Ricci Flow & Theoretical Physics

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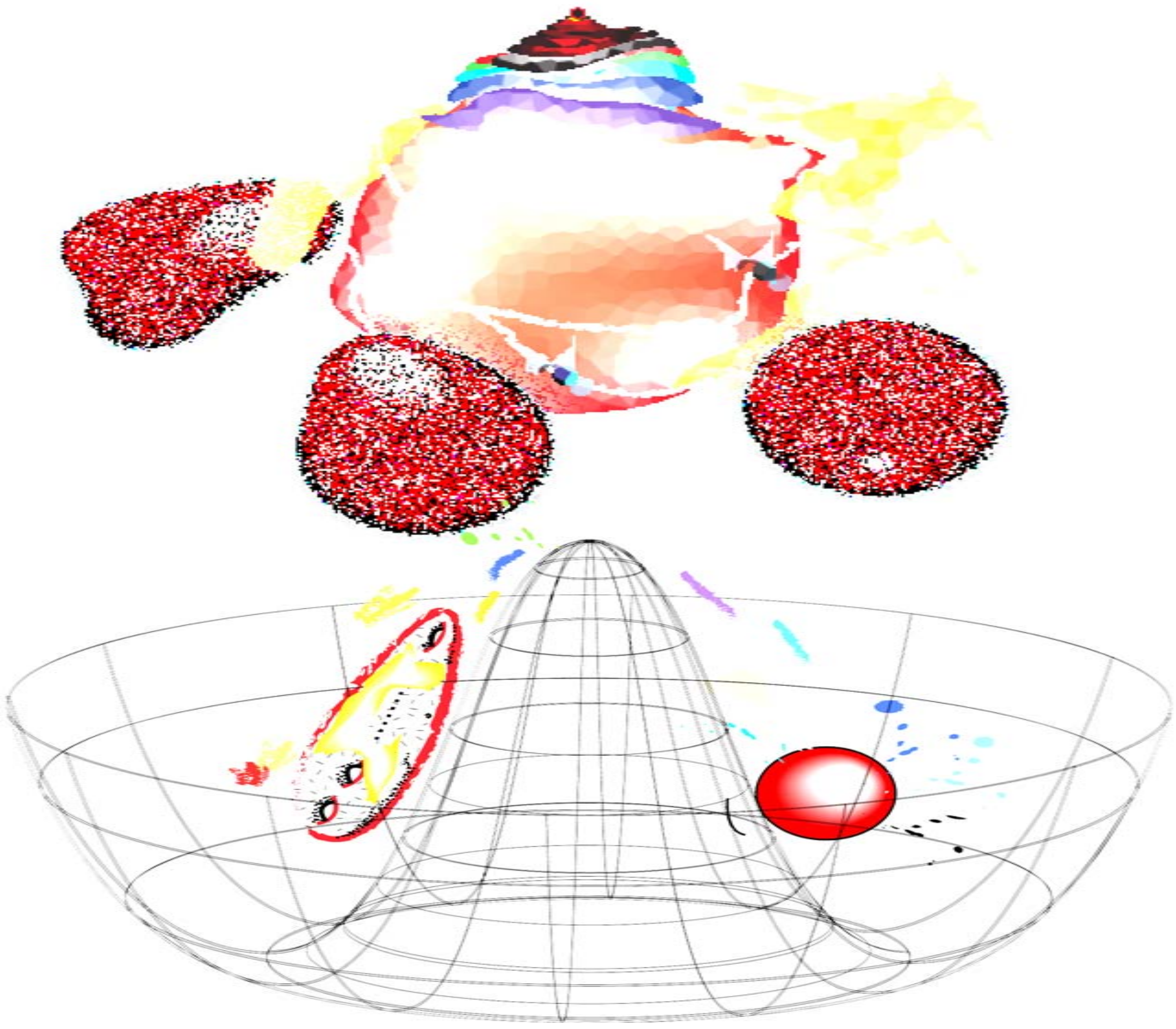
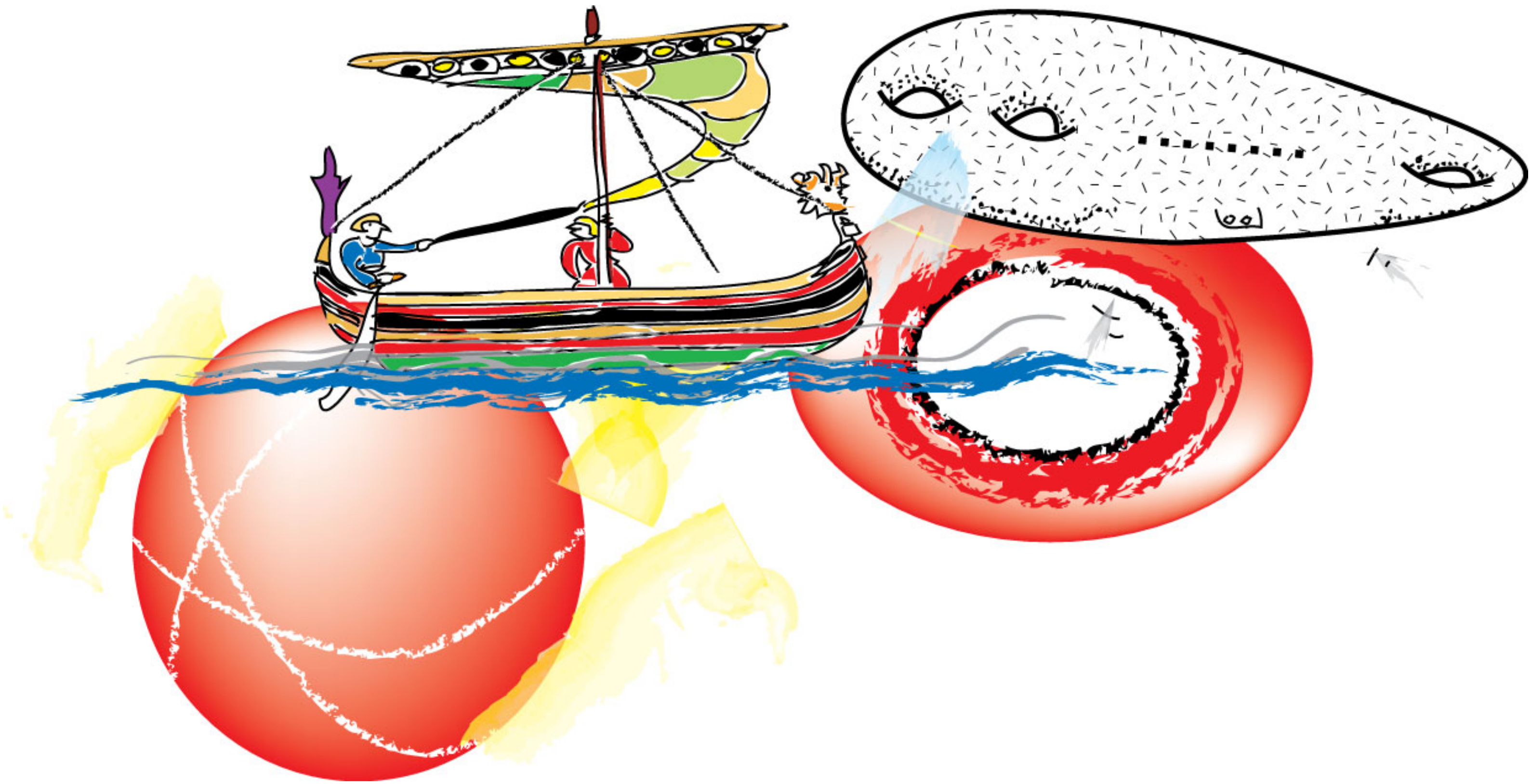


Warming up: The case of Surfaces

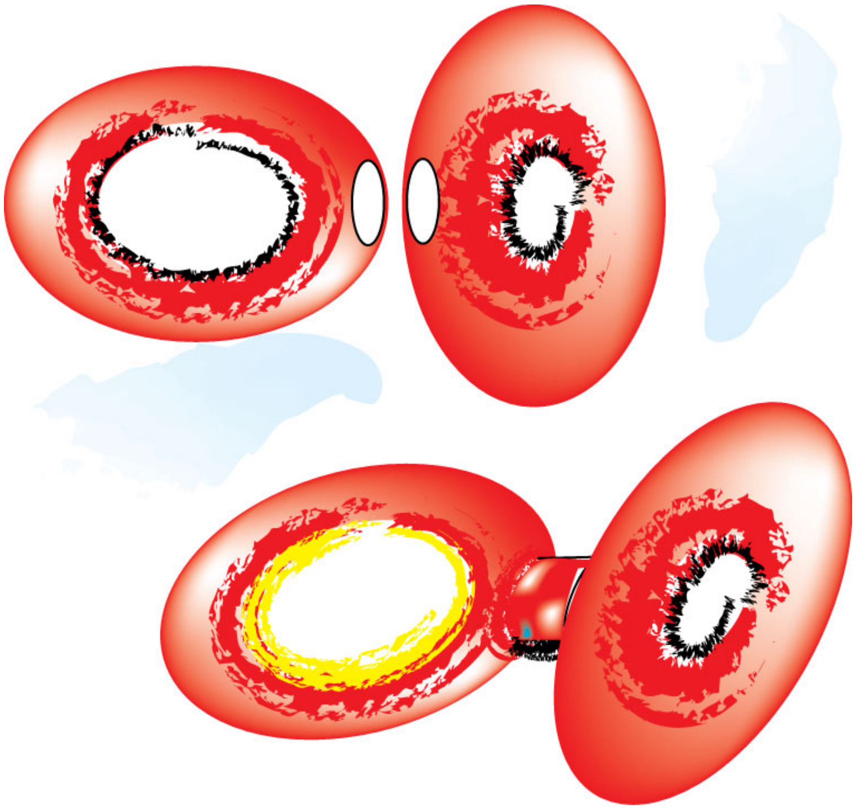
In many applications of geometry and PDE theory to theoretical physics we need to understand and classify the Geometric structure associated with Surfaces

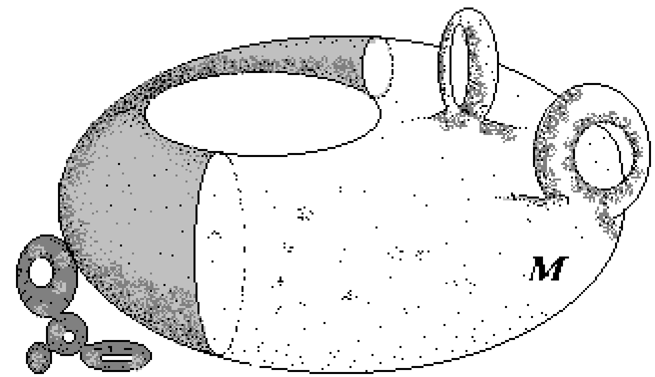
Basic examples: Statistical mechanics, CFT, String theory, General Relativity.



Topologically, surfaces are classified by their genus: the number of handles of the surface.

Surface can be constructed by a connected sum surgery .
Delete the interior of disks and glue the resulting
surfaces by an orientation-reversing homeomorphism.





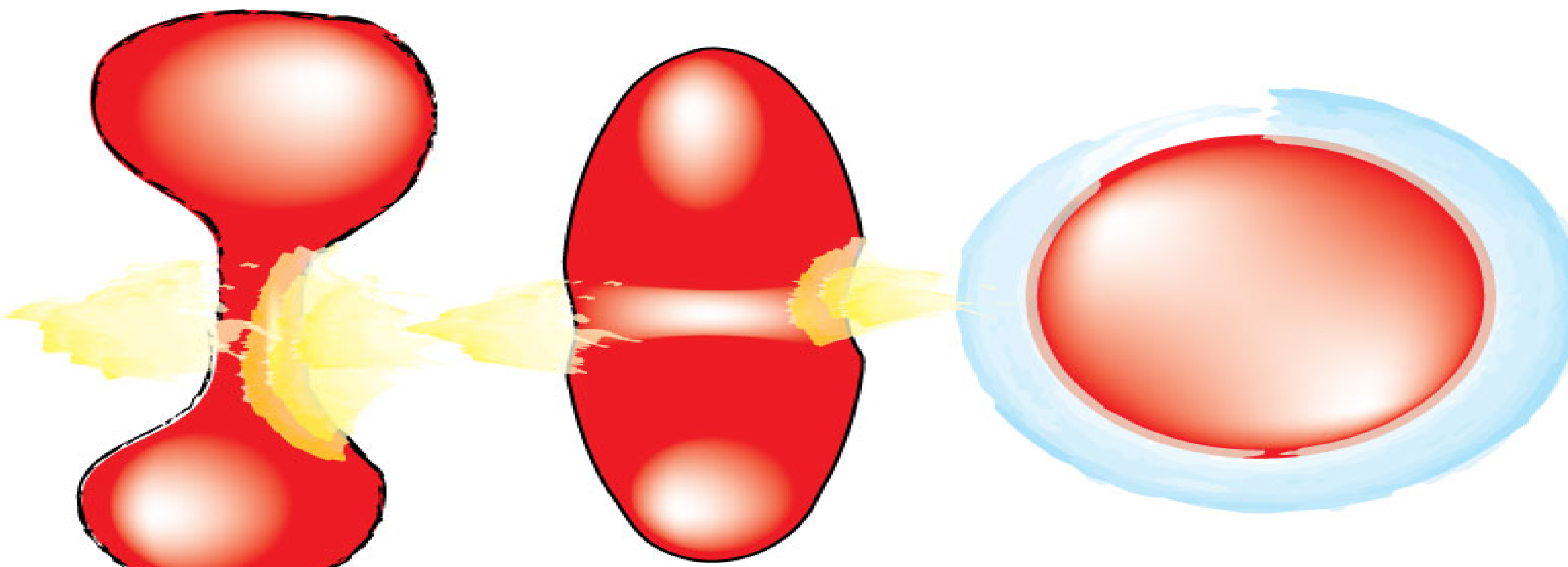
This implies the Topological Classification theorem for Surfaces
The set of Closed connected orientable surfaces consist of: Spheres, Tori, and finite number of connected sum of tori .

A deeper understanding of surfaces requires investigating the interplay between their topology and their metric geometry: Gaussian Curvature and all that.

If a surface Σ carries a Riemannian metric $g = g_{ab}dx^a \otimes dx^b$ then we have :

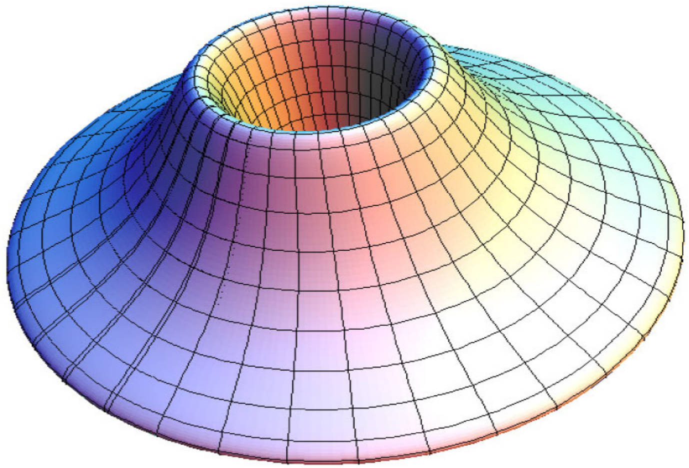
Gauss-Bonnet theorem

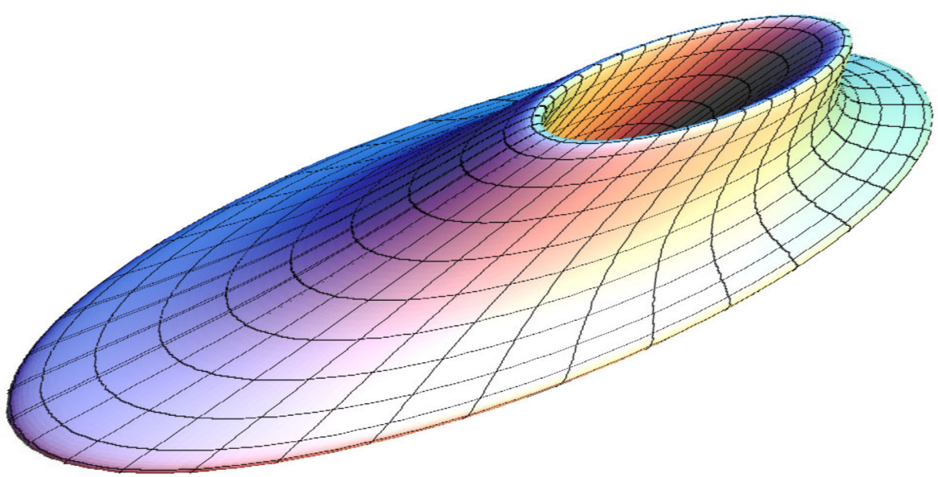
$$\int_\Sigma K(g)\, d\mu = 2\pi\,(2 - 2\,h)$$



$(\Sigma,\ g_{(1)})$ $(\Sigma,\ g_{(2)})$ $(\Sigma,\ g_{(3)})$

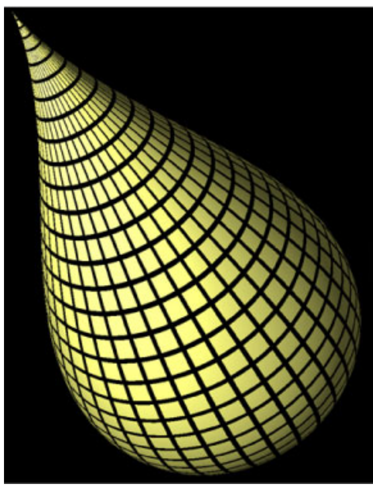
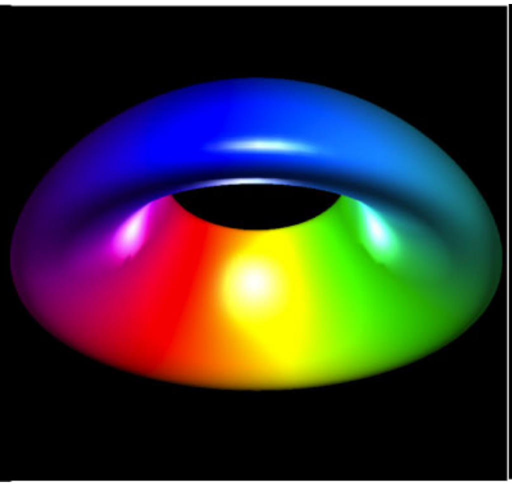
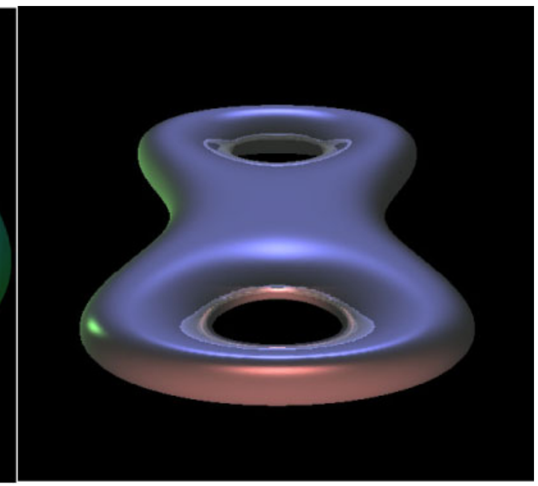
CONFORMAL GEOMETRY is the relevant structure
This is the structure that implies that we can draw :
Latitude and Longitude lines on every surface.
Conformal deformations preserve this network of lines

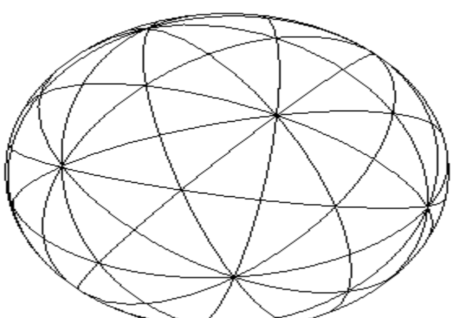
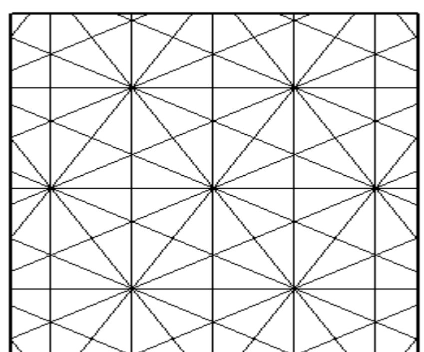
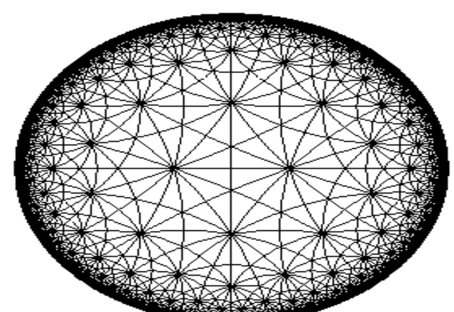




UNIFORMIZATION THEOREM (Poincaré, Koebe)

Any closed orientable surface is conformal to a surface with constant Gaussian curvature:.
The round sphere if the genus $h = 0$,
The Euclidean plane if the genus $h = 1$,
The hyperbolic plane if the genus $h \geq 2$.

