qquad A\xi_1 + B\xi_2 + C\xi_3 = 0}$$

Exemple 4

$\sim SL(2,\mathbb{R})$: $X_1 = \partial_y$, $X_2 = x\partial_x + y\partial_y$, $X_3 = 2xy\partial_x + y^2\partial_y$

Differential invariants:

$$\boxed{I_1 = \frac{2x\,y'' + y'}{(y')^3} \qquad I_2 = \frac{x^2\,(y'\,y''' - 3y''^2)}{y'^3} \qquad I_2 = F(I_1)}$$

$GL(2,\mathbb{R})$: add $X_4 = x\partial_x$ $\Rightarrow$ $F(z) = A\,z^{3/2}$

$$\boxed{x^2\,(y'\,y''' - 3(y'')^2) = A\,(y')^{1/2}\,(2x\,y'' + y')^{3/2}}$$

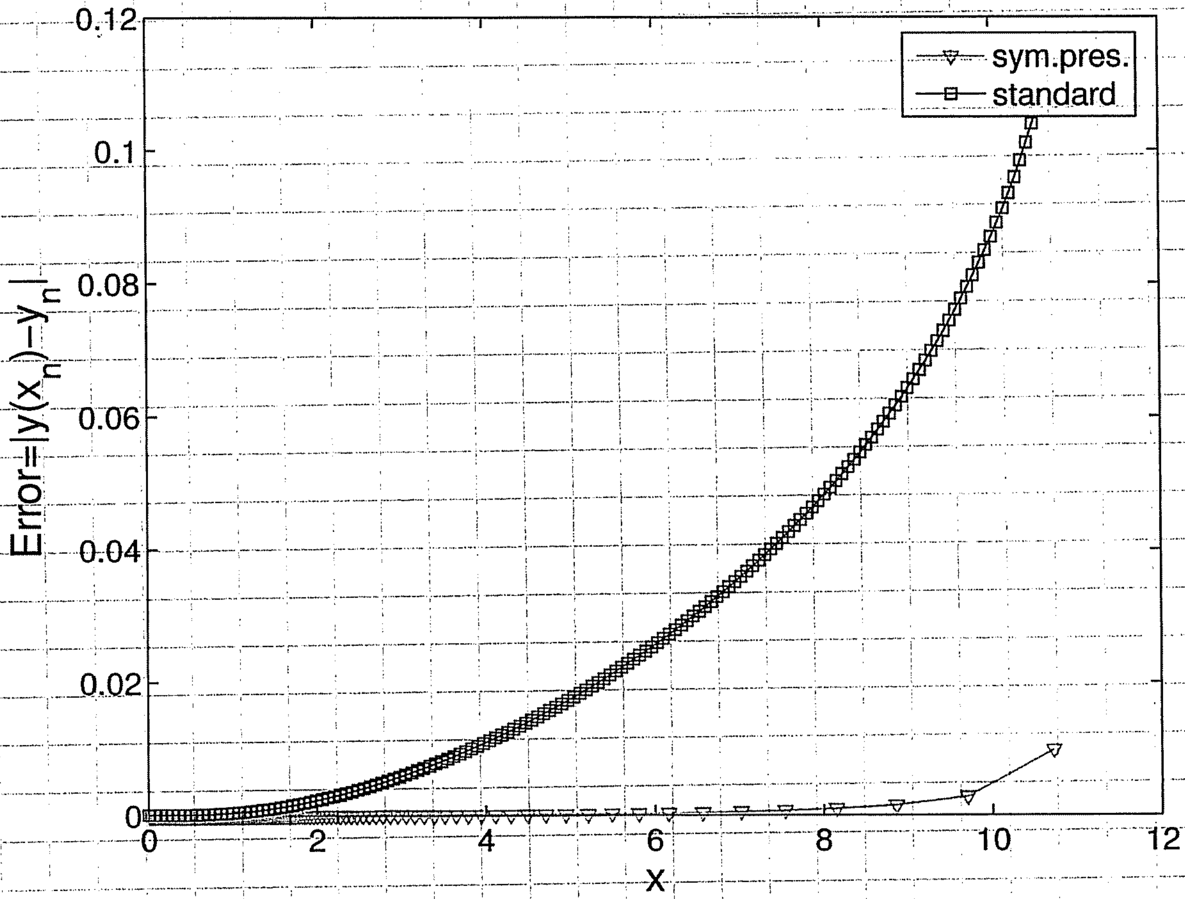


Figure 3. Discretization errors for the symmetry preserving scheme and the standard scheme, Example 3, for the case with blow-up.

$$(1+x^2)y''' + 3x y'' = (y'')^2 (1+x^2)^{3/2} \qquad F(I_1) = I_1^2$$

$$\frac{\xi_3}{\xi_4} = \frac{\xi_4}{\xi_5} \qquad h_0 = 0.01$$

Both schemes are first order ones

Invariant one: errors smaller by $10^{-1} - 10^{-2}$

Examples: with and without blow-up.