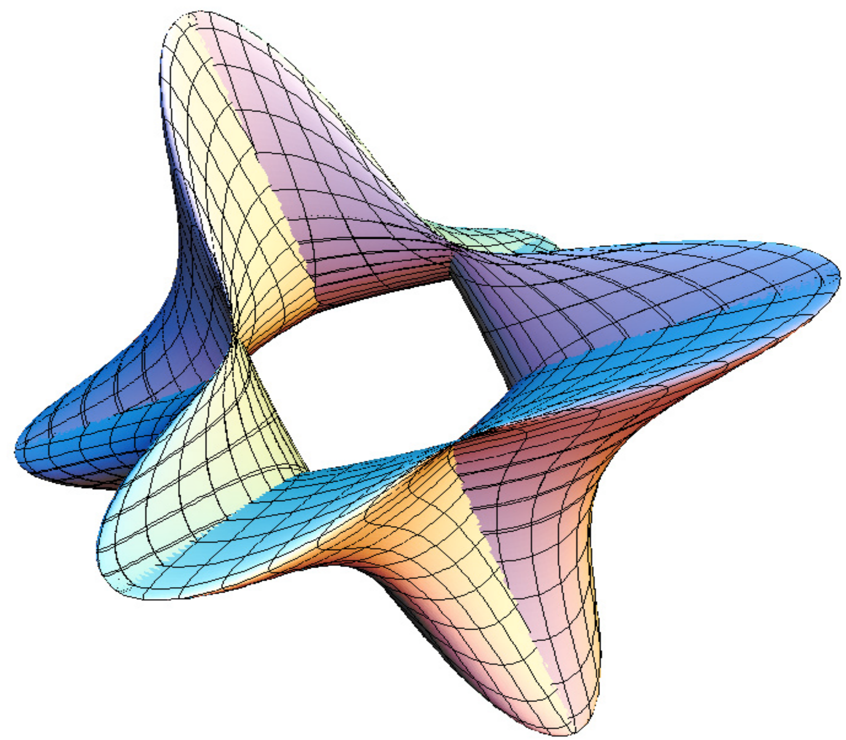




A TRIUMPH OF COMPLEX ANALYSIS

Riemann :

If Ω is a simply connected subset of the complex plane, not equal to C, then there exists a holomorphic homeomorphism from the unit disc onto Ω

Poincaré- Koebe: (Up to top. identifications) Any Riemann surface is biholomorphically equivalent to the Riemann sphere, the complex plane or the unit disc.
These three surfaces are not conformally equivalent. .

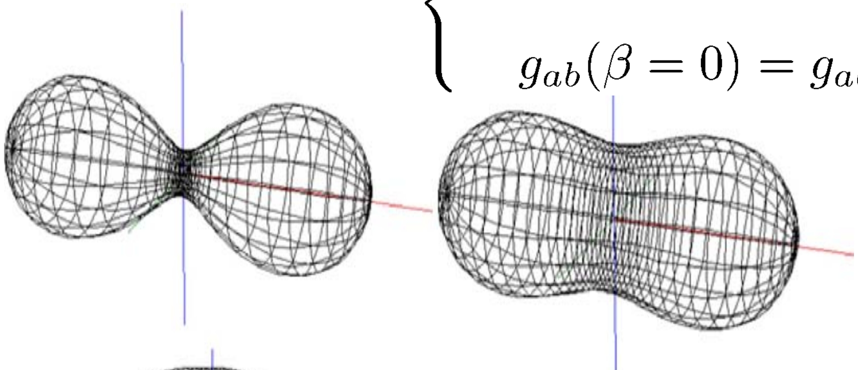


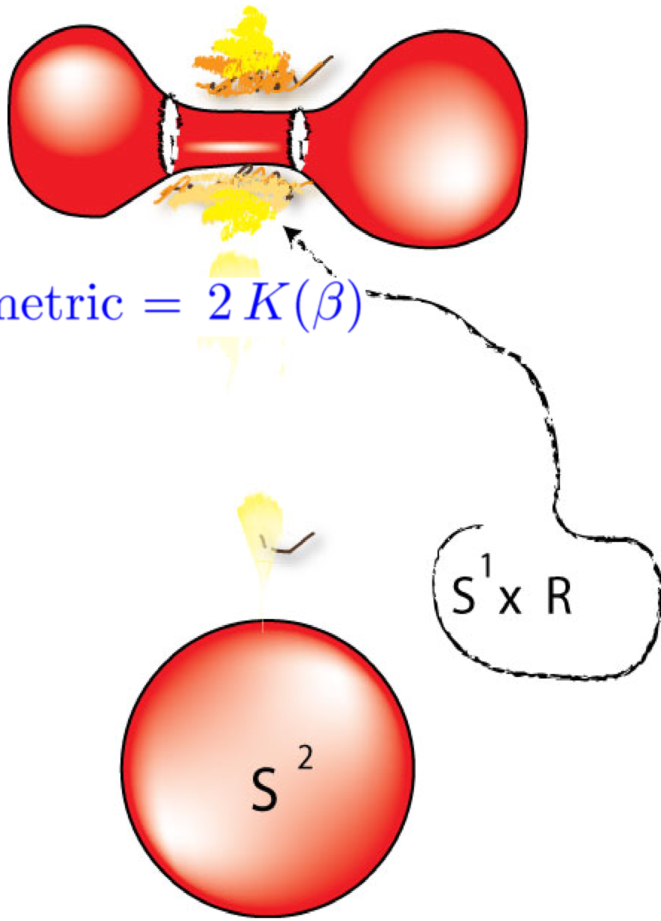
A MORE GEOMETRICAL APPROACH

(actually suggested after the introduction of the 3-dim Ricci flow!)

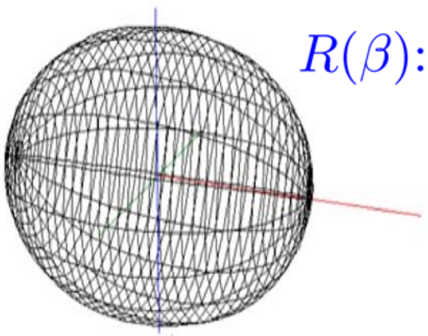
R. Hamilton : Deform the metric on the surface in the direction of its Gaussian curvature

$$\begin{cases} \frac{\partial}{\partial \beta} g_{ab}(\beta) = (r - R(\beta))\, g_{ab}(\beta) \\ \\ g_{ab}(\beta = 0) = g_{ab}, \end{cases} \qquad (1)$$

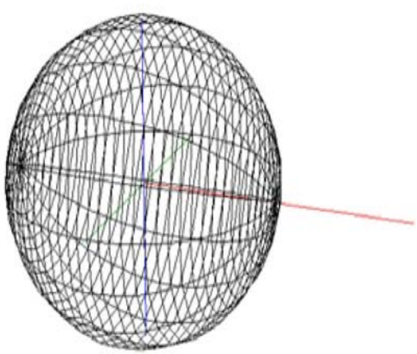




$R(\beta)$: the scalar curvature of the metric $= 2\, K(\beta)$



$$r \doteq \frac{\int_{\Sigma_\beta} R(\beta) d\mu_{g(\beta)}}{[Vol(\Sigma, g_{ab}(\beta))]}$$

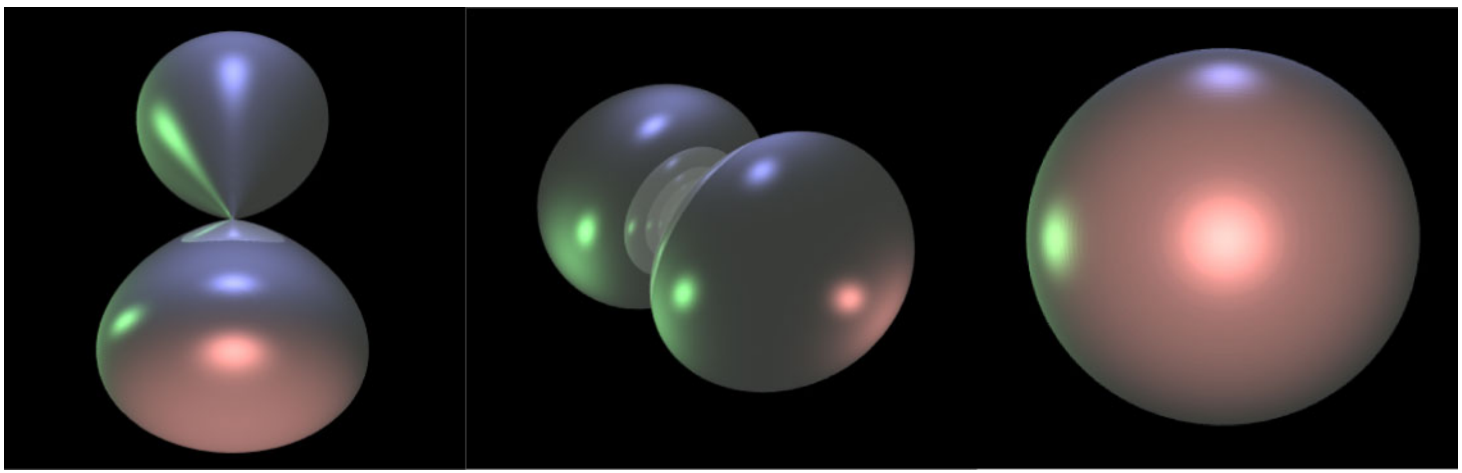


The Ricci flow induces a heat-like equation diffusing curvature.

$$\frac{\partial R(\beta)}{\partial\beta} = \Delta_{g(\beta)} R(\beta) + R(\beta)(R(\beta) - r(\beta)),$$

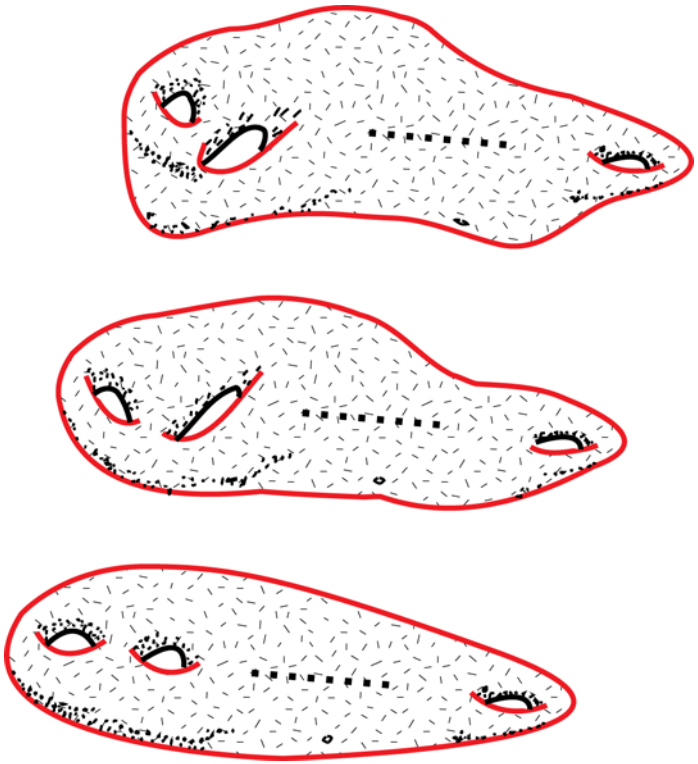
Actually a reaction-diffusion equation .

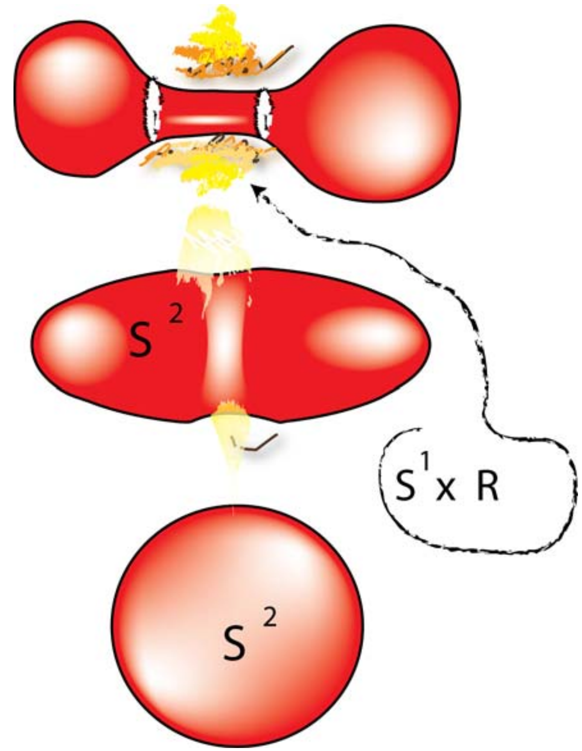
Thus, it is reasonable to expect (but not trivial to prove) that the equilibrium state for such a deformation will be a metric with constant curvature.



If (Σ, g_0) is a closed Riemannian surface, there exists a unique solution of the normalized Ricci flow. The solution exists for all time. As $t \to \infty$, the metrics $g(t)$ converge uniformly (in any C^k-norm) to a smooth metric g_∞ of constant curvature.

Geometric rationale underlying the possible large β behavior of the Ricci flow on surfaces:

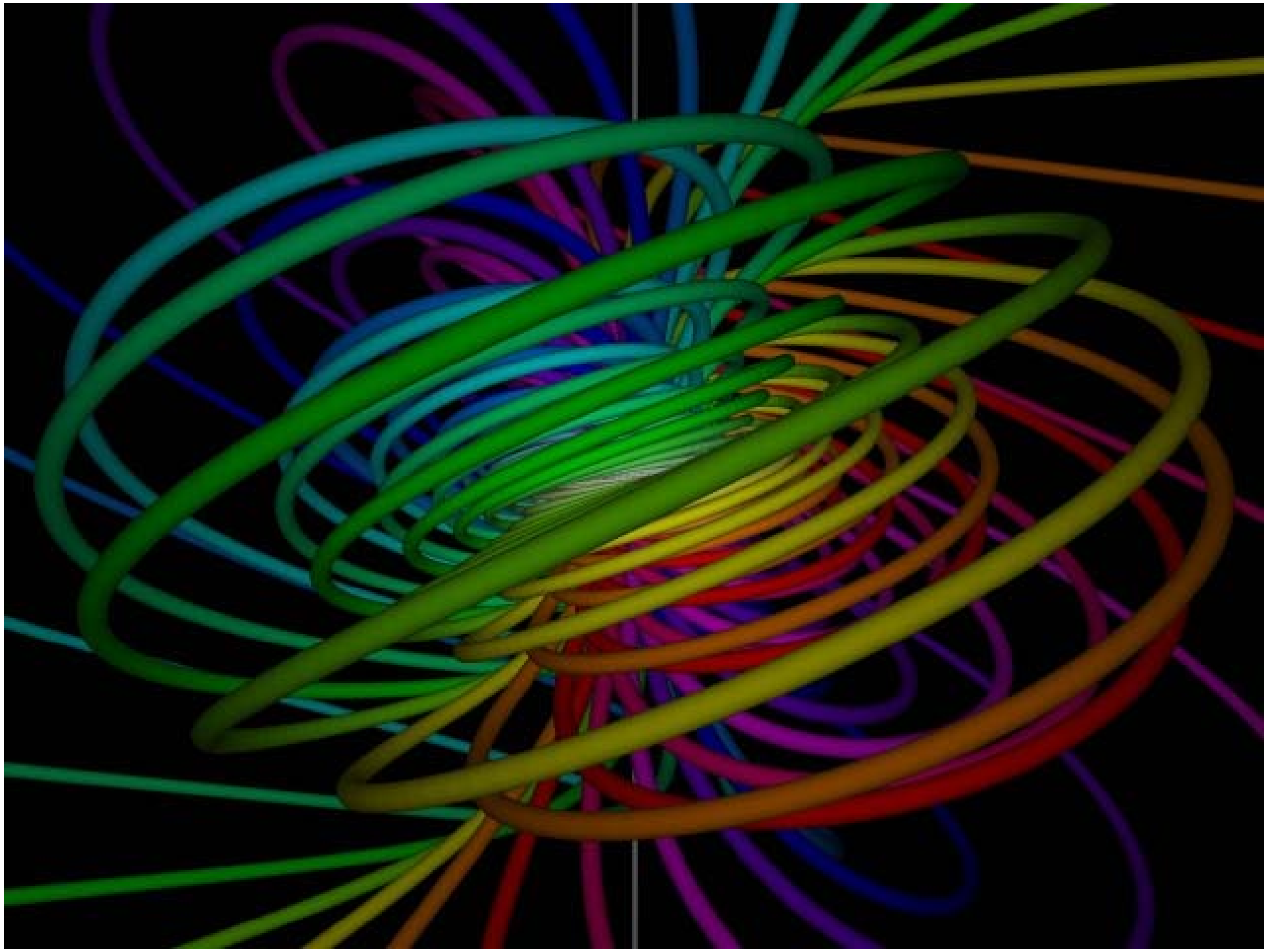




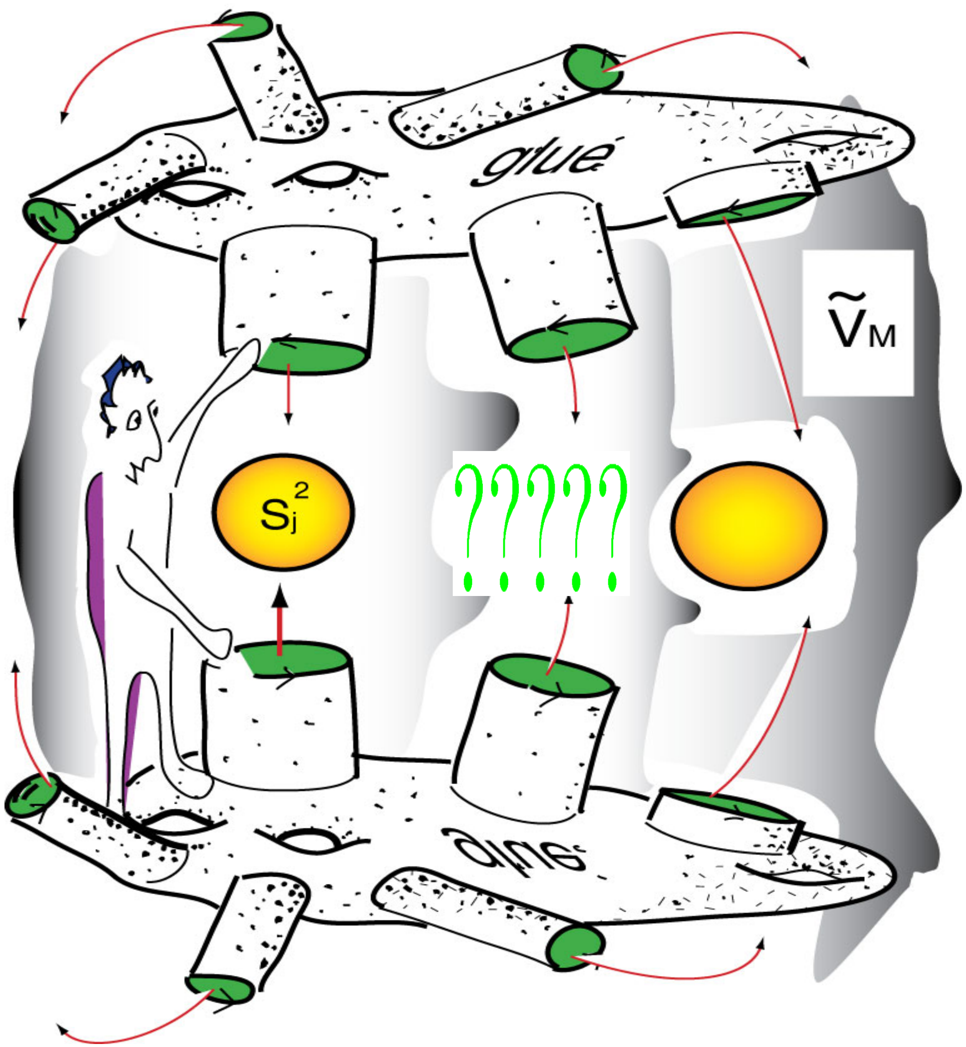
(i). If the Euler characteristic of Σ is non-positive, then the solution metric $g(\beta)$ converges to a constant curvature metric as $\beta \to \infty$.