On Fig 3 : blow-up at $x_B = 11.25$

Fig 4 : No blow up

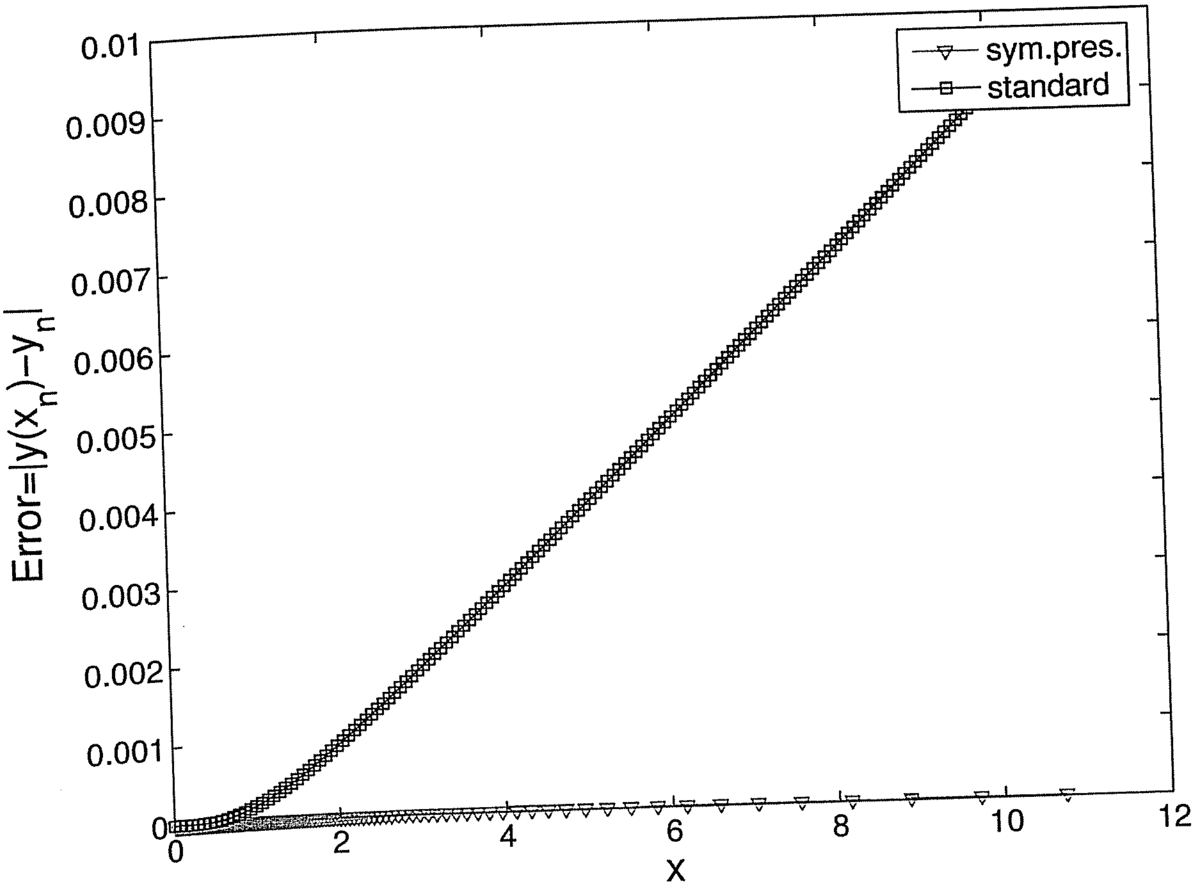


Figure 4. Discretization errors for the symmetry preserving scheme and the standard scheme, Example 3, for the case without blow-up.

Invariance under $SL(2,\mathbb{R})$

4 inequivalent realizations of $sl(2,\mathbb{R})$ exist
Here: We consider two of them

1) $\{X_1 = \partial_y,\ X_2 = x\partial_x + y\partial_y,\ X_3 = 2xy\partial_x + y^2\partial_y\}$
... $sl(2,\mathbb{R})$
Add $X_4 = x\partial_x \quad \Rightarrow \quad gl(2,\mathbb{R})$

Differential invariants of $SL(2,\mathbb{R})$

$$I_1 = \frac{2xy'' + y'}{(y')^3} \qquad I_2 = \frac{x^2(y'y''' - 3y''^2)}{y'^5}$$

Invariant ODEs:

Second order

$$I_1 = \gamma \qquad \boxed{2xy'' + y' = \gamma y'^3}$$

Solution: $y_{1,2} = \begin{cases} y_0 \pm \frac{2}{c}\sqrt{c - \gamma x} & c \neq 0 \\ y_0 \pm \frac{1}{\sqrt{\gamma}}\, x & c = 0 \end{cases}$

Third order

$$I_2 = F(I_1)$$

Invariant under $GL(2,\mathbb{R})$: $F(z) = \alpha z^{3/2}$

$$\boxed{x^2(y'y''' - 3y''^2) = \alpha(2xy'' + y')^{3/2}\, y'^{1/2}}$$

Exact solution: implicit

Difference invariants

$$\underset{x_{n-1}}{|}\overset{h_n}{\longrightarrow}\underset{x_n}{|}\overset{h_{n+1}}{\longrightarrow}\underset{x_{n+1}}{|}\overset{h_{n+2}}{\longrightarrow}\underset{x_{n+2}}{|}$$

$$I_1^n = \frac{y_n - y_{n-1}}{\sqrt{x_n x_{n-1}}} \qquad I_1^{n+1} = \frac{y_{n+1} - y_n}{\sqrt{x_{n+1} x_n}}$$

$$I_2^{n+1} = \frac{y_{n+1} - y_{n-1}}{\sqrt{x_{n+1} x_{n-1}}}$$

3 points x_{n-1}, x_n, x_{n+1}

$$I_1^{n+2} = \frac{y_{n+2} - y_{n+1}}{\sqrt{x_{n+2} x_{n+1}}} \qquad I_2^{n+2} = \frac{y_{n+2} - y_n}{\sqrt{x_{n+2} x_n}}$$

3 point x_n, x_{n+1}, x_{n+2}

~~invariant~~

Difference scheme for second order ODE.

$$J_1^{n+1} = \gamma \qquad I_1^{n+1} = I_1^n \equiv I_1 = \text{const}$$ (depends on initial conditions)

$$J_1^{n+1} = 8\,\frac{I_2^{n+1} - (I_1^n + I_1^{n+1})}{I_1^n I_1^{n+1}(I_1^n + I_1^{n+1})} = \frac{2xy'' + y'}{y'^3} +$$

$$+ 2(h_{n+1} - h_n)\,x\,\frac{-3y''^2 + y'y'''}{3y'^4} + O(h^2)$$

$$\left.\begin{aligned} I_1^{n+1} &= I_1^n \\ h_i &= \alpha_i \varepsilon \end{aligned}\right\} \Rightarrow h_{n+1} - h_n \sim \varepsilon^2$$

Rewrite scheme: solve for x_{n+1}, y_{n+1}

$$x_{n+1} = x_{n-1}\,\frac{y_n - y_{n-1}}{\beta x_{n-1} - (y_n - y_{n-1})}$$

$\beta = I_1\left(\frac{\gamma}{4} I_1^2 + 2\right) =$ constant

$$y_{n+1} = \frac{\beta x_{n-1} y_n - (y_n - y_{n-1})\, y_{n-1}}{\beta x_{n-1} - (y_n - y_{n-1})}$$

Properties of scheme

1. Explicit and linear : $x_{n+1} = \cdots$
$y_{n+1} = \cdots$

2. Order ε^2 for equation ($\not\Rightarrow$ order ε^2 for solution)

Standard scheme : Either implicit (3rd order algebraic equation for y_{n+1}) or first order in ε

<u>Difference scheme for third order ODE</u>

$$K^{n+2} = F(J_1^{n+1}) \qquad I_1^{n+2} = I_1^{n+1} = I_1^n \equiv I_1$$

$$K^{n+2} = \frac{3}{2}\left(\frac{J_1^{n+2} - J_1^{n+1}}{I_1^n + I_1^{n+1} + I_1^{n+2}}\right) = \frac{x^2}{y^5}(y'y''' - 3y''^2)$$
$$+ (h_{n+2} - h_n) R(x, y, y', y'', y''', y^{(4)}) + O(\varepsilon^2)$$

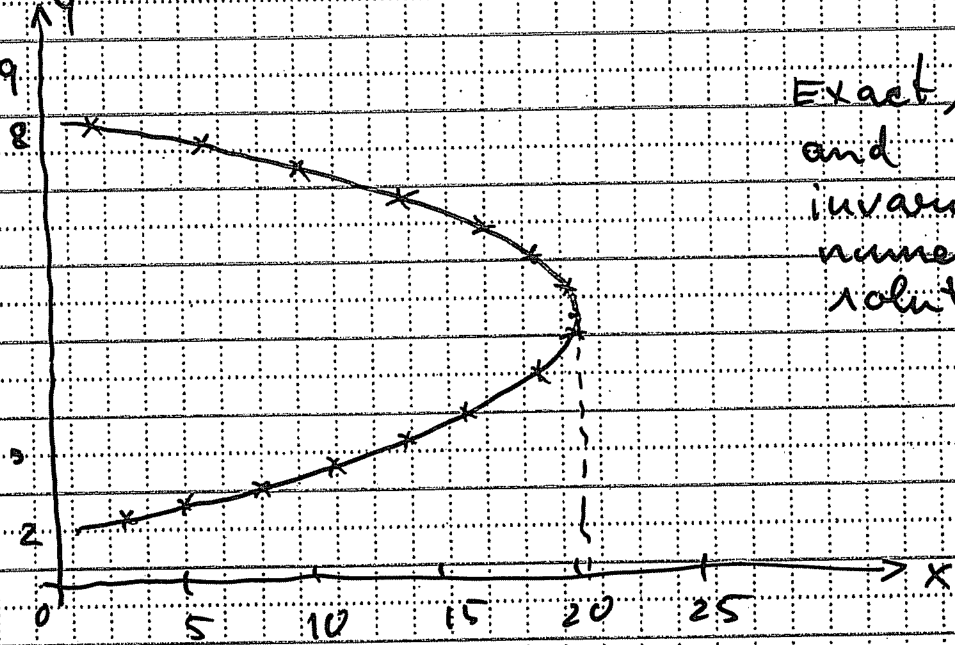
Again a linear explicit second order scheme

$$x_{n+2} = \frac{x_n}{(1-\omega_{n+1})^2} \qquad y_{n+2} = \frac{y_n - \omega_{n+1} y_{n+1}}{1-\omega_n}$$

$$\omega_{n+1} = \frac{I_1^2}{4}[2 I_1 F(J_1^{n+1}) + J_1^{n+1}] \sqrt{\frac{x_n}{x_{n+1}}}$$

$$J_1^{n+1} = \frac{4}{\sqrt{x_{n+1}x_{n-1}}\, I_1^3}\left[y_{n+1} - y_{n-1} - 2 I_1 \sqrt{x_{n+1}x_{n-1}}\right]$$