(ii). If the scalar curvature $R(g_0)$ of the initial metric g_0 is positive, then the solution metric $g(\beta)$ converges to a positive constant curvature metric as $\beta \to \infty$.

What about the 3-dimensional case?

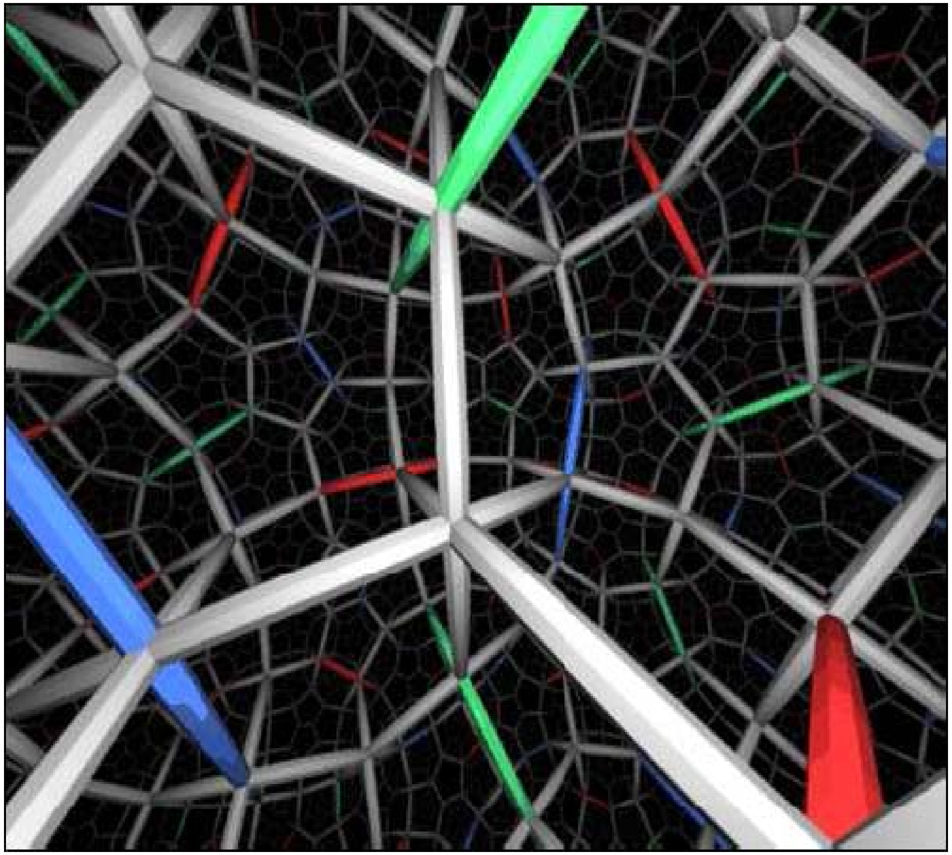


Standard Uniformization techniques do not work in dimension 3



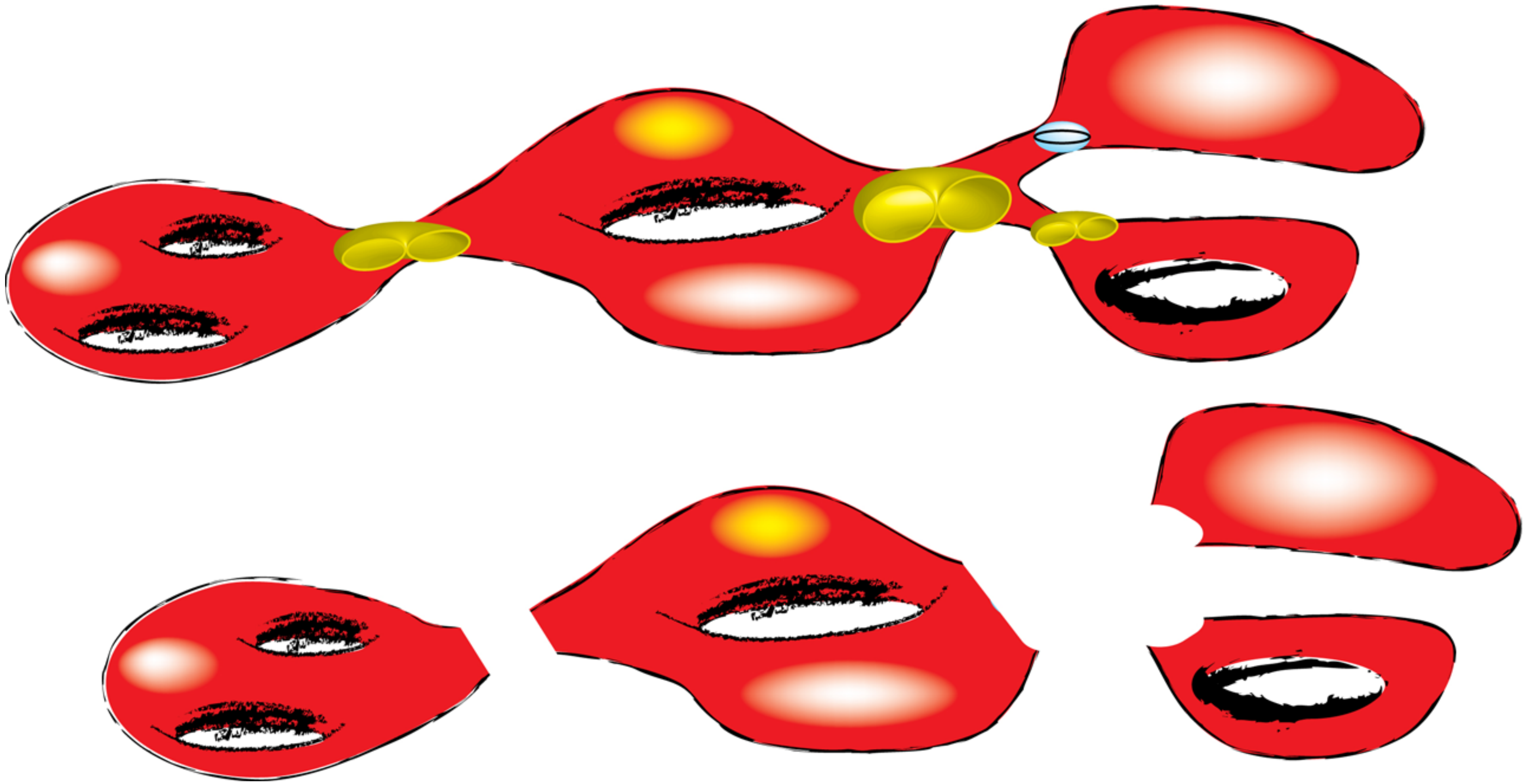
However, in many applications it would be very useful
to have some sort of uniformization theorem for 3-manifolds

Actually 3-dimensional Riemannian manifolds
are very rigid: strict relations between geometry and topology:



What can we say on the structure of Three dimensional manifolds? :

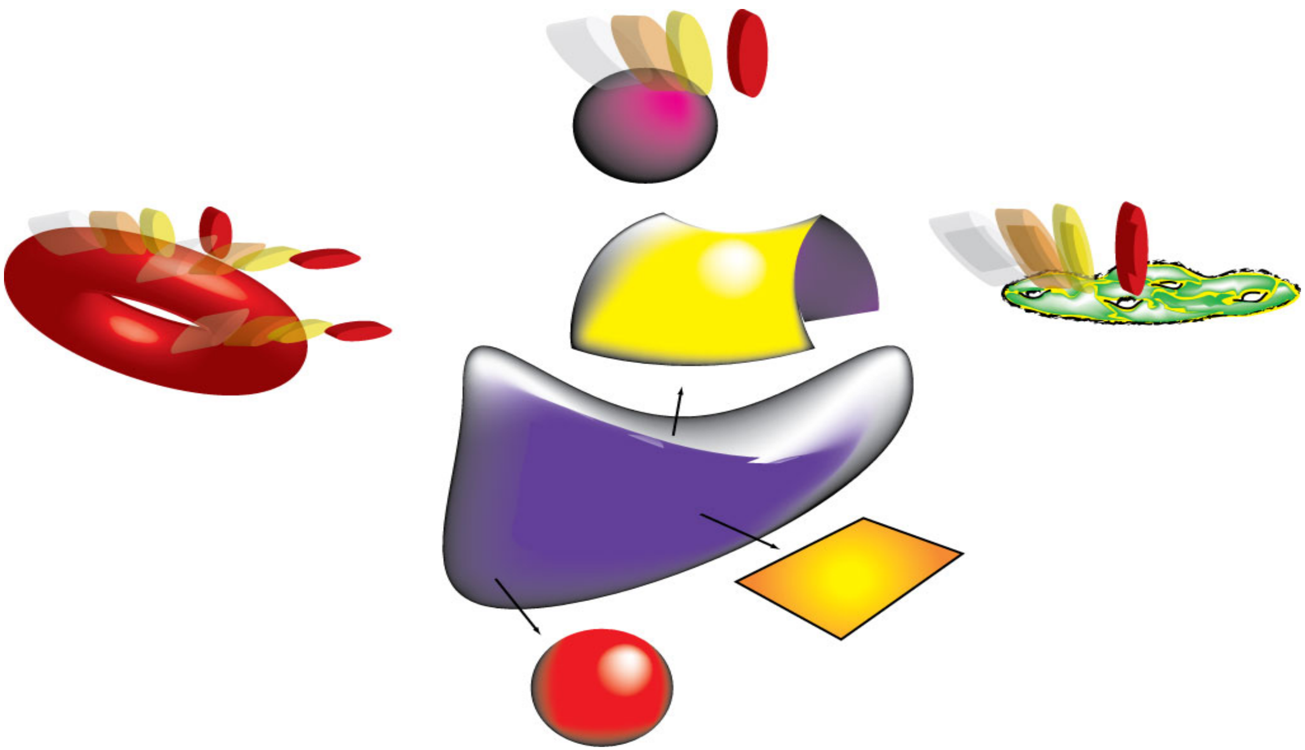
Geometrization Conjecture (W. Thurston)
(now a theorem- G. Perelman) . Let M be a closed, oriented 3-manifold. Then each component of the sphere and torus decomposition admits a geometric structure



In particular:(Poincaré, 1904).
If M is simply connected then it is diffeomorphic to the three-sphere S^3.

The strategy is to cut the 3-manifold
into a connected sum of (prime) manifolds
then cut each such a piece along (incompressible) tori.
The remaining components admit a geometry out of the following list

Euclidean geometry
Hyperbolic geometry
Spherical geometry
The geometry of S^2 x R
The geometry of H^2x R
The geometry of SL_2R
Nil geometry , or
Sol geometry .

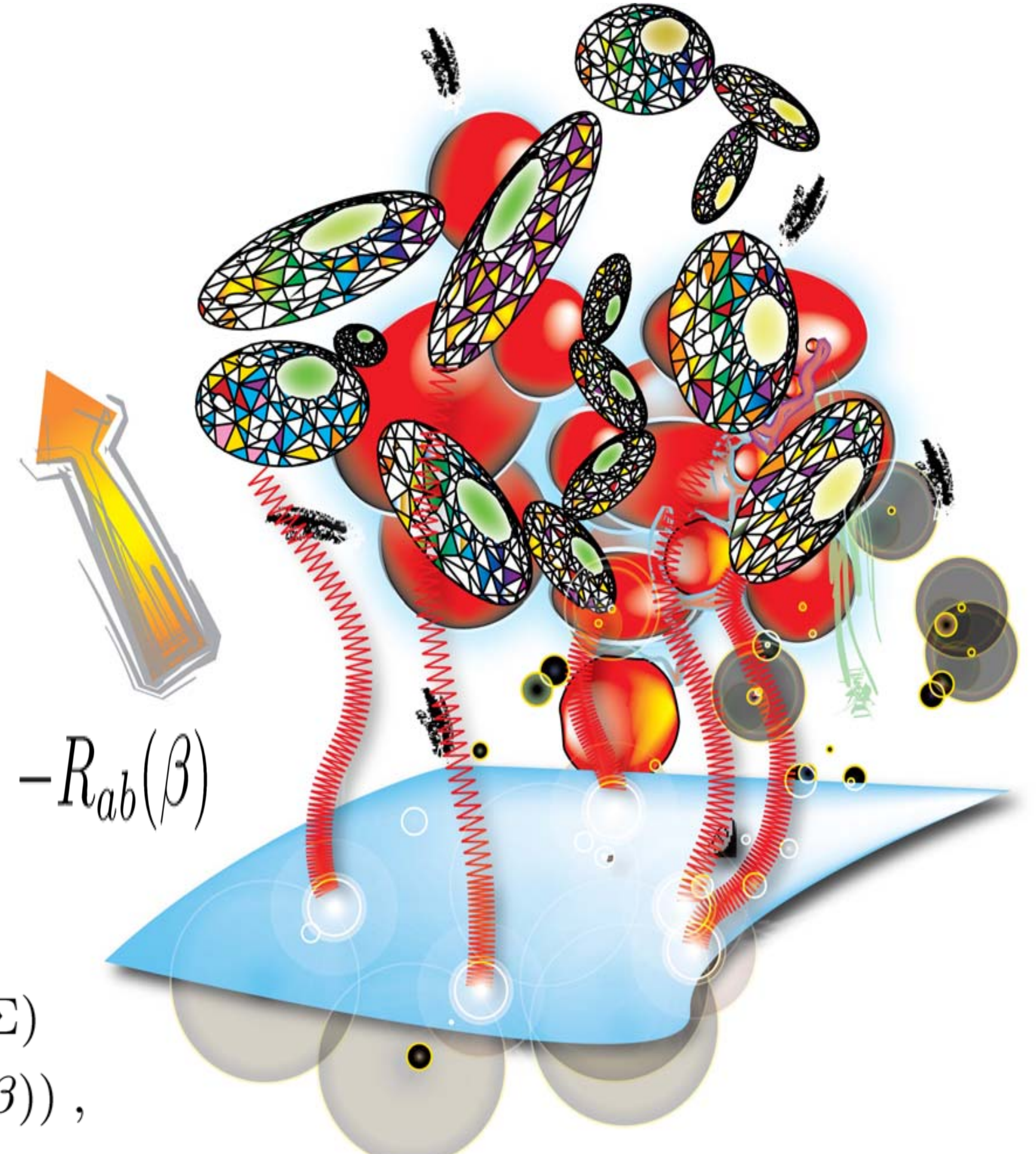


The situation is very much like dimension 2
A 3-manifold can be classified by a decomposition in geometric pieces
However, now different pieces of the same manifold
generally have different geometric structures.

Thurstons conjecture tends to classify three dimensional geometries by exploiting group theory. However this group-theoretical approach is difficult to be used for general 3-dimensional Riemannian manifolds.