Second order ODE



Exact solutions and invariant numerical solution

x_s – singularity

$$y_{1,2} = y_0 \pm \frac{2}{c}\sqrt{c - \gamma x} \qquad x_s = \frac{c}{\gamma} \qquad y_1 = y_2$$

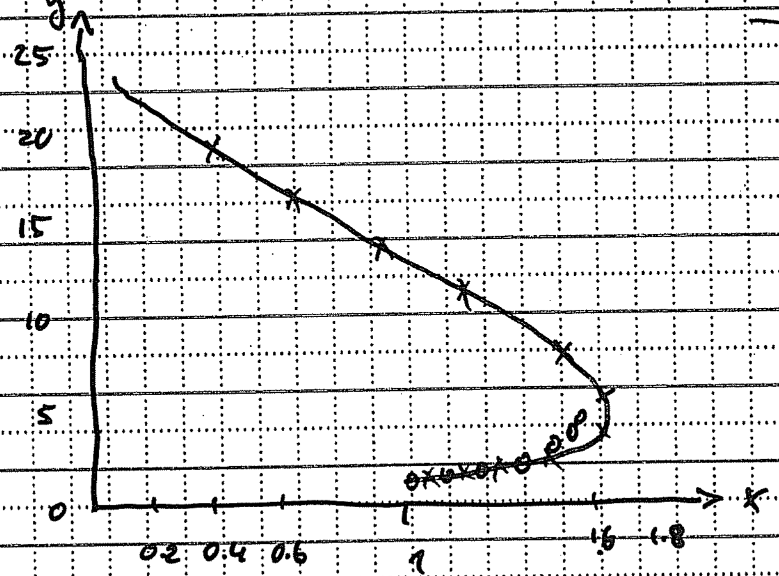
Standard numerical solution: stops before singularity

Singularity: $y_s = y_0$ finite

$$y'(x_s) \to \infty$$

Third order ODE

$GL(2,R)$ invariant



— reference solution

* invariant method

o standard method

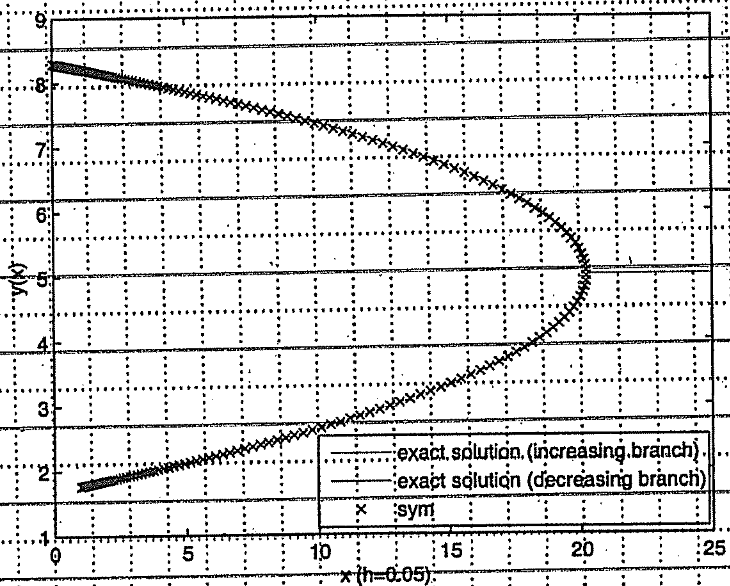


Figure 1. Behaviour of the symmetry preserving scheme near the singularity for eq. (2.4)

28c

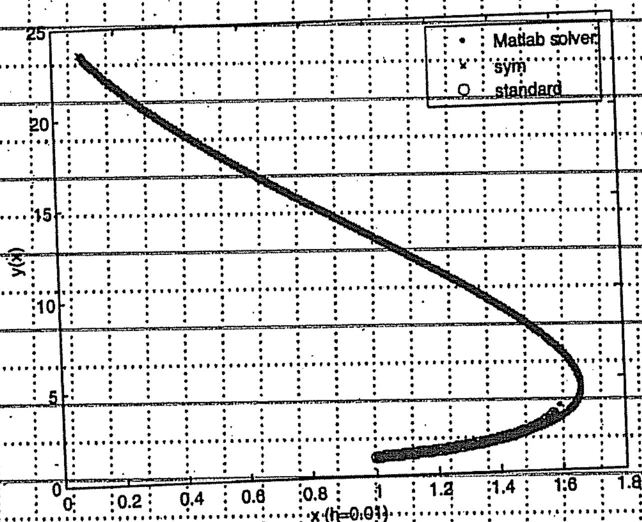
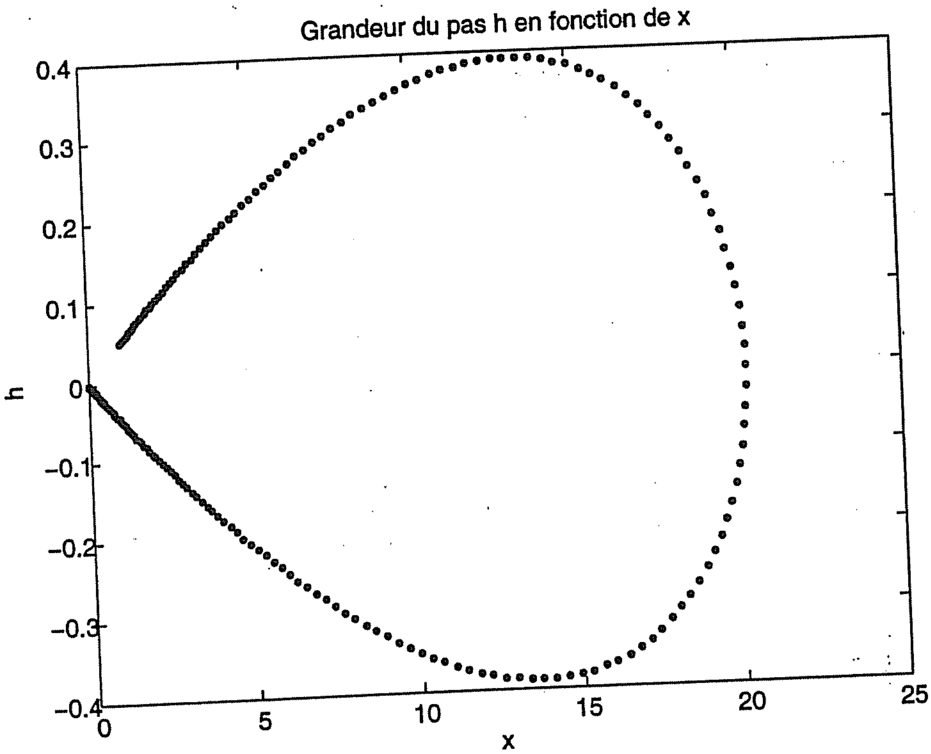


Figure 3. Behaviour of the symmetry preserving scheme near a singularity for eq. (2.6)



$$h_{n+1} = x_{n+1} - x_n = \frac{(y_n - y_{n-1})(x_{n-1} + x_n) - \beta x_n x_{n-1}}{\beta x_{n-1} - (y_n - y_{n-1})}$$

$h_{n+1} > 0 \qquad x < x_{sing}$

$h_{n+1} < 0 \qquad x > x_{sing}$

$2 x y'' + y' = \gamma y'^3 \quad , \quad sl(2, \mathbb{R})$

29a

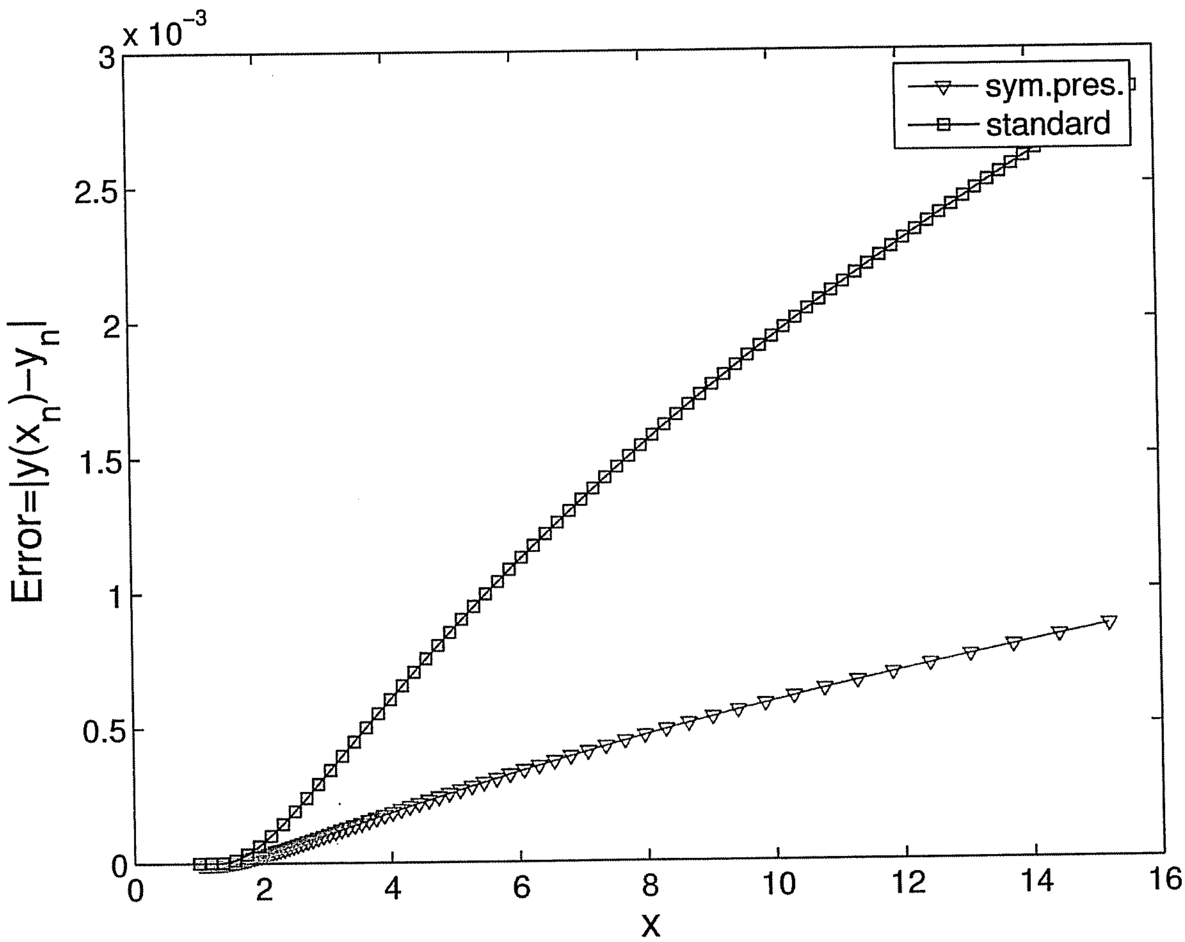


Figure 5. Discretization errors for the symmetry preserving scheme and the standard scheme, Example 4.

$$x^2(y' y''' - 3y''^2) = A\, y'^{1/2}(2xy'' + y')^{3/2} \qquad A = -1$$

$$\frac{\xi_1}{\xi_2} = \gamma = \sqrt{\frac{x_n x_{n+1}}{x_{n+1} x_{n+2}}}\ \frac{y_{n+2} - y_{n+1}}{y_{n+1} - y_n}$$

Mesh depends on solution!