It turns out that a curvature flow is again the natural technique for proving Thurston's conjecture:

The Ricci flow: Deforming the metric of the 3-manifold in the direction of its Ricci tensor



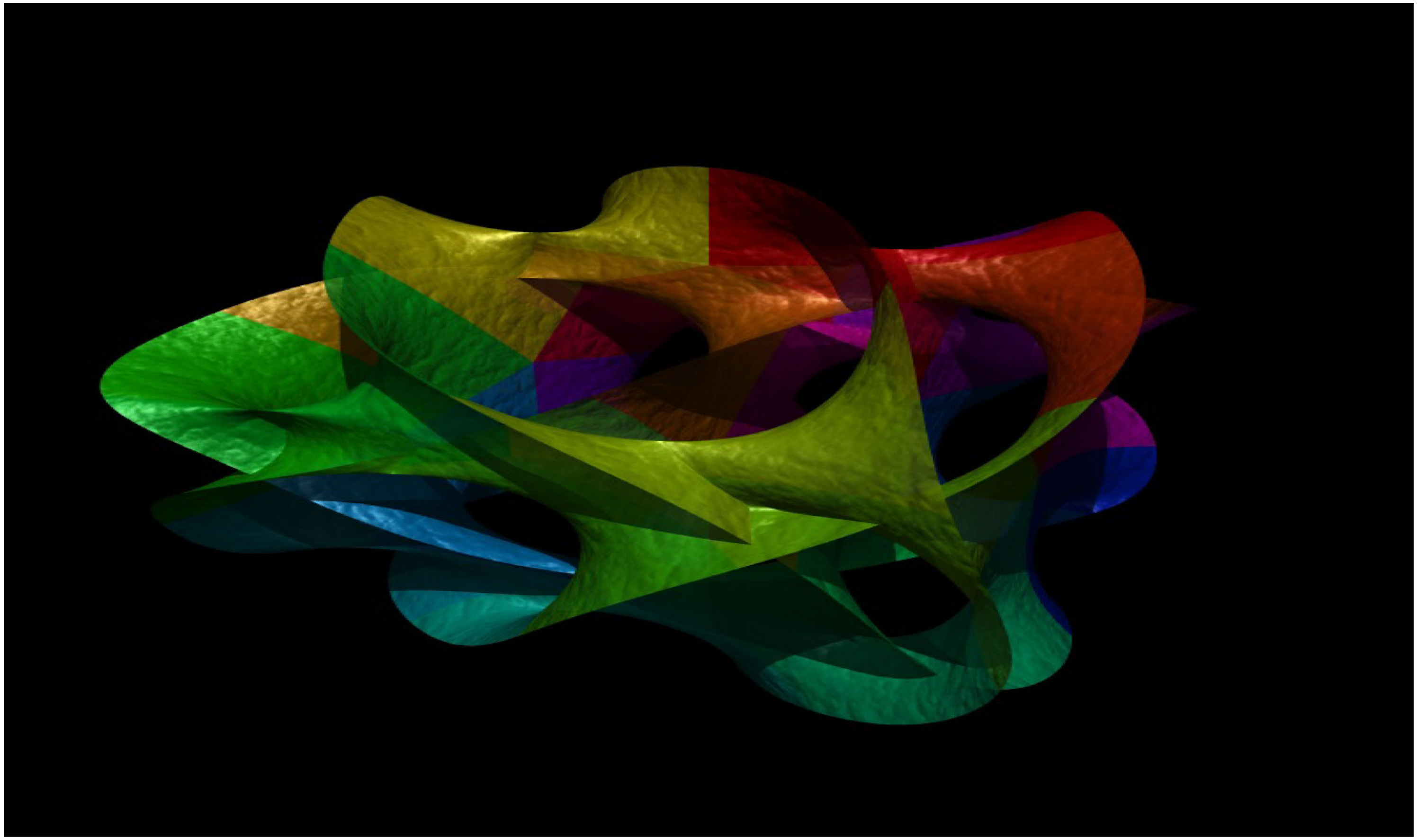
$$
\begin{array}{ccc}
\mathcal{M}et(\Sigma) & \longrightarrow & \mathcal{M}et(\Sigma) \\
(\Sigma, g) & \mapsto & (\Sigma, g(\beta))\ ,
\end{array}
$$

(M, g) three-dimensional compact Riemannian Manifold without boundary

In a three dimensional Riemannian manifold, the curvature can be different when measured along different sections, and is characterized by the Ricci tensor

$$R_{ab} = \partial_c \Gamma^c_{ab} - \partial_{ab} \ln \sqrt{g} - \Gamma^d_{ac} \Gamma^c_{db} + \Gamma^c_{ab} \partial_c \ln \sqrt{g},$$

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{ji} - \partial_l g_{ij})$$



The Ricci tensor . Passive view: Fix the manifold.
Change of local Riemannian measure $d\mu$ in passing from a metric ball to a nearby one.

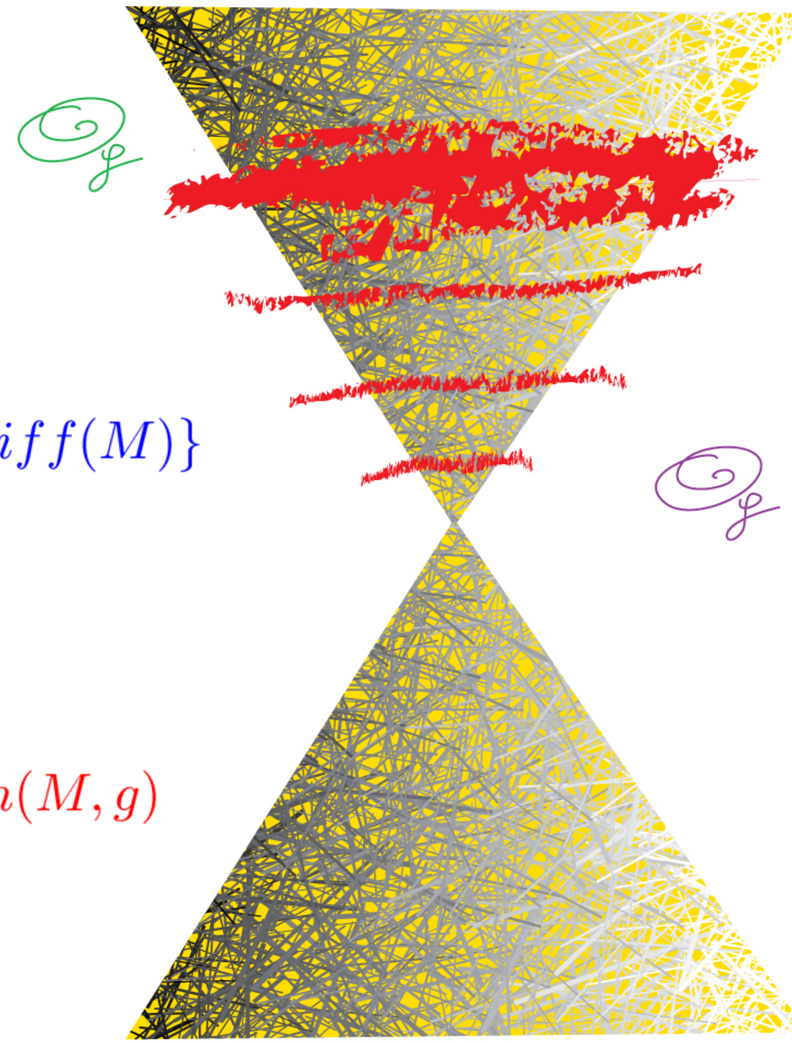
(M, g)

$$\frac{\int_{B(p,r)} f \ d\mu}{Vol\,[B(p,r)]}$$

The Ricci tensor . Active view: Fix the Ball.
Change of Riemannian measure $d\mu$ in the ball in passing from a Riemannian manifold to a nearby one.

Properties of the space of Riemannian structures $\frac{Riem(M)}{Diff(M)}$

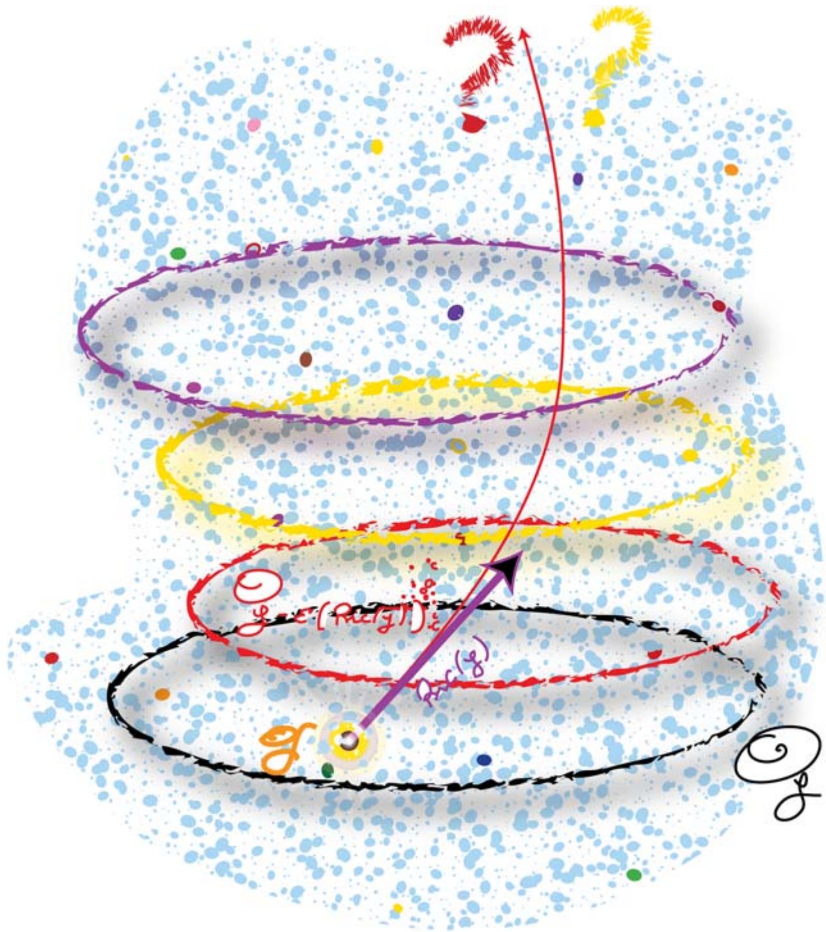
A stratified manifold of Orbits O_g of $g \in Riem(M)$ under the action of $Diff(M)$. The O_g are labelled by the isotropy group I_g of the given metric g.



$$O_g = \{\tilde{g} \in Riem(M) | \tilde{g} = \phi^* g,\, \phi \in Diff(M)\}$$

$$I_g = \{\psi \in Diff(M) |\, \psi^* g = g\} = Isom(M, g)$$

J.-P. Bourguignon noticed that, unless (M, g) is an Einstein Manifold, the Ricci tensor $Ric(g)$ (thought of as an element of $T_g Riem(M)$) provides a non-trivial vector field on $Riem(M)$.



$$-R_{ik}(g) = h_{ik} \in T_g Riem(M)$$

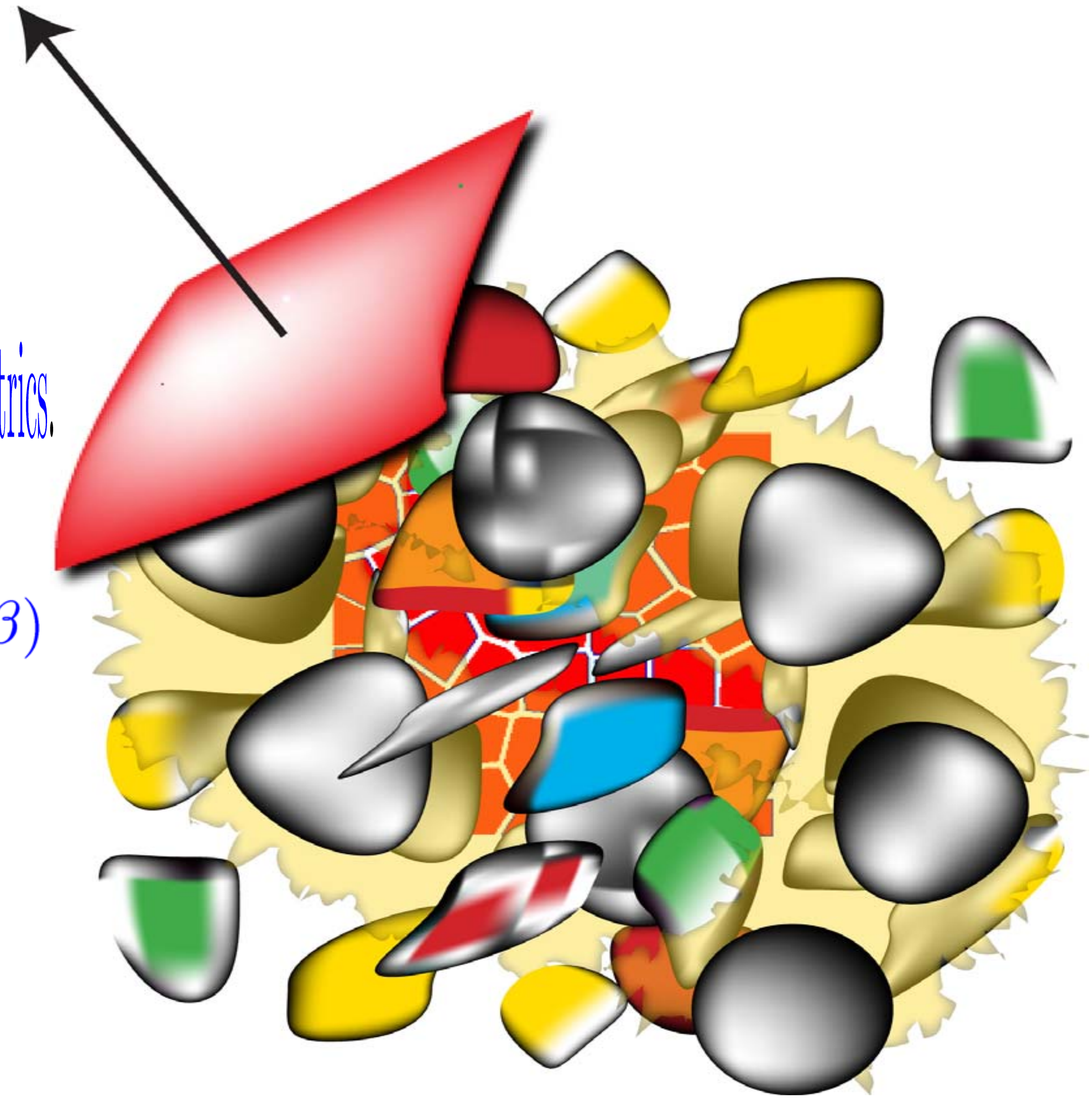
It is not evident, a priori, that with such a vector field we can associate smooth trajectories.

The Ricci flow (Hamilton (1982)): the simplest deformation of a Riemannian metric which is invariant under the action of the diffeomorphism group.

The Ricci flow can be thought of as a

dynamical system on the space of Riemannian metrics.

$$\begin{cases} \frac{\partial}{\partial \beta} g_{ab}(\beta) = -2R_{ab}(\beta) \\ g_{ab}(\beta = 0) = g_{ab}, \end{cases}$$



What is the geometrical meaning of the flow $\beta \to g_{ab}(\beta)$ associated with the vector field $g \to -2Ric(g)$?

Locally, (pointwise in normal geodesic coordinates; or, for maximal regularity, in harmonic coordinates)

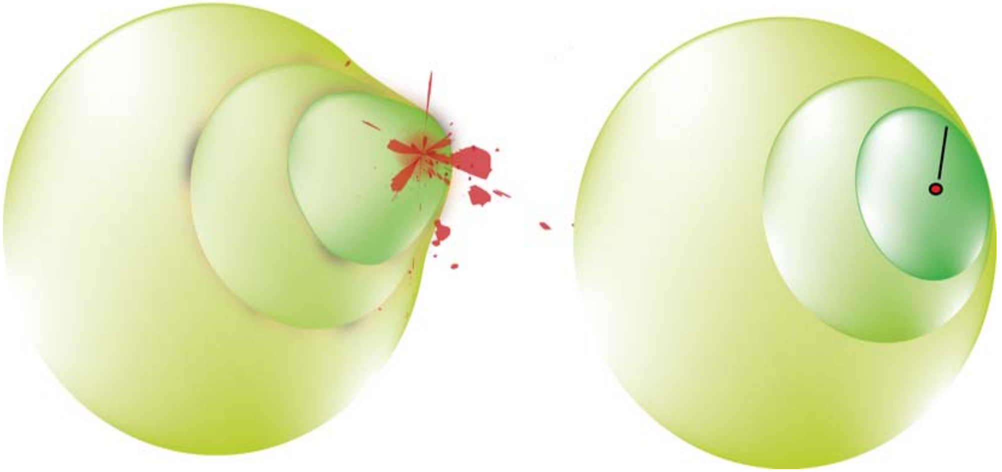
$$g_{ik}(x) = \delta_{ik} - \frac{1}{3} R_{ipkq}\, x^p x^q + O(|x|^3)$$

we have that the flow

$$\frac{\partial}{\partial\beta}\, g_{ab}(\beta) = -2R_{ab}(g(\beta))$$

reads

$$\frac{\partial}{\partial\beta}\, g_{ab}(\beta) = 2\Delta\, g_{ab}(\beta) + Q(\partial_k g, g)$$

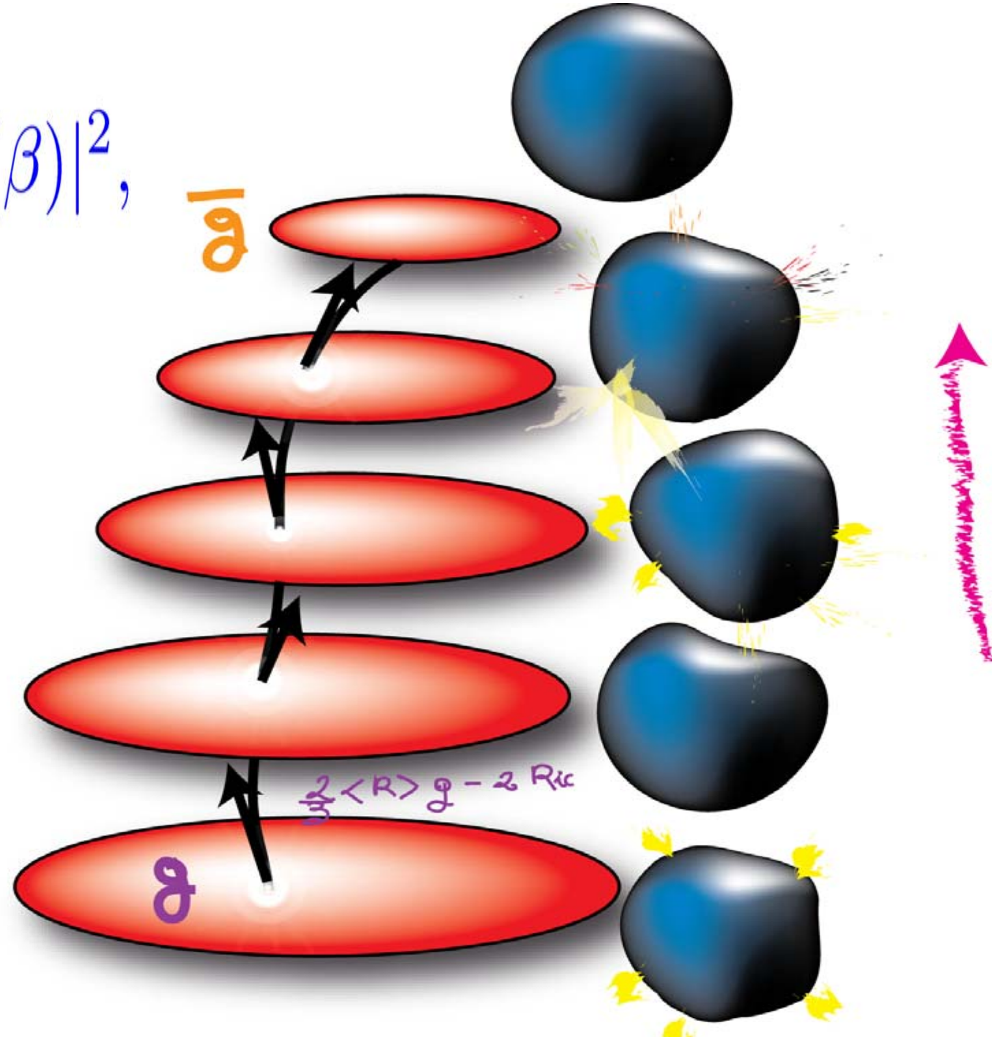


Heat equation for the components of the metric tensor.

Thus, we expect that along the solution of the Ricci flow, the metric structure improves, at least for small β.

The flow always exists for small β (Hamilton, Nash-Moser th.; simpler proof by D. DeTurck). Under suitable hypotheses on the curvature of the initial metric, the flow is global.

$$\frac{\partial R(\beta)}{\partial \beta} = \Delta_{g(\beta)} R(\beta) + 2|Ric(\beta)|^2,$$



Diffusion-Reaction equation

Singularities
are caused by curvature blowup.