Both schemes: second order

Invariance under $SL(2,\mathbb{R})$

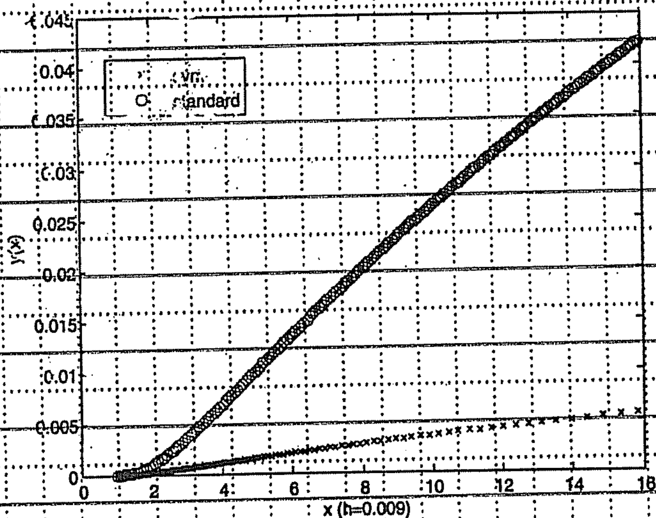


Figure 2. Discretization errors for standard and symmetry preserving schemes for eq. (2.6), $\alpha = -1$ for a regular solution

Exemple 5: Another $sl(2,\mathbb{R})$ algebra.

$$X_1 = \frac{\partial}{\partial y} \qquad X_2 = y\frac{\partial}{\partial y} \qquad X_3 = y^2\frac{\partial}{\partial y}$$

Can be embedded into $sl_y(2,\mathbb{R}) \oplus sl_x(2,\mathbb{R})$

$$X_4 = \frac{\partial}{\partial x} \qquad X_5 = x\partial_x \qquad X_6 = x^2\partial_x$$

$SL_y(2,\mathbb{R})$: Two differential invariants in $\{x, y, y', y'', y'''\}$.

$$I_1 = \frac{1}{y'^2}\left(y'y''' - \frac{3}{2}y''^2\right), \quad I_2 = x$$

I_1 = Schwarzian derivative

Invariant ODE:

$$\boxed{\frac{1}{y'^2}\left(y'y''' - \frac{3}{2}y''^2\right) = F(x)}$$

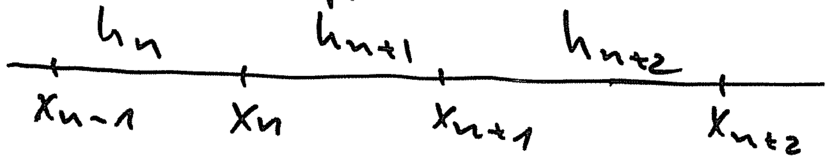
Add X_4 $\Rightarrow$ $\frac{1}{y'^2}\left(y'y''' - \frac{3}{2}y''^2\right) = K$

Add also X_5 $\Rightarrow$ $K = 0$

Then: invariant under $SL(2,\mathbb{R})_y \times SL(2,\mathbb{R})_x$

$$y'y''' - \frac{3}{2}y''^2 = 0$$

$SL_y(2,\mathbb{R})$ difference invariants



$$R = \frac{(y_{n+2}-y_n)(y_{n+1}-y_{n-1})}{(y_{n+2}-y_{n+1})(y_n-y_{n-1})}, \quad x,\ h_{n+2},\ h_{n+1},\ h_n$$

Put

$$J = \frac{6 h_{n+2} h_n}{h_{n+1}(h_{n+1}+h_{n+2})(h_n+h_{n+1})(h_n+h_{n+1}+h_{n+2})} \left[\frac{(h_{n+1}+h_{n+2})(h_n+h_{n+1})}{h_n h_{n+1}} - R \right].$$

$$J = \frac{1}{y'^2}\left[y' y''' - \frac{3}{2} y''^2 \right] + O(\varepsilon)$$

Invariant scheme approximating the Schwarzian equation $I_1 = F(x)$

$$J_1 = F(x_n, h_n, h_{n+1}, h_{n+2})$$

$$\phi(x_n, h_n, h_{n+1}, h_{n+2}) = 0$$

$$F(x_n, 0, 0, 0) = F(x)$$

$$\phi(x_n, 0, 0, 0) = 0$$

Add $X_4 = \partial_x \quad \Rightarrow \quad F(x) = K$

$$\Phi(\cancel{x_n}, h_n, h_{n+1}, h_{n+2})$$

OΔS invariant under $SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R})$

$$\frac{(x_{n+2}-x_n)(x_{n+1}-x_{n-1})}{(x_{n+2}-x_{n+1})(x_n-x_{n-1})} = \frac{(y_{n+2}-y_n)(y_{n+1}-y_{n-1})}{(y_{n+2}-y_{n+1})(y_n-y_{n-1})} = (\quad (*)$$

$$\frac{(x_{n+2}-x_n)(x_{n+1}-x_{n-1})}{(x_{n+2}-x_{n+1})(x_n-x_{n-1})} = K_0 \qquad (**)$$