If the solution to the Ricci flow exists for $[0, T)$ but does not extend to a larger β interval, then there is a point p in M for which the curvature $Rm(p, \beta)$ is unbounded as β approaches T

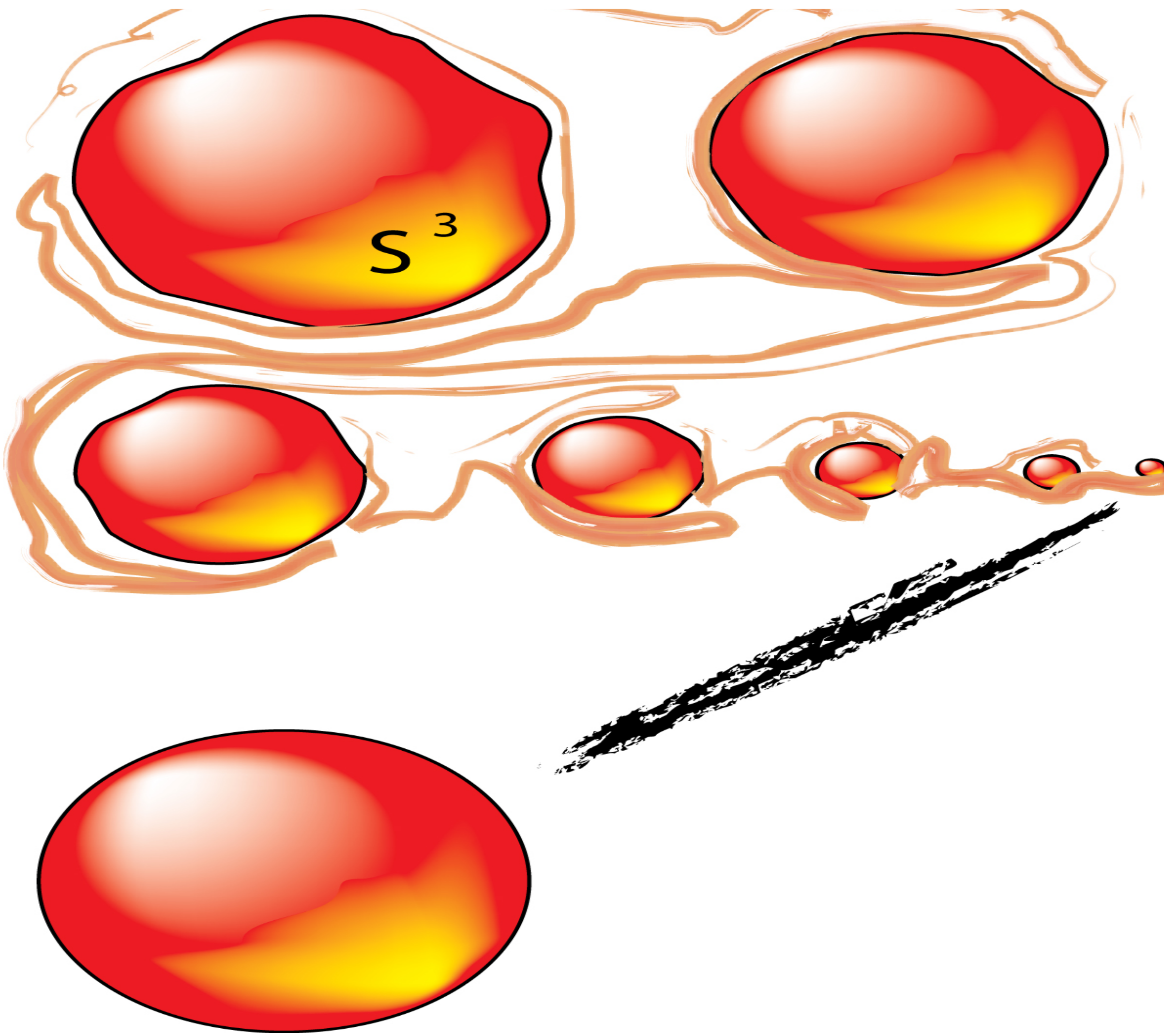
Hamilton's theorem (JDG '82) : The Ricci flow maps a given initial metric g, with $Ric(g) > 0$, to the round metric on S^3/Γ.

Finite subgroups $\Gamma \subset SO(4)$

For 3-manifolds (closed), the flow (not volume-normalized) smoothes and collapses positive curvature regions.

Up to global rescalings, it improves the geometry of positively curved portions of a 3-manifold.

Shrinking sphere solutions defined on the interval $-\infty < \beta < \frac{r_0}{2(n-1)}$:



$$g(\beta) = r^2(\beta) g_{can}$$

$$r(\beta) = \sqrt{r_0 - 2(n-1)\beta}$$

In general, near points where $Ric(g) < 0$ the Ricci flow expands the metric, whereas near points where $Ric(g) > 0$ it tends to collapse the metric.

If we set $Ric(g_0) = k(n-1)g_0$, k constant (consequence if $n \geq 3$), and seek solutions of the form $g(\beta) = r^2(\beta)g_0$ one finds

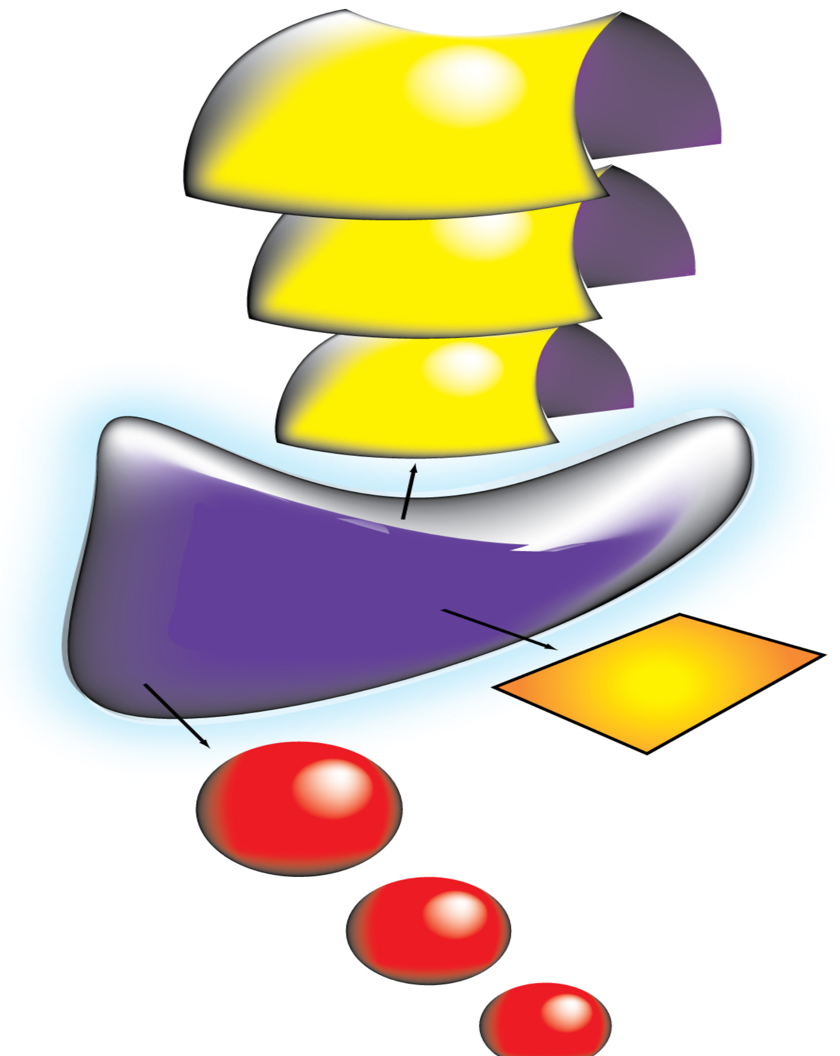
$$r^2(\beta) = 1 - 2k(n-1)\beta.$$

$r^2(\beta) > 0$ implies:

Antique solutions: $k > 0$,
$g(\beta) = 2k(n-1)(T-\beta)g_0$,
with $-\infty < \beta < T$, $T = \frac{1}{2k(n-1)}$.

Eternal solutions: $k = 0$, $g(\beta) = g_0$,
with $-\infty < \beta < +\infty$.

Dilatative solitons: $k < 0$,
$g(\beta) = -2k(n-1)(-T+\beta)$,
with $T < \beta < +\infty$.
Curvature $\to -\infty$ as $\beta \to T$,
and approaches 0 as $\beta \to +\infty$.



These solutions describe the non-trivial action of the diffeomorphisms group and rescalings, $Diff(M) \times R^+$.

$$\frac{\partial}{\partial\beta}\, g_{ab}(\beta) = -2\, R_{ab}(\beta)$$

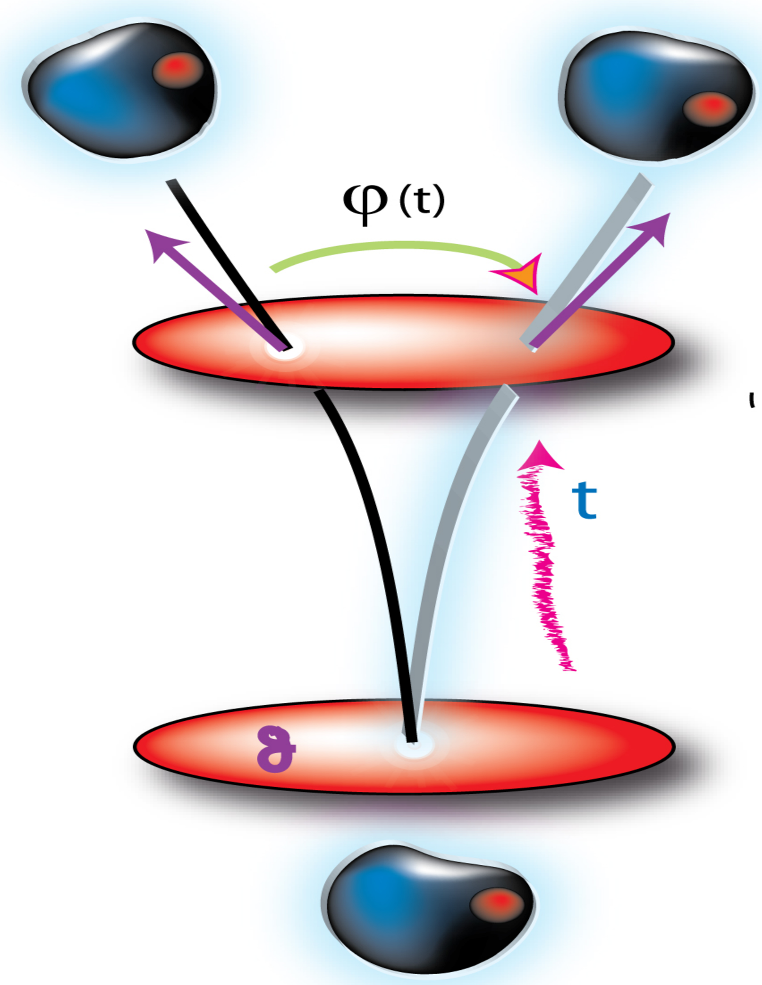
$$\frac{\partial}{\partial\beta}\, \widetilde{g}_{ab}(\beta) = -2\, \widetilde{R}_{ab}(\beta) + L_{w(\beta)}\, \widetilde{g}_{ab}(\beta)$$

equivalent under $\varphi(\beta) \in Diff(M)$ generated by the vector field $w(\beta)$,

$$\frac{\partial}{\partial\beta}\varphi(\beta) = w(\beta)$$

Related to the notion of Ricci Soliton

$$\widetilde{g}_{ab}(\beta) = (\varphi(\beta)^* g)_{ab}(\beta)$$

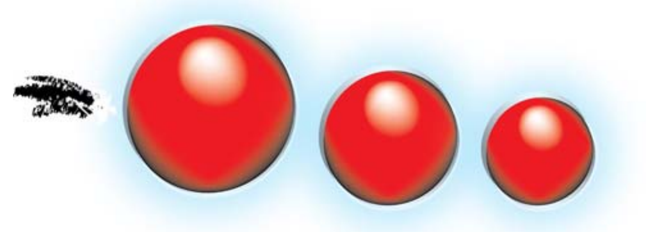


A Riemannian manifold (M, g) is a gradient Ricci soliton if its Ricci tensor is such that $R_{ab} - \frac{<R>}{n} g_{ab} + \nabla_a \nabla_b f = 0$ for some function f.

A Ricci soliton satisfies the normalized Ricci flow

$$\frac{\partial}{\partial \beta} g_{ab}(\beta) = -2 R_{ab}(\beta) + \frac{2}{3} < R(\beta) > g_{ab}(\beta) = 2 \nabla_a \nabla_b f$$

i.e. $$\frac{\partial}{\partial \beta} g_{ab}(\beta) = L_{\nabla f} \, g_{ab}(\beta)$$



These solutions are self-similar.

According to the sign of $< R(\beta) >$ we have:

Shrinking soliton ($< R(\beta) > > 0$)
Steady soliton ($< R(\beta) > = 0$)
Expanding soliton ($< R(\beta) > < 0$).

Round S^3 is a trivial shrinking soliton
$S^2 \times I$ (the cylinder) is a shrinking soliton too.

Shrinking solitons: models for singularity formation in Ricci flow

Cigar soliton (∞tely long cigar with tip located at $r = 0$)

$$g_0 = \frac{dr^2 + r^2 d\vartheta^2}{1 + r^2}$$

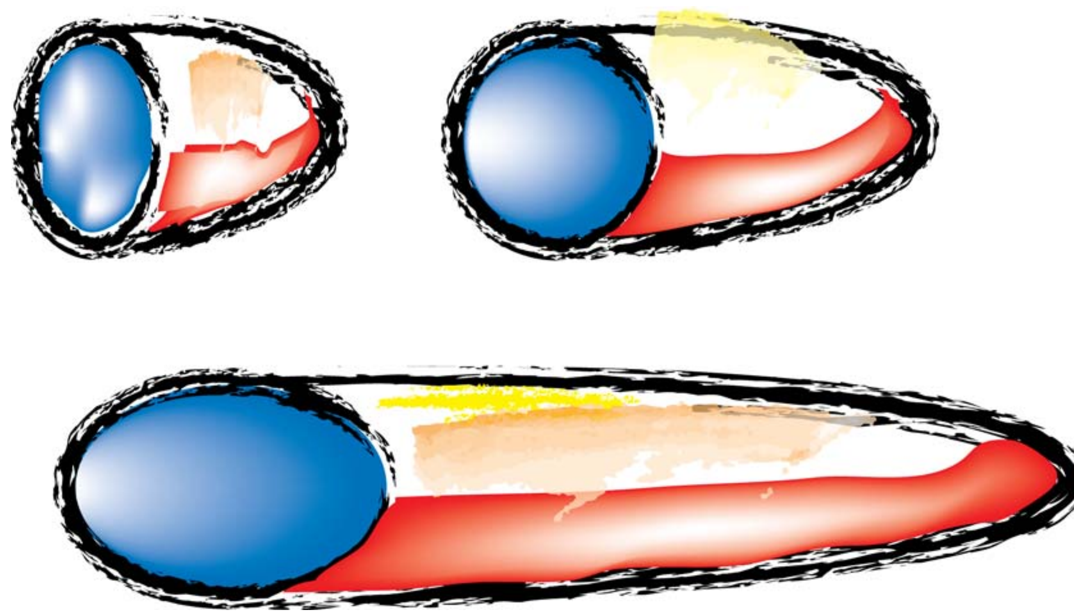
Define $g(\beta) = \varphi_\beta^* g_0$, $-\infty < \beta < +\infty$, where $\varphi_\beta \in Diff(R^2)$ is provided by

$$Ric_{g_0} = \frac{2}{1 + r^2} g_0$$

$$\varphi_\beta(x, y) = e^{-2\beta}(x, y)$$

Set $V = \frac{d}{d\beta}\varphi_\beta|_0 = -2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$, then $L_V\, g = -2\, Ric(g)$

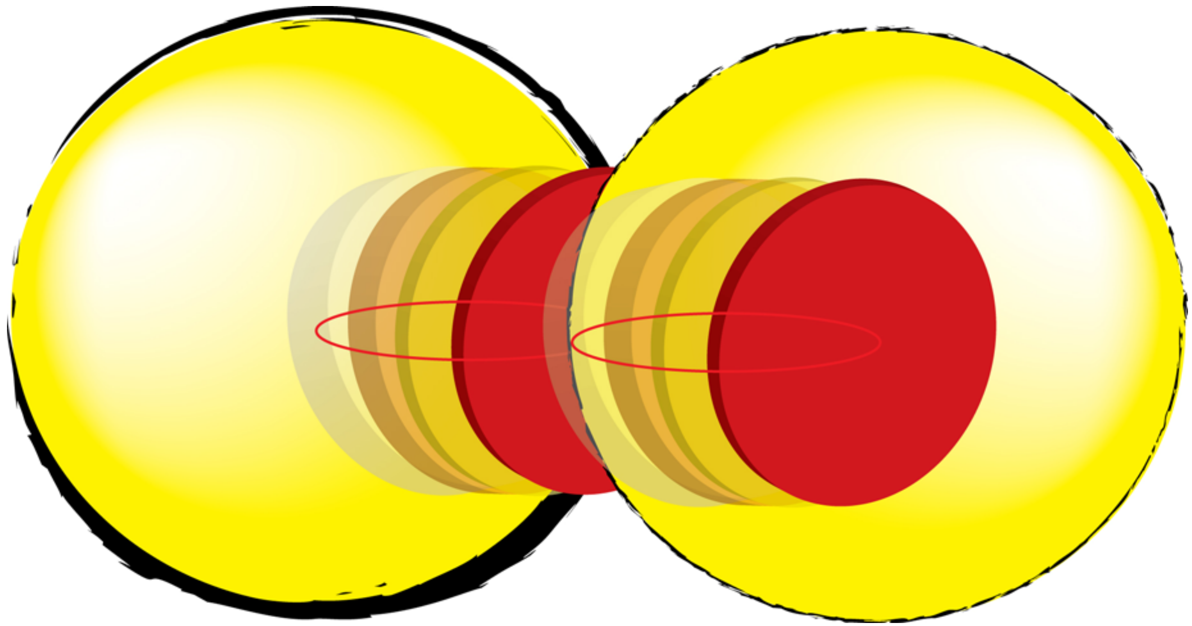
$$\frac{d}{d\beta}\varphi_\beta^* g_0 = -2Ric(\varphi_\beta^* g_0)$$



Gradient soliton with $V = \nabla f$,
$f = -\frac{1}{2}\ln(1 + x^2 + y^2)$.
Note that $g_0 = e^{2f}(dx^2 + dy^2)$.

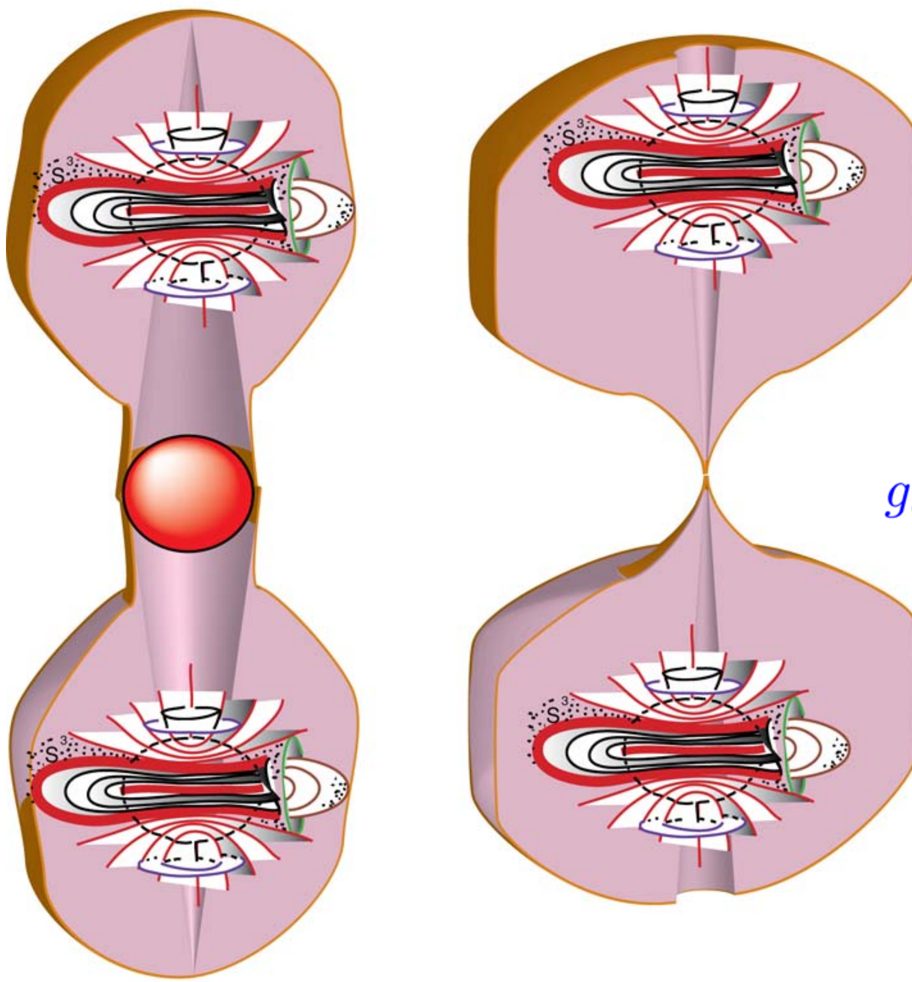
In general, there are topological restrictions to the globalization of the Ricci flow because not all 3-manifolds admit Einstein metrics. The flow must develop singularities.

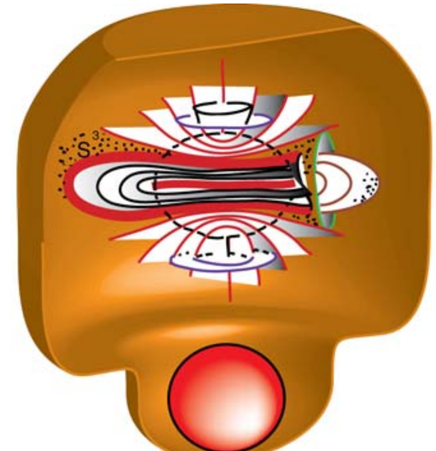
Singularities of various types. Interesting behavior on necks:



The Ricci flow tends to perform surgery on $S^2 \times I$ necks.

Since the segment I has no curvature, the flow collapses the S^2 factor and for a suitable region of the neck we have a collapsing cylindrical metric

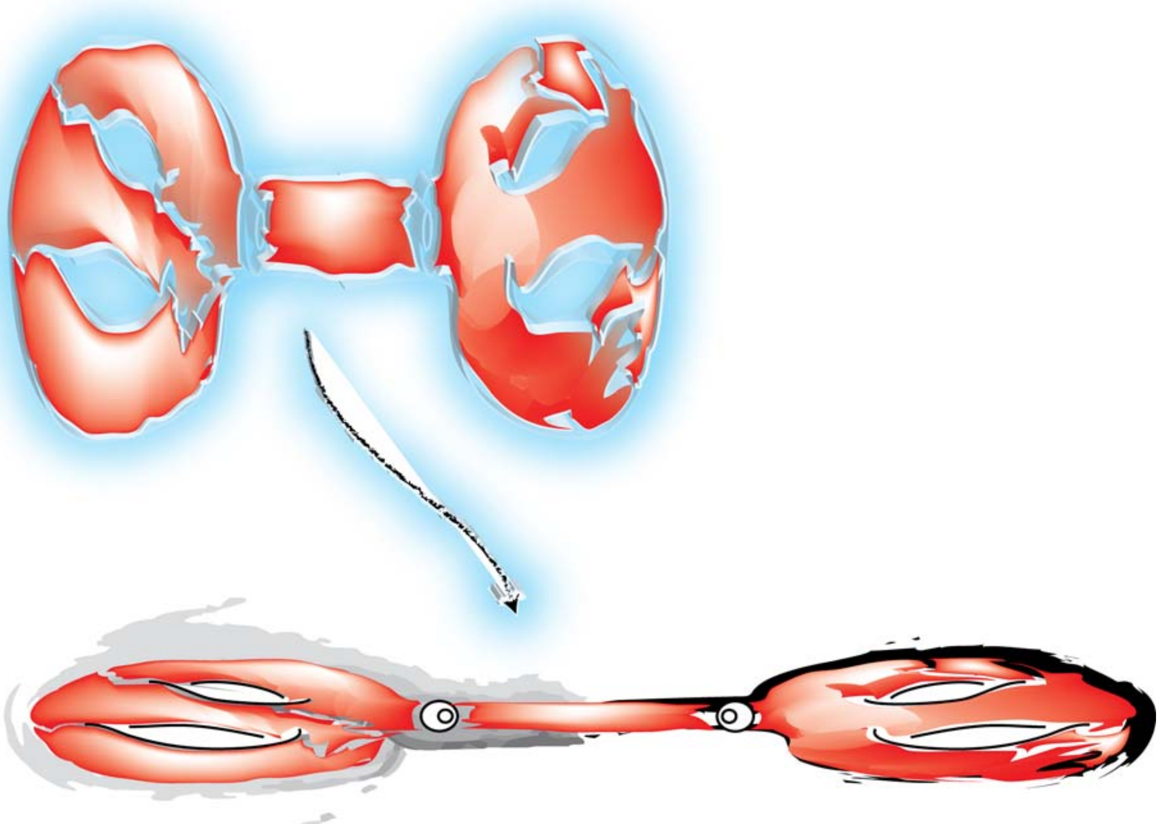




$$g_{cyl}(\beta) = r^2(\beta) g_{can} + dy^2$$

(Flat) Tori-necks $T^2 \times I$ connecting hyperbolic 3-manifolds.

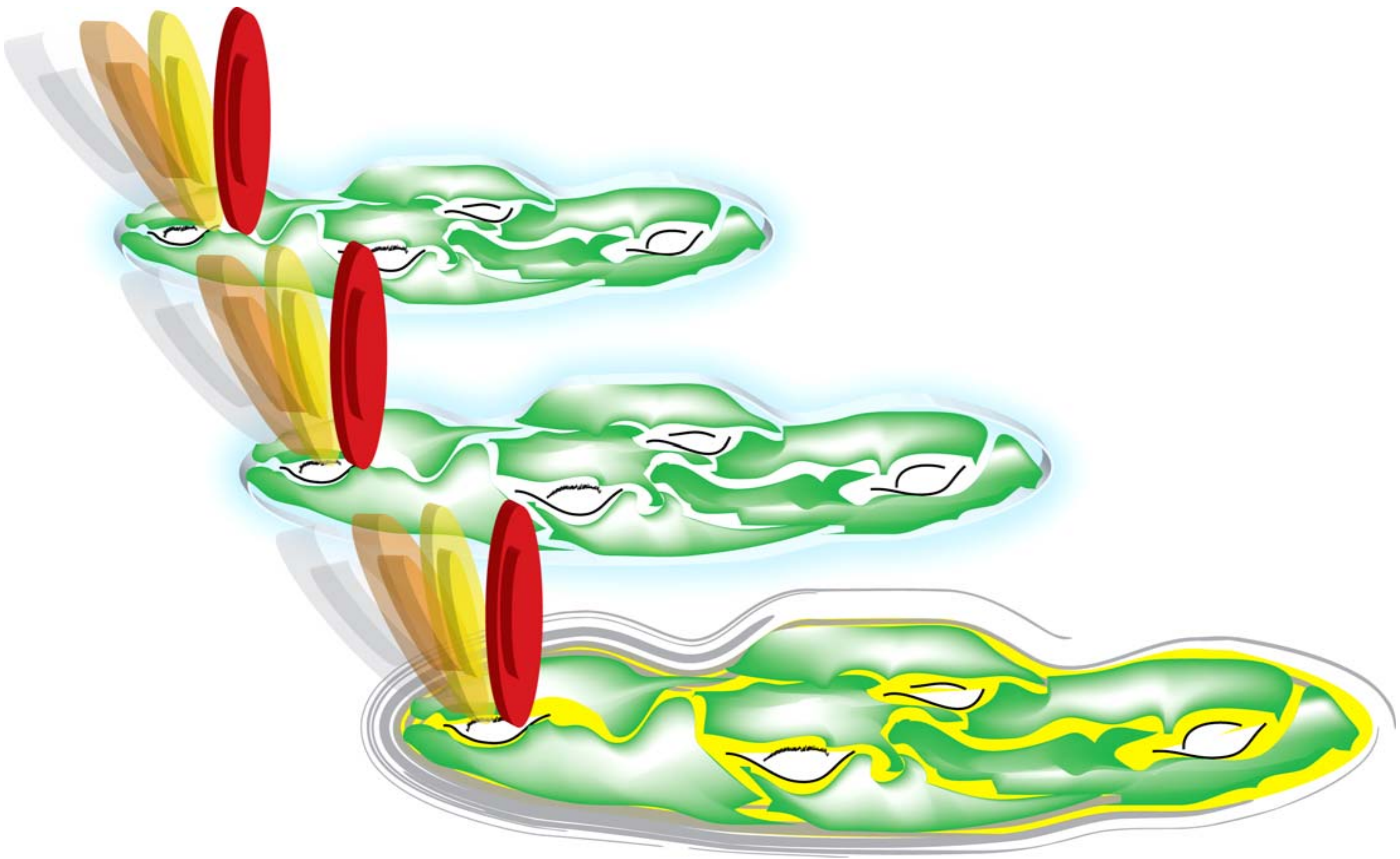
Since the 2-torus is flat, the neck is flat or has slightly negative curvature.

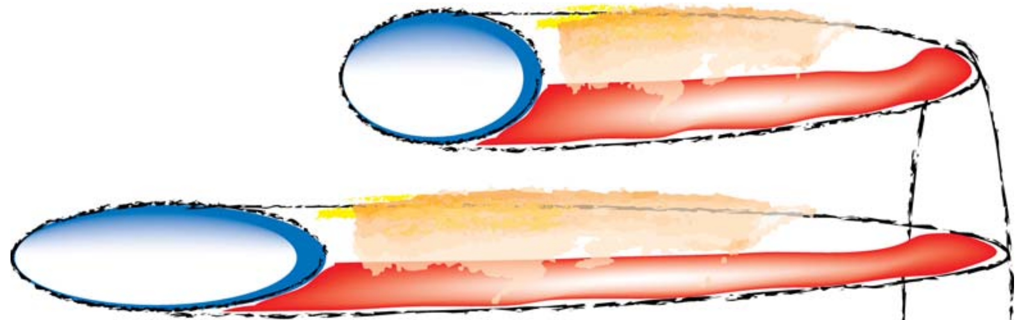


The neck expands slowly in comparison to the hyperbolic pieces which expands much more rapidly. One expects the solution to exists for all β, with the manifold approaching flatness at a rate $\propto \frac{1}{\beta}$. Neck pinches if we keep the volume of the hyperbolic ends constant.

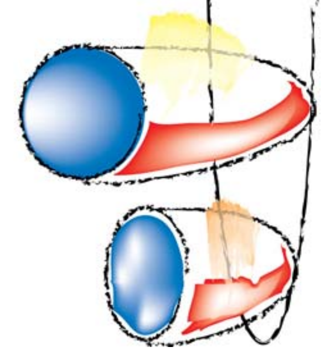
Product metrics on circle bundles $S^1 \times \Sigma$ over surfaces Σ.

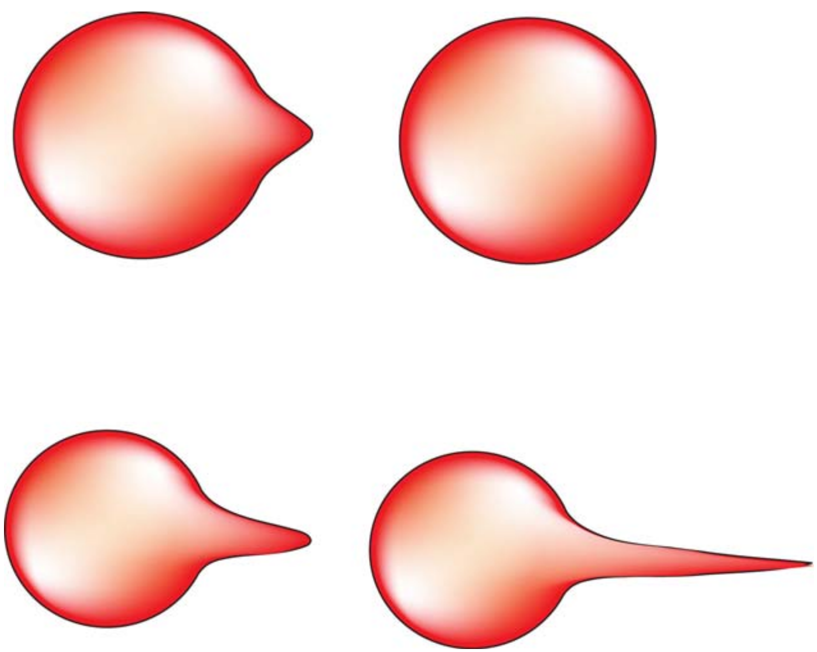
The length of the circle factor stays constant (is flat) while the surface factor expands or contracts according to the sign of surface curvature

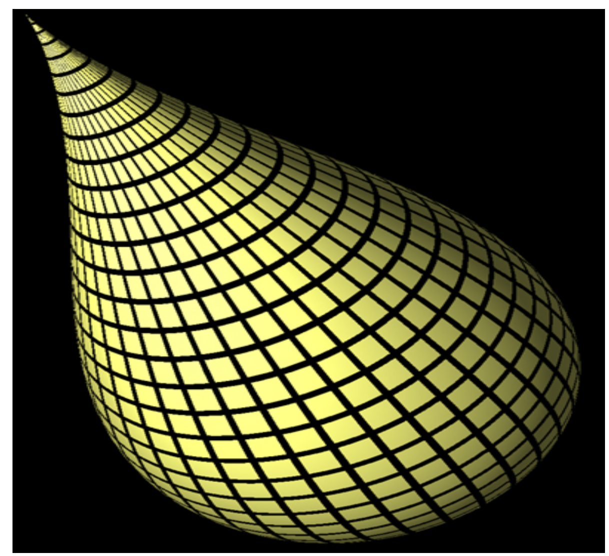




Possible and potentially dangerous singular solutions.

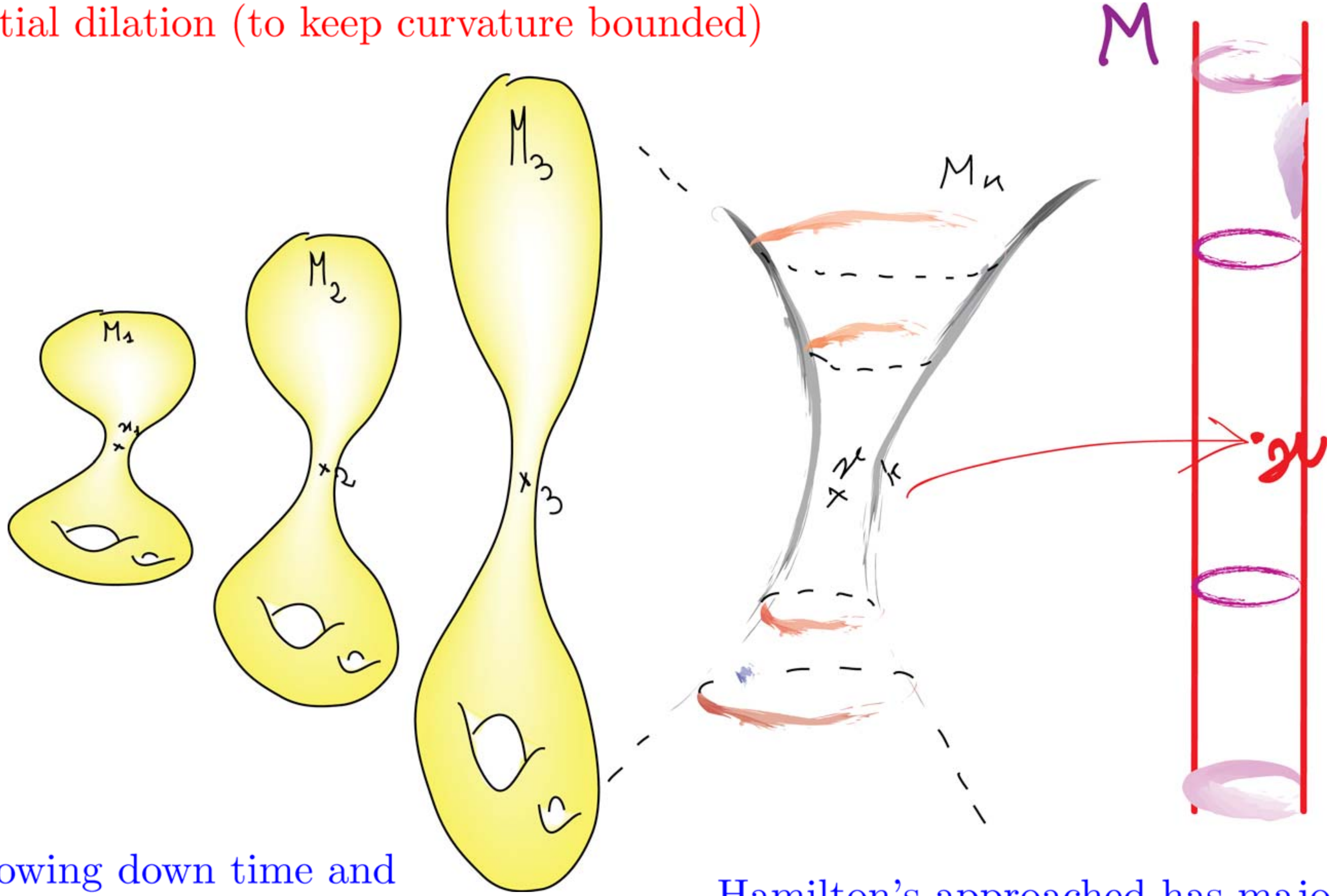






Singularity models are discussed by means of Parabolic Rescaling

Spatial dilation (to keep curvature bounded)



Slowing down time and its reinitialization to 0

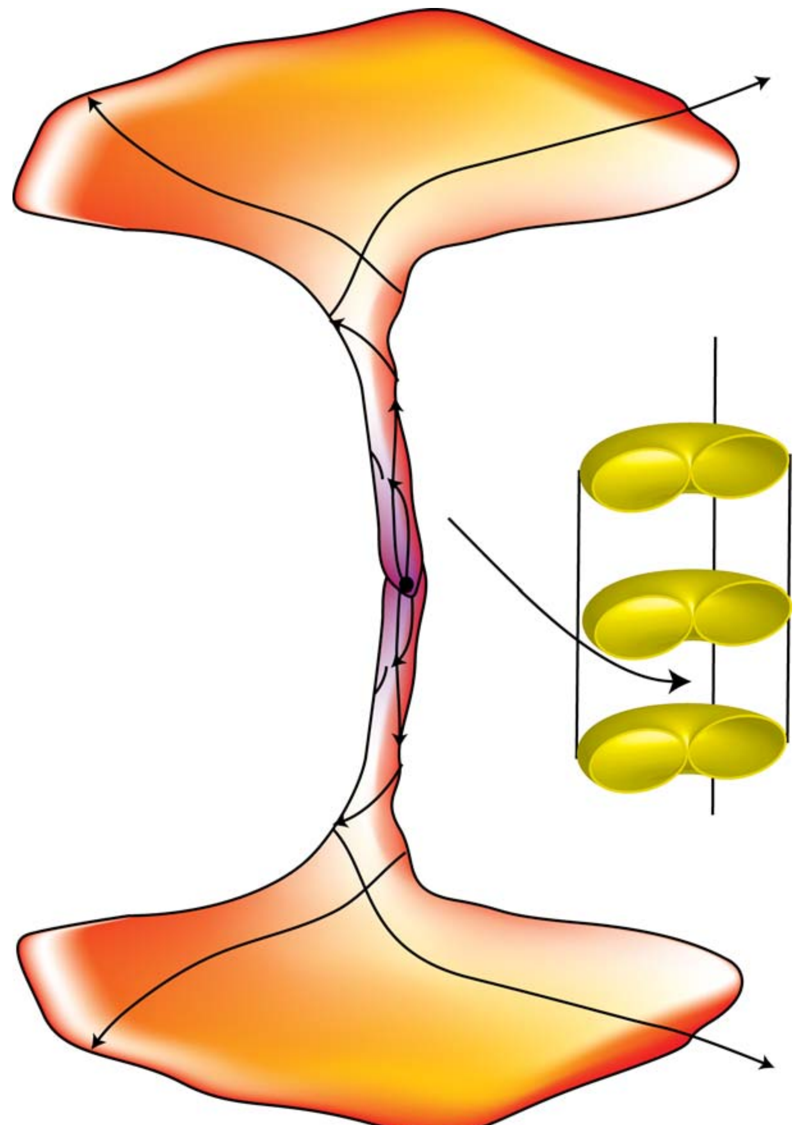
Hamilton's approached has major difficulties in controlling the limit of parabolic rescalings

What goes wrong?