For $K_0 = 4$: Exact scheme:

$$\dot{y}\,\dddot{y} - \frac{3}{2}\ddot{y}^2 = 0$$

Two families of solutions:

$$y = \frac{1}{ax+b} + c \qquad y = \alpha x + \beta \qquad (***)$$

Solve $(**)$: for $K_0 = 4$

$$x_n = \frac{1}{an+b} + c \qquad x_n = \alpha n + \beta$$

On these lattices: $(***)$ solves $(*)$ exactly!

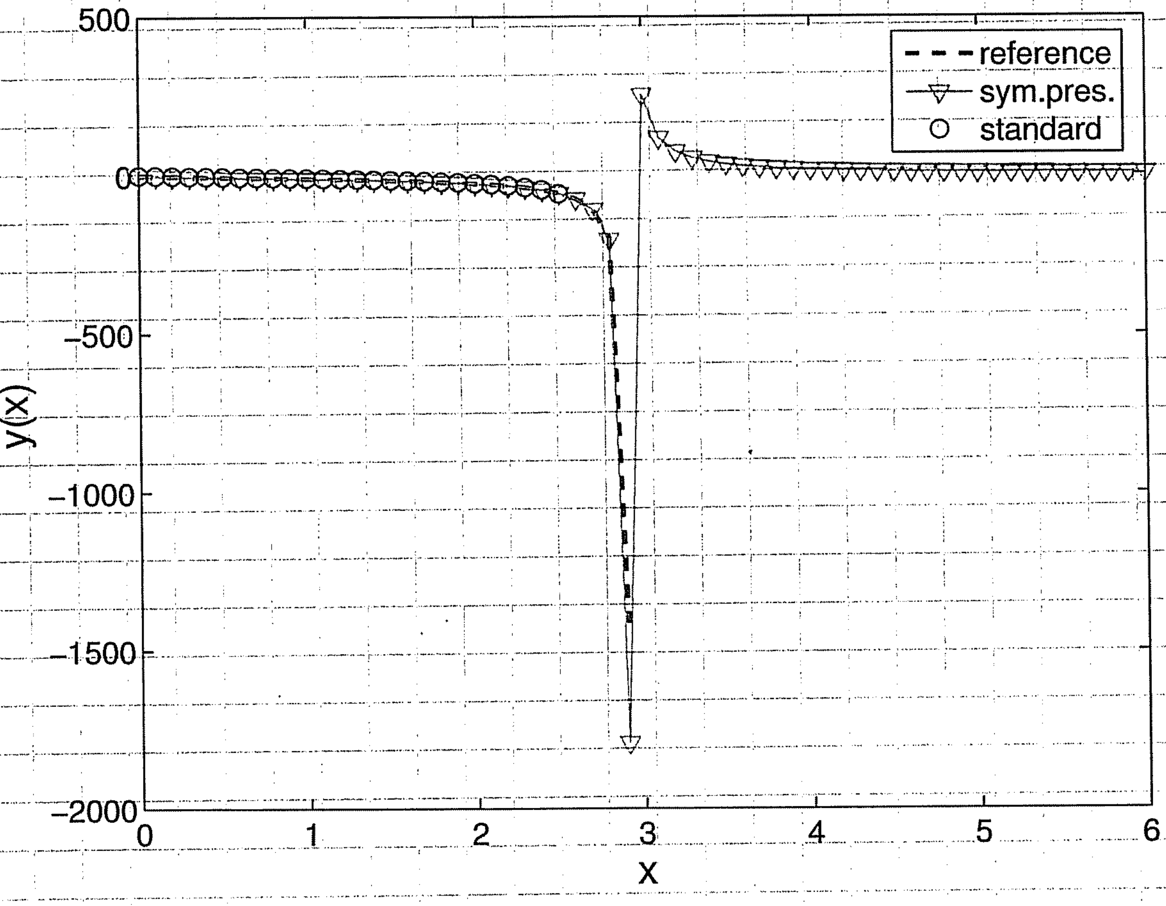


Figure 6. Solution for the symmetry preserving scheme and the standard scheme, h=0.1, Example 5, on [0, 6].

$$\frac{1}{y'^2}\left(y' y''' - \frac{3}{2} y''^2\right) = F(x) \qquad F(x) = \sin x$$