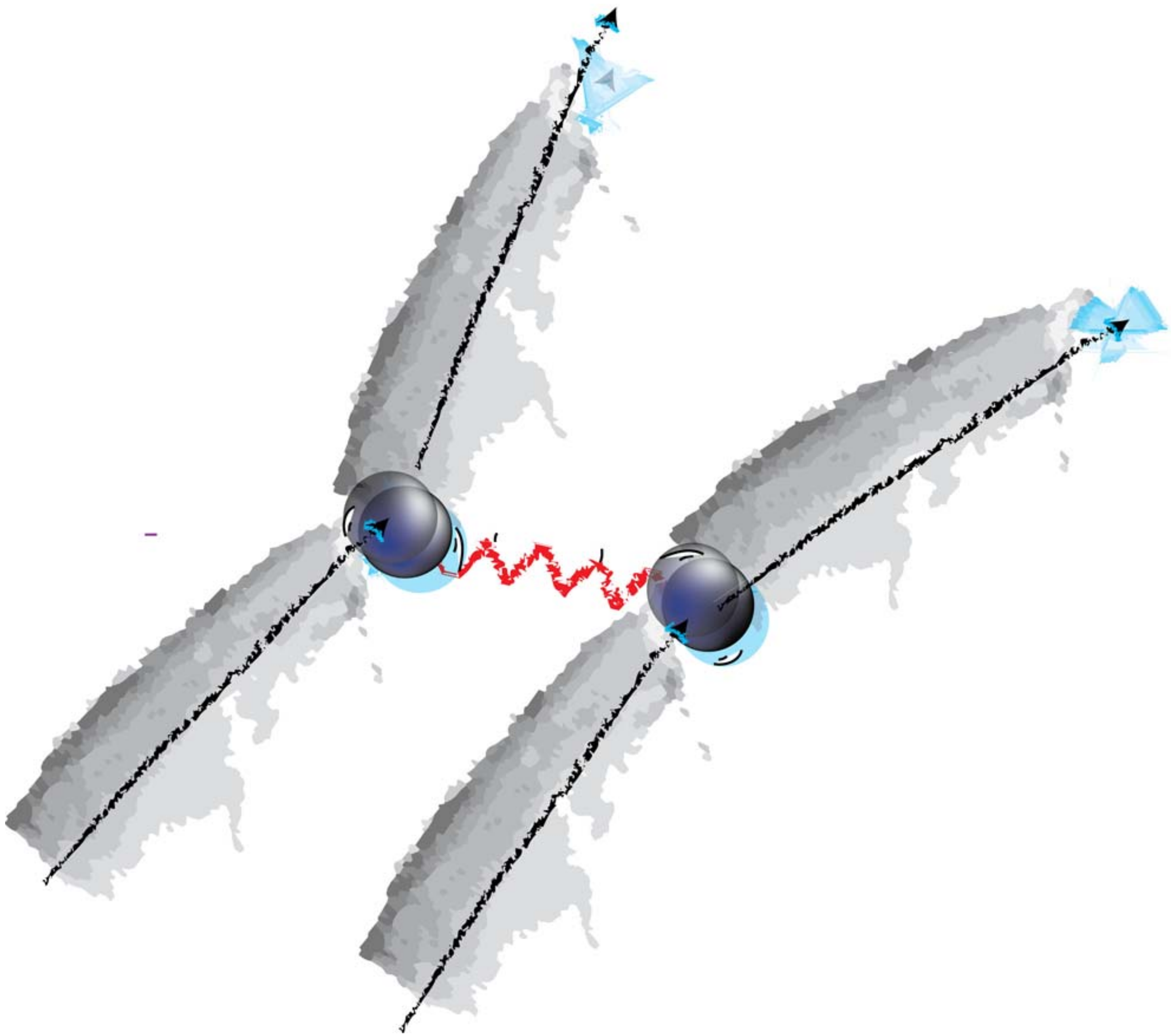
The sequence of parabolically rescaled Ricci flows can collapse to a lower dimensional object. The scale-invariant quantity

$$inj(x_i)^2 |Riem(x_i)|$$

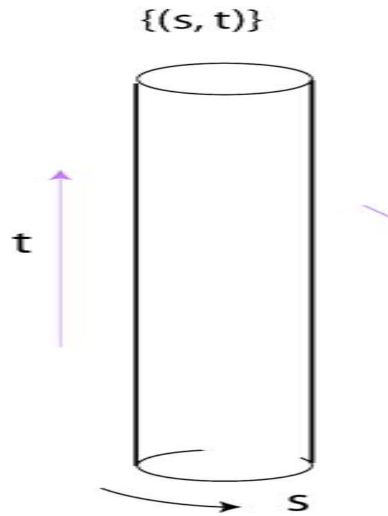
may not be bounded away from 0. With respect to the size of the local curvature, the radius over which geodesics start crossing goes to zero too fast. The manifold is not locally modelled after R^3 any longer.

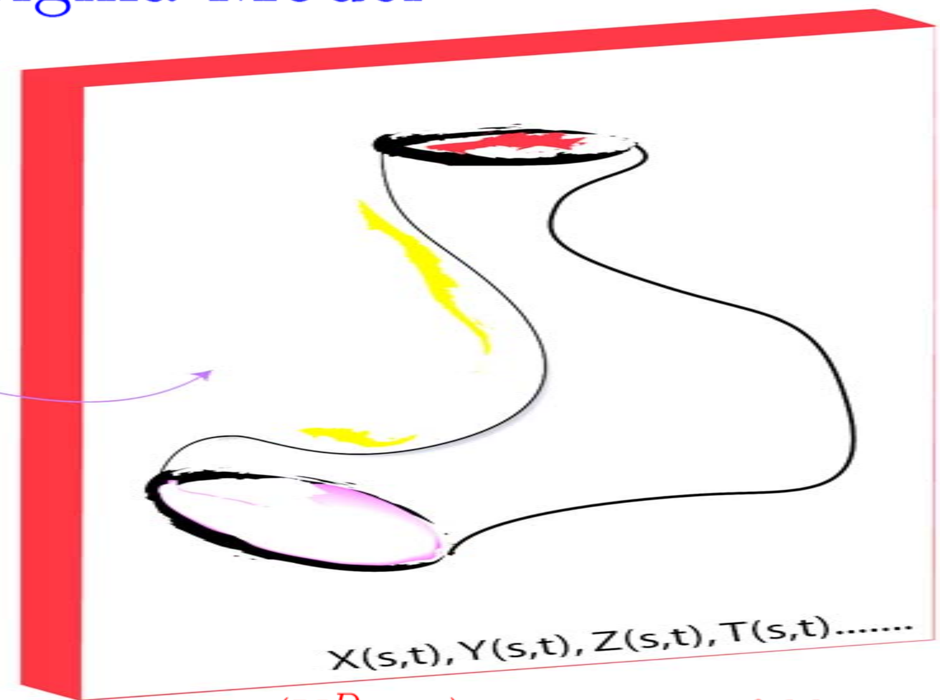


Suggestions from Theoretical Physics



Non linear Sigma Model





$(M, h_{\alpha\beta})$: surface M worldsheet.

$h_{\alpha\beta}$: metric on the surface M.

(V^D, g_{ab}): target manifold.

(Bosonic) WORLDSHEET ACTION

$$S = \frac{1}{4\pi l_s^2} \int_M d^2\xi \sqrt{h} \left(h^{\alpha\beta} \partial_\alpha X^a \partial_\beta X^b g_{ab} + \varepsilon^{\alpha\beta} \partial_\alpha X^a \partial_\beta X^b B_{ab} + l_s^2 R^{(2)}(h) \Phi(\xi) \right)$$

Geometrical Coupling "Constants"

g_{ab}: Target manifold metric

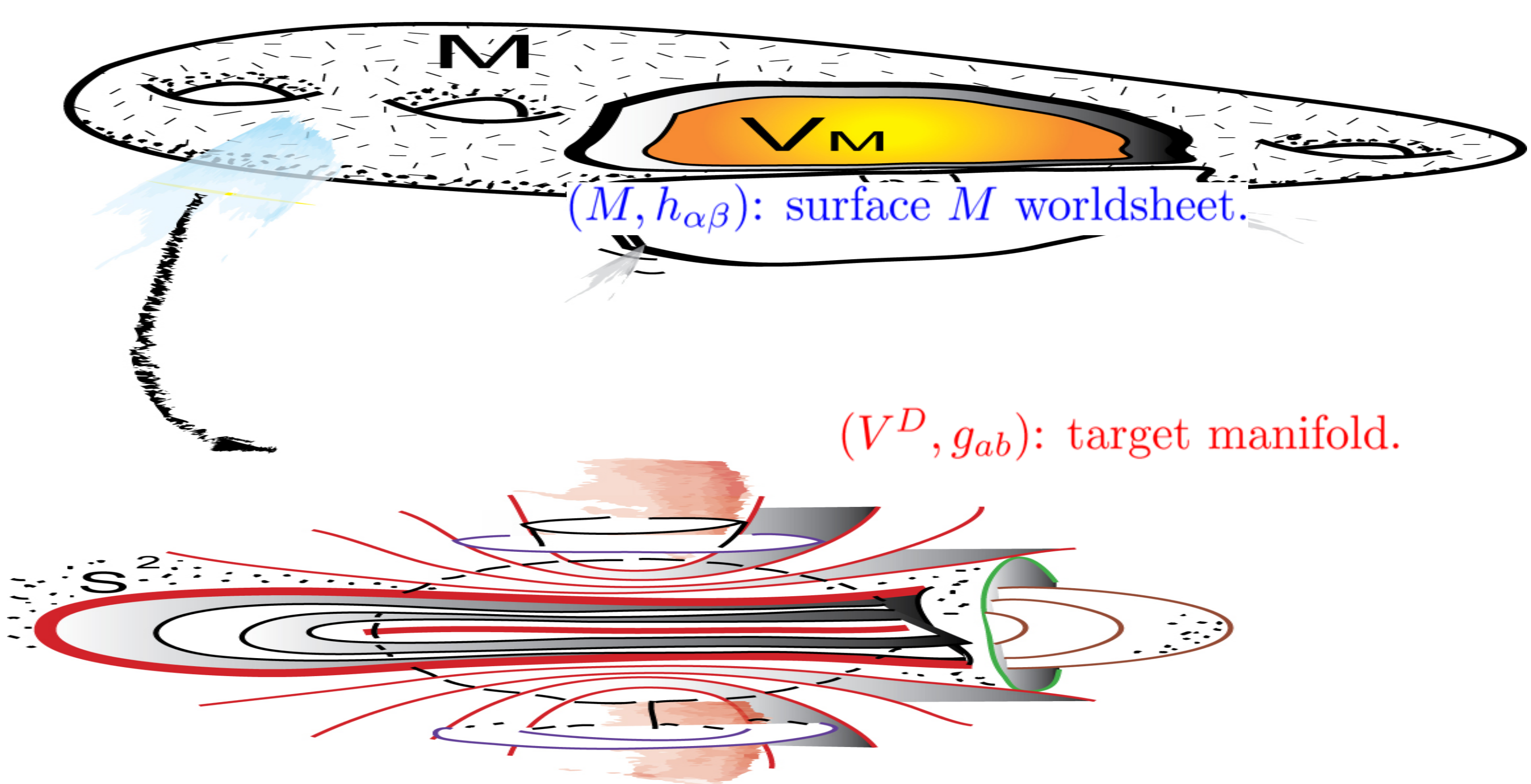
B_{ab}: B–field, a 2-form

Φ: The dilaton field

l_s^2: length scale associated with the physics of the problem e.g. string tension

Dan Friedan (1985) showed that sigma models have a renormalization group flow which is the Ricci flow.

When curvature of target ($\simeq l^{-2}$) is small, (and h flat), S is perturbatively renormalizable in terms of the parameter $(l_s/l)^2$.

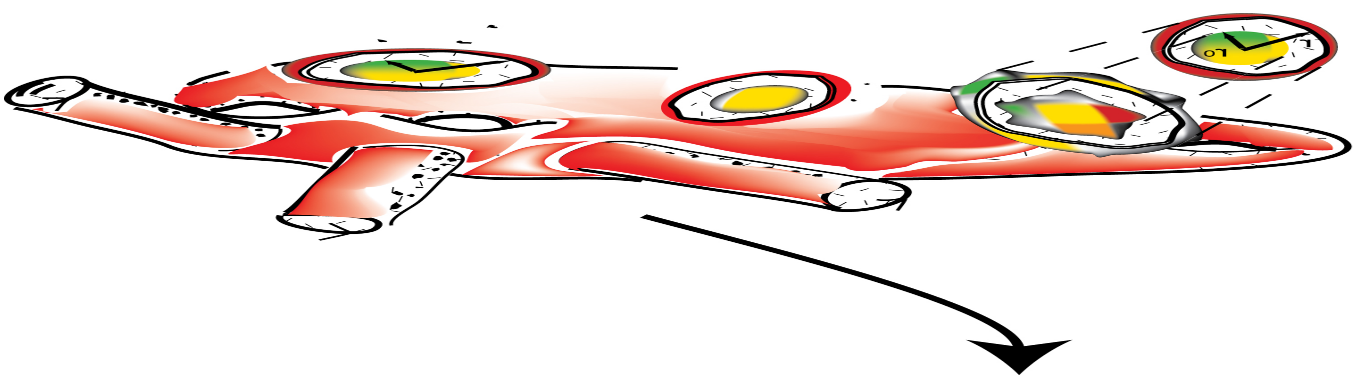


If we set $B = 0$, $\Phi = 0$, then the RG equations provide

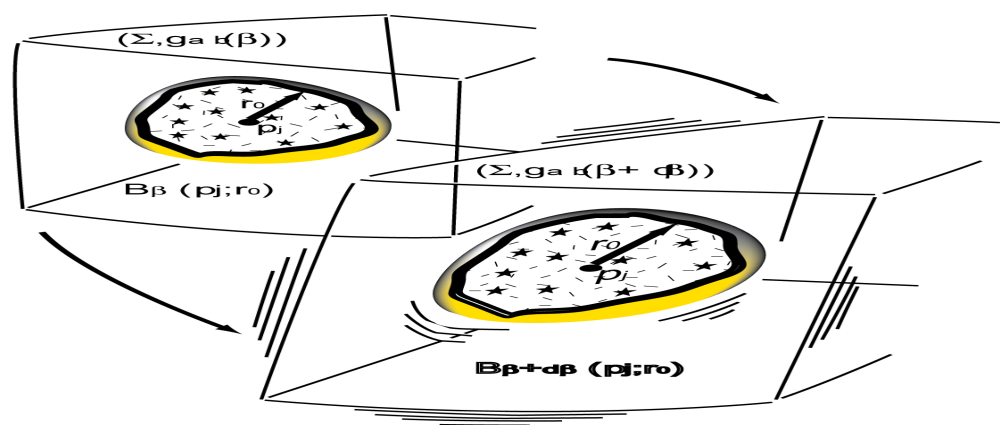
$$L^{-1}\frac{\partial}{\partial L^{-1}} g_{ab} = -\beta(g_{ab}) = -l_s^2 R_{ab} - \frac{1}{2} l_s^4 (R_{acde} R_b^{cde}) + \ldots \qquad L \doteq (l_s/l)^2$$

In logarithmic scale $t = \log L^{-1}$: $t \to -\infty$ UV, $t \to +\infty$ IR.

To lowest order in perturbation theory, changes of target space geometry induced by changes of the world-sheet logarithmic scale are described by

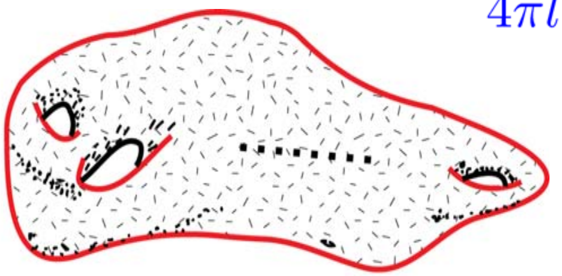


$$l_s^{-2}\frac{\partial}{\partial t} g_{ab} = -(2) R_{ab}$$

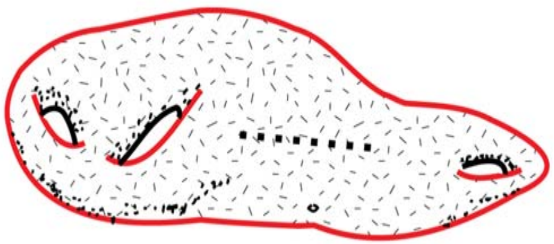


If we consider also
the DILATON field $\Phi(\xi)$:

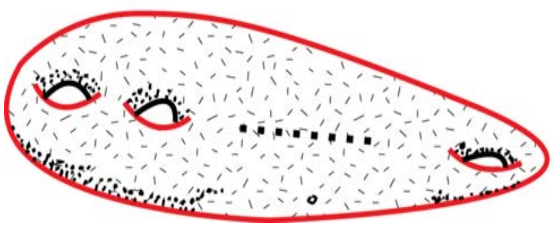
$$S = \frac{1}{4\pi l_s^2} \int_M d^2\xi \sqrt{h} \left(h^{\alpha\beta} \partial_\alpha X^a \partial_\beta X^b g_{ab} + l_s^2 R^{(2)}(h) \Phi(\xi) \right)$$



The RG flow for the metric
and dilatonic couplings is:



$$l_s^{-2} \frac{\partial}{\partial t} g_{ab} = -2R_{ab} + \nabla_a \nabla_b \Phi$$

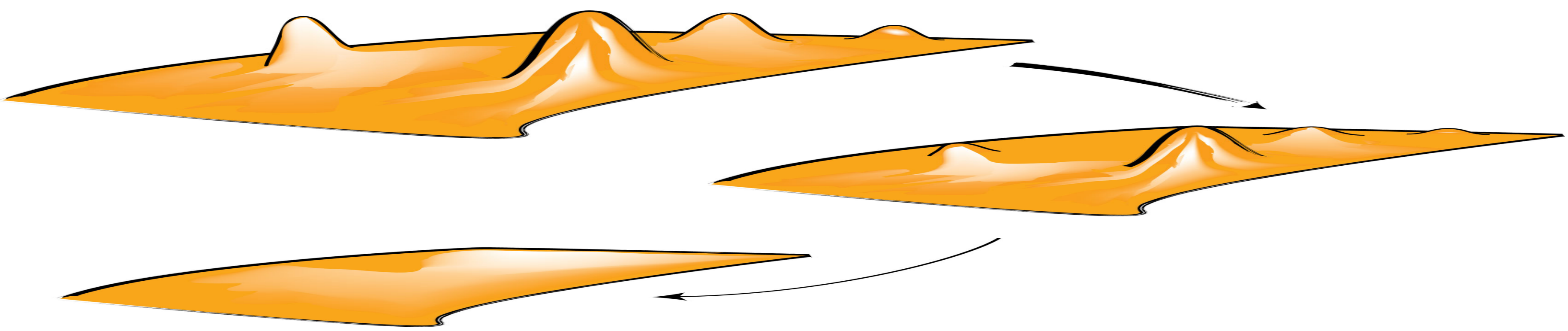


$$l_s^{-2} \frac{\partial}{\partial t} \Phi = -\Delta\Phi + |\nabla\Phi|^2$$

RG Flow Wisdom : Since RG averages out degrees of freedom one expects that the RG flow is dissipative and, hopefully, gradient flow of a suitable c–entropy functional .
(A.B. Zamolodchikov for unitary 2d QFT) .

Natural candidate for c–entropy:

$$\int_{\Sigma} (const. + R(\beta) + 4|\nabla\Phi|^2) e^{-2\Phi} d\mu_g,$$



Impact on Ricci flow theory:
Ricci flow as a gradient flow associated with a suitable entropy functional?
(Yes! G. Perelman) .

(Apparently) inspired by the dilatonic low-energy effective action in string theory, G. Perelman has introduced the functionals

$$F[g; f] \doteq \int_{\Sigma} (R(\beta) + |\nabla f|^2) e^{-f} d\mu_g,$$

$$W[g; f_\beta, \tau] \doteq \int_{\Sigma} \left[\tau \left(|\nabla f_\beta|^2 + R(\beta) \right) + f_\beta - 3 \right] (4\pi\tau(\beta))^{-\frac{3}{2}} e^{-f_\beta} d\mu_{g(\beta)},$$

$$f_\beta : R \longrightarrow C^\infty(\Sigma_\beta, R)$$

$$d\varpi(\beta) \doteq (4\pi\tau(\beta))^{-\frac{3}{2}} e^{-f(\beta)} d\mu_{g(\beta)},$$

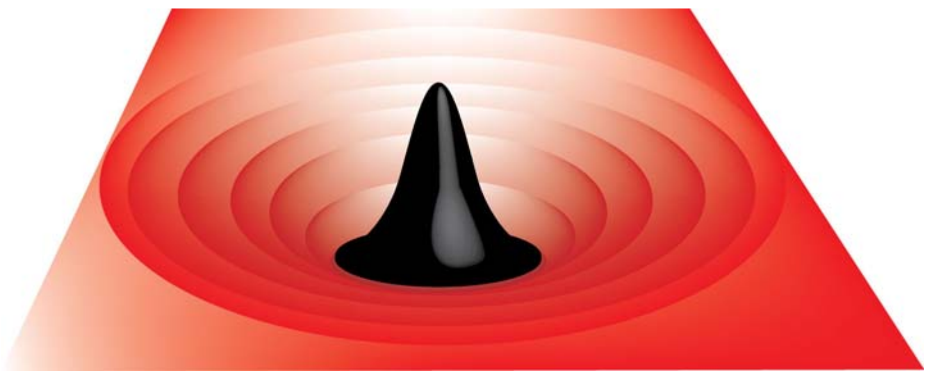
Perelman's Coupling

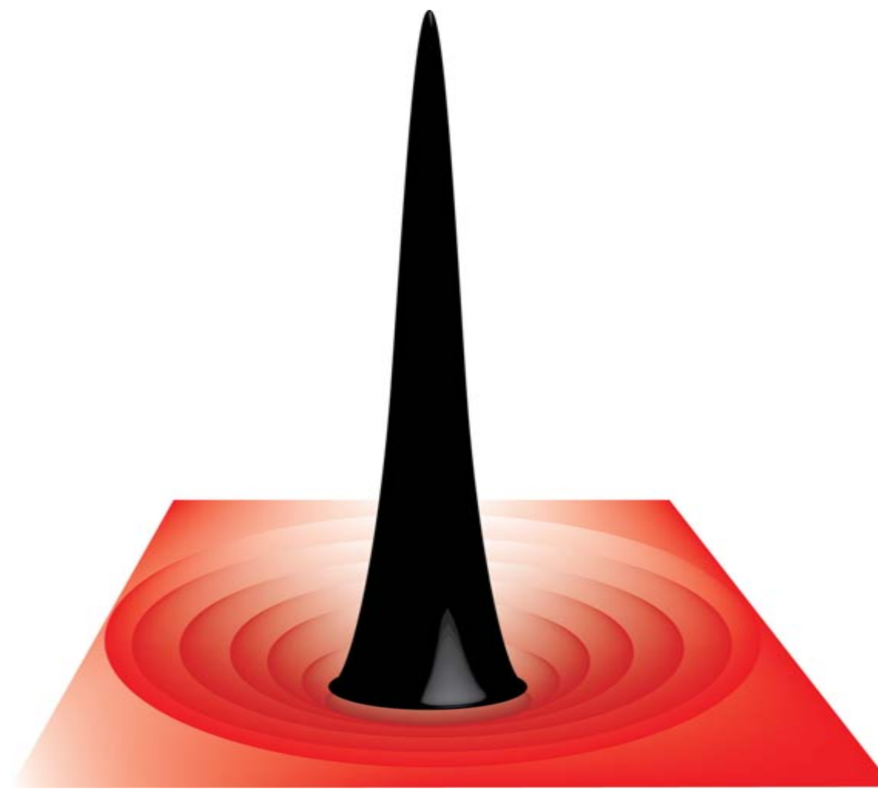
$$\int_{\Sigma_\beta} d\varpi(\beta) = 1.$$

$$\frac{\partial}{\partial\beta} \left[(4\pi\tau(\beta))^{-\frac{3}{2}} \int_{\Sigma_\beta} e^{-f(\beta)} d\mu_{g(\beta)} \right] = 0,$$

$W[g; f_\beta, \tau]$ is invariant under the $d\varpi$-measure preserving diffeomorphisms $Diff(M; d\varpi)$.

Philosophy: Use the dilatonic field f_β (or rather the associated probability measure) to probe the Ricci flowed geometry at a given scale.





These functionals can be used to produce a gradient flow which is the Ricci flow altered by a diffeomorphism, plus a (backward) diffusion for the measure.

$$\begin{cases} \frac{\partial}{\partial\beta} g_{ab}(\beta) = -2R_{ab}(\beta) - 2\nabla_a\nabla_b f, \\ g_{ab}(\beta = 0) = g_{ab}, \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial\lambda} d\varpi(\lambda) = \Delta_{g(\lambda)}\left(d\varpi(\lambda)\right), \\ d\varpi(\lambda = 0) = d\varpi_0. \end{cases}$$

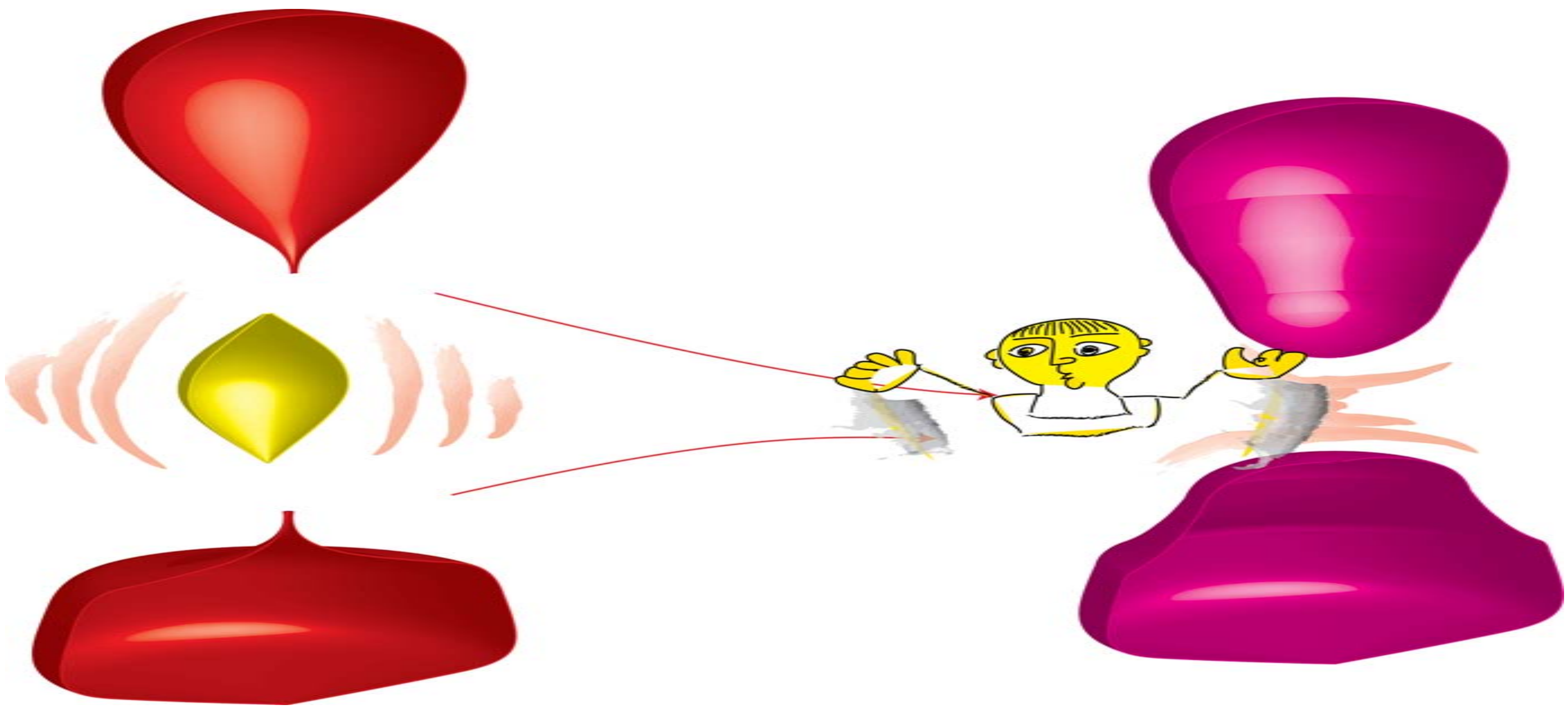
$$\lambda \doteq \beta^* - \beta$$

Since the Dilaton has such an important role in the Ricci flow yielding for Perelman's breakthrough, we wonder if the B–field

may have a similar role in further improving Ricci flow theory?

$$S = \frac{1}{4\pi l_s^2} \int_M d^2\xi \sqrt{h} \left(h^{\alpha\beta} \partial_\alpha X^a \partial_\beta X^b g_{ab} + \varepsilon^{\alpha\beta} \partial_\alpha X^a \partial_\beta X^b B_{ab} + R^{(2)}(h) \Phi(\xi) \right)$$

For instance in improving the surgery around the singularities of the flow.



The insertion of Branes and the associated gauge fields indicates the possibility of providing a framework for blending Ricci flow with the Mean Curvature Flow (G. Huisken)

Dirichlet σ–model

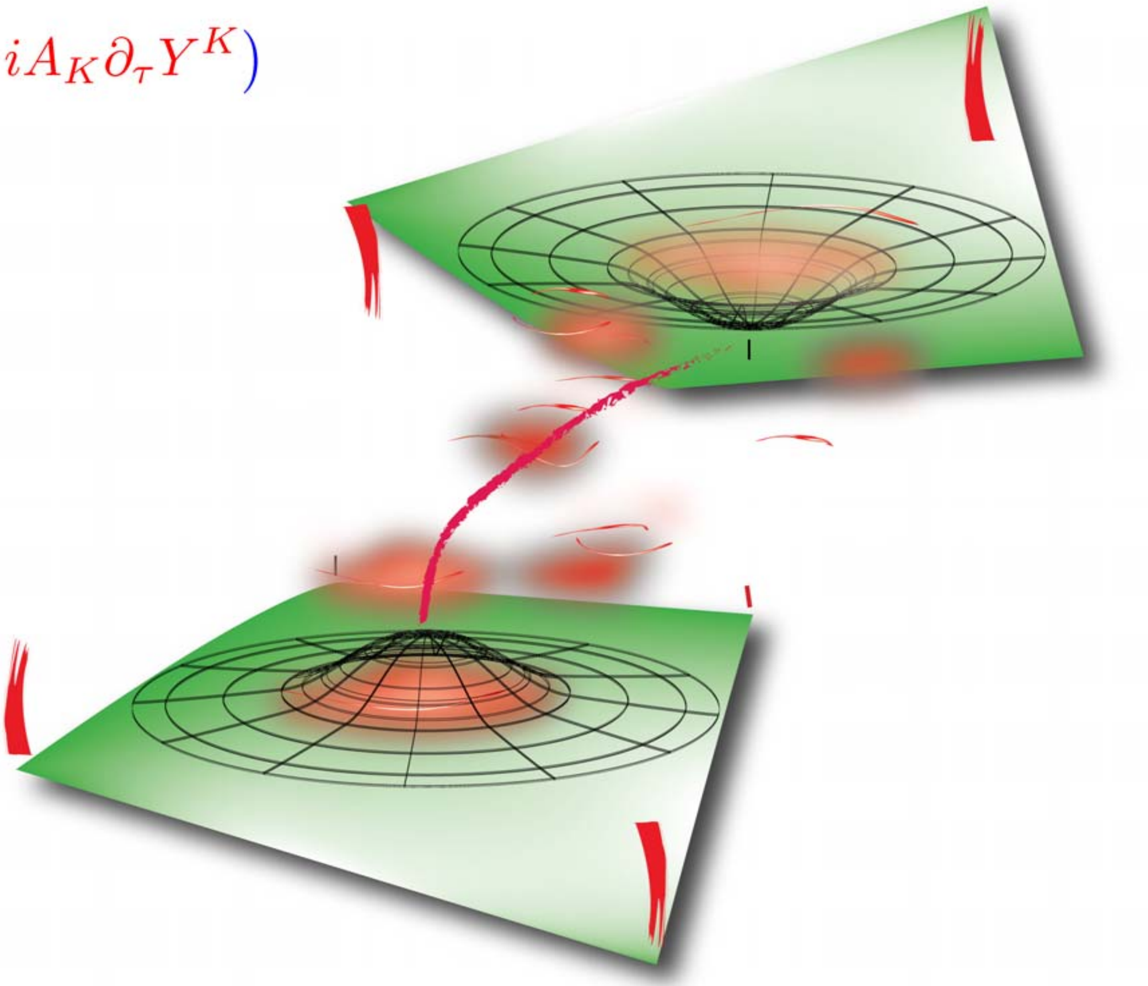
$$S_b = \frac{1}{2\pi l_s^2} \int_{\partial M} d\tau \left(\partial_n X^a v_a - i A_K \partial_\tau Y^K \right)$$

S_b: Boundary action

A: $U(1)$–gauge fields tangent to the boundary submanifold Σ

Y^a: coordinates in Σ

v_a: fields perpendicular to Σ



As $\beta^* \searrow 0^+$, we have the asymptotics

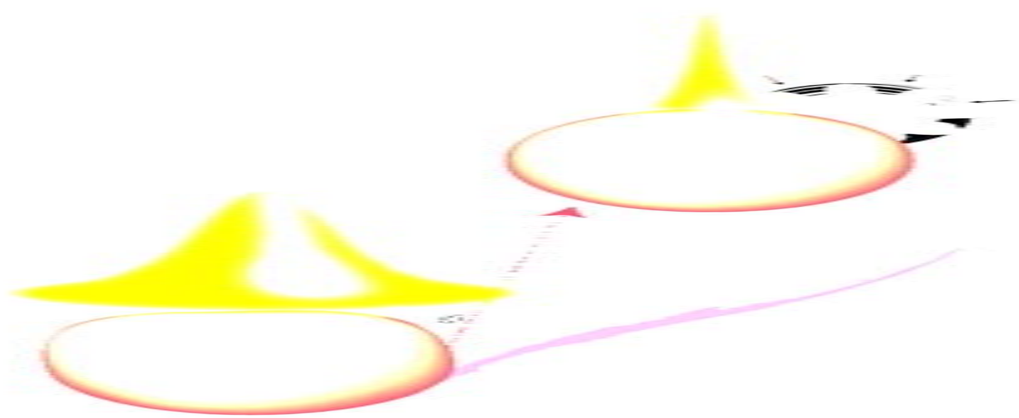
$$g_{i'k'}(y, \beta^*) = \tag{1}$$

$$= \frac{1}{(4\pi\,\beta^*)^{\frac{3}{2}}}\int_\Sigma e^{-\frac{d_*^2(y,x)}{4\beta^*}}\,\tau^{ab}_{i'k'}(y,x;\beta^*)\,\left[g^{(0)}_{ab}(x) - 2\beta^*\,\mathcal{R}^{(0)}_{ab}(x)\right]\,d\mu_{g(x)}$$

$$+\sum_{h=1}^{N}\frac{(\beta^*)^h}{(4\pi\,\beta^*)^{\frac{3}{2}}}\int_\Sigma e^{-\frac{d_*^2(y,x)}{4\beta^*}}\,\Phi[h]^{ab}_{i'k'}(y,x;\beta^*)\left[g^{(0)}_{ab}(x) - 2\beta^*\,\mathcal{R}^{(0)}_{ab}(x)\right]\,d\mu^{(0)}_{g(x)}$$

$$+O\left((\beta^*)^{N-\frac{1}{2}}\right)\,,$$

where $d_*^2(y,x)$, $\tau^{ab}_{i'k'}(y,x;\beta^*)$, and $\Phi[h]^{ab}_{i'k'}(y,x;\beta^*)$ are evaluated on $(\Sigma, g(\beta^*))$, whereas the superscript $^{(0)}$ refers to $(\Sigma, g(\beta = 0))$.



CONCLUSIONS

Partly inspired by a renormalization group approach, Perelman has proposed a solution to Thurston's geometrization conjecture by providing an innovative description of the structure of the singularities of the Ricci flow by exploiting the entropic properties of the dilatonic functionals $F[g; f_\beta]$, $W[g; f_\beta, \tau]$, suggested by NLσM.

It is not yet clear if there is a role for the B–field and of Dirichlet–NLσM.

FURTHER IMPORTANT ISSUES SUGGESTED BY NLσM

(i) Ricci solitons and UV regime in RG flow for σ-models
(ii) Ancient solutions, Perelman's compactness theorem and space of local QFT: models for backward evolution, without singularities, of RG flow.
(iii) In general it is important, from the point of view of QFT, to understand geometric flows which extend backward in time.

For more details see A. Tseytlin, A. Zamolodchikov, E.Woolgar, M.C., and J. Bakas' talks at http://www.aei.mpg.de/ olito/WS/Talks/

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