Mesh: $h_n = h_{n+1} = h_{n+2} = h$

$x_0 = x_n + \frac{h}{2}$

x_{n-1} x_n x_{n+1} x_{n+2}

$$\boxed{y_{n+2} = \frac{(y_{n+1}-y_{n-1})\,y_n - K_n\,(y_n - y_{n-1})\,y_{n+1}}{(y_{n+1}-y_{n-1}) - K_n\,(y_n - y_{n-1})}}$$

$$K_n = 4\left(1 - \frac{h^2}{2} F\left(x_n + \frac{h}{2}\right)\right)$$

Standard scheme: nonlinear eq. for y_{n+2}

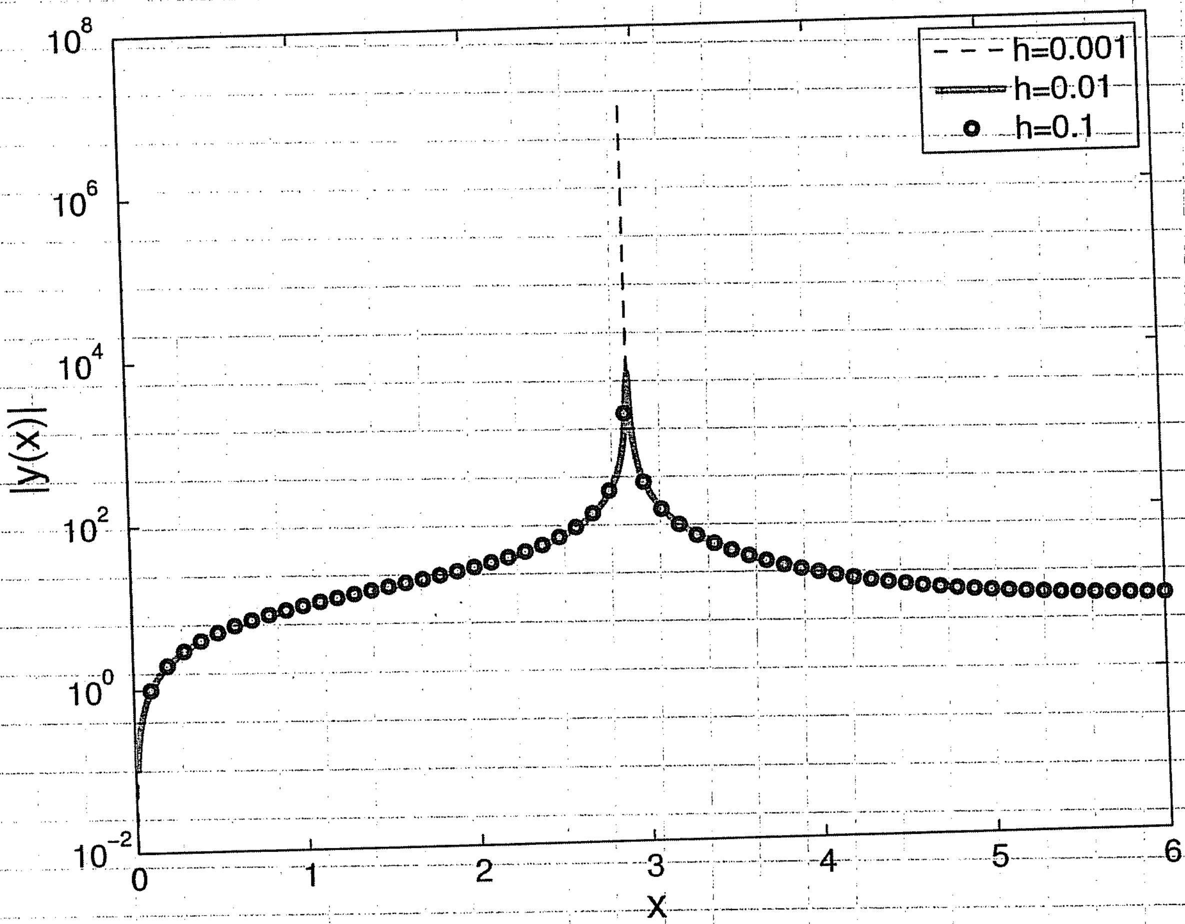


Figure 7. Absolute value of the solution with the symmetry preserving scheme, h=0.1, 0.01, 0.001. Example 5, on [0, 6].

Conclusions.

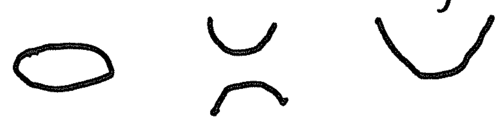
1. Symmetry preserving discretization: example of geometric integration of ODEs and PDEs

More generally: identify "global" qualitative features of an ODE or PDE, then create numerical scheme preserving them. Here: point symmetries. Other features: Hamiltonian structures, Lagrangians, conservation laws, integrability (Lax pairs)

Example: Kepler problem:

$$\left.\begin{array}{lll} \vec{L} & \Rightarrow & \text{in plane} \\ \vec{A} & \Rightarrow & \text{on cone} \end{array}\right\} \Rightarrow \text{conic section}$$

$E \Rightarrow$



2. ODE of order 2 : $\dim \mathcal{L} \geq 2$ $\Rightarrow$ we can integrate in quadratures

ODE of order 3 : $\dim \mathcal{L} \geq 3$: we greatly improve approximation and capture qualitative behavior.

3. ODE of order ≥ 4, $\dim \mathcal{L}=3$ even in principle not enough to solve analytically.
Symmetry adapted numerical methods get more important.
However: the methods are not "universal"
The equation must have enough symmetry in the first place

Open problems and work in progress

1. Systematic investigation of ODEs of all orders. Qualitative properties of invariant difference schemes. Further numerics
2. Systems of ODEs. Initial and boundary value problems
3. PDEs with finite and infinite dimensional symmetry groups (work with D. Levi (Roma III) and L. Martina, (Lecce))
4. Contact symmetries: Dorodnitsyn
5. Related work: symmetries of difference equations on given lattices (transforming or fixed.