REDUCED HAMILTONIAN FOR INTERSECTING SHELLS

F. Fiamberti and P. Menotti

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Introduction

Semiclassical quantum gravity

Previous work: W. Fischler, D. Morgan, J. Polchinski; E. Farhi, A. Guth, J. Guven; P. Kraus F. Wilczek; M. Parikh, F. Wilczek

Related work: S. Ansoldi, A. Aurilia, R. Balbinot, E. Spallucci; M. Nadalini, L. Vanzo, S. Zerbini; M. Angheben, M. Nadalini, L. Vanzo, S. Zerbini

Main connection with: P. Kraus F.Wilczek; J. Friedman , J. Louko S. Winters-Hilt; J. Louko, B. Whiting, J. Friedman

Shell dynamics: P. Hajicek; J. Kijowski, I. Kouletsis

Related results: H. Redmount; T. Dray-G. 't Hooft

Suggestions:

K-W: Extend to two or more shells; correlations ? Information exchange?

Critics:

1. **Very complicated**
2. **Limit gauge**
3. **Extension to two or more shells appear singular**

The action

ADM form

$$ds^2 = -N^2dt^2 + L^2(dr + N^r dt)^2 + R^2 d\Omega^2 \tag{1}$$

We shall work on a finite region of space time $(t_i, t_f) \times (r_0, r_m)$**. On the two initial and final surfaces we give the intrinsic metric by specifying** $R(r, t_i)$ **and** $L(r, t_i)$ **and similarly** $R(r, t_f)$ **and** $L(r, t_f)$**. The complete action in hamiltonian form, boundary terms included is (Hawking-Hunter)**

$$S = S_{shell} + \int_{r_0}^{r_m} dr(\pi_L \dot{L} + \pi_R \dot{R} - N\mathcal{H}_t - N^r\mathcal{H}_r) + \tag{2}$$

$$\int dt\, (-N^r\pi_L L + \frac{NRR'}{L})\Big|_{r_0}^{r_m}$$

where

$$S_{shell} = \int_{t_i}^{t_f} dt \left(\hat{p}\dot{\hat{r}} - N(\hat{r}, t)\sqrt{\frac{\hat{p}^2}{L^2} + m^2} + N^r(\hat{r}, t)\hat{p}\right) \tag{3}$$

immediately generalized to a finite number of shells

The constraints are given by [?, ?, ?]

$$\mathcal{H}_r = \pi_R R' - \pi_L' L - \hat{p}\ \delta(r - \hat{r}), \tag{4}$$

$$\mathcal{H}_t = \frac{RR''}{L} + \frac{R'^2}{2L} + \frac{L\pi_L^2}{2R^2} - \frac{RR'L'}{L^2} - \frac{\pi_L \pi_R}{R} - \frac{L}{2} + \sqrt{\hat{p}^2 L^{-2} + m^2}\ \delta(r - \hat{r}). \tag{5}$$

To go straight to the Kraus-Wilczek-Parikh result

$$S = \int p_c d\hat{r} + b.t.$$

and (the mass is taken 0**)**

$$p_c = \sqrt{2M\hat{r}} - \sqrt{2H\hat{r}} - \hat{r}\log\frac{1-\sqrt{\frac{2H}{\hat{r}}}}{1-\sqrt{\frac{2M}{\hat{r}}}}$$

Thus

$$i\mathrm{Im}\int p_c d\hat{r} = i\pi\int_{2M}^{2H}\hat{r}d\hat{r} = 2\pi i(H^2-M^2) = 4\pi i(M+\frac{\omega}{2})\omega$$

$$e^{-4\pi(H^2-M^2)} = e^{-8\pi(M+\frac{\omega}{2})\omega} =$$

and thus

$$\frac{1}{k_B T} = 8\pi(M+\frac{\omega}{2})$$

Hawking temperature with back reaction correction.

In terms of semiclassical modes and Bogoliubov transformation

E. Keski-Vakkuri, P. Kraus, Nucl.Phys.B491(1997) 249

From the constraints [?]

$$\mathcal{M} = \frac{\pi_L^2}{2R} + \frac{R}{2} - \frac{R\,(R')^2}{2L^2}, \tag{6}$$

is constant in the regions of r where there are no sources as

$$\mathcal{M}' = -\frac{R'}{L}\mathcal{H}_t^0 - \frac{\pi_L}{RL}\mathcal{H}_r^0 = 0 \tag{7}$$

The equations of motion for the gravitational field are [?]

$$\dot{L} = N\left(\frac{L\pi_L}{R^2} - \frac{\pi_R}{R}\right) + (N^r L)', \tag{8}$$

$$\dot{R} = -\frac{N\pi_L}{R} + N^r R' \tag{9}$$

$$\dot{\pi}_L = \frac{N}{2}\left[-\frac{\pi_L^2}{R^2} - \left(\frac{R'}{L}\right)^2 + 1 + \frac{2\ \hat{p}^2\ \delta(r-\hat{r})}{L^3\sqrt{\hat{p}^2\hat{L}^{-2}+m^2}}\right] - \frac{N'R'R}{L^2} + N^r\pi_L', \tag{10}$$

$$\dot{\pi}_R = N\left[\frac{L\pi_L^2}{R^3} - \frac{\pi_L\pi_R}{R^2} - \left(\frac{R'}{L}\right)'\right] - \left(\frac{N'R}{L}\right)' + (N^r\pi_R)'. \tag{11}$$

Remark: The equations of motion for $\dot{\hat{r}}$ and $\dot{\hat{p}}$ follow from the above equations for R, L, π_R, π_L.

$$\dot{\hat{r}} = \frac{\hat{N}\hat{p}}{\hat{L}^2\sqrt{\hat{p}^2\hat{L}^{-2}+m^2}} - \hat{N}^r \tag{12}$$

$$\dot{\hat{p}} = \frac{\hat{N}\hat{p}^2\overline{L'}}{\hat{L}^3\sqrt{\hat{p}^2\hat{L}^{-2}+m^2}} - \overline{N'}\sqrt{\hat{p}^2\hat{L}^{-2}+m^2} + \hat{p}\ \overline{(N^r)'} \tag{13}$$

Not from a true variational procedure. In the massless case $m = 0$ no discontinuity

From the equations of motion it follows

$$\frac{d\mathcal{M}}{dt} = -N\frac{R'}{L^3}\mathcal{H}_r^0 - N^r\frac{R'}{L}\mathcal{H}_t^0 - N^r\frac{\pi_L}{RL}\mathcal{H}_r^0 = 0 \tag{14}$$

in the region free of sources M is constant both in r and in t

Eq.(??) allows to solve for the momenta [?, ?]

$$\pi_L = R\sqrt{\left(\frac{R'}{L}\right)^2 - 1 + \frac{2\mathcal{M}}{R}} \equiv RW \tag{15}$$

$$\pi_R = \frac{L[(R/L)(R'/L)' + (R'/L)^2 - 1 + \mathcal{M}/R]}{W} \tag{16}$$

In the one shell problem we shall call the value of such $\mathcal{M}$ M for $r < \hat{r}$ and H for $R > \hat{r}$. The function

$$F = RL\sqrt{\left(\frac{R'}{L}\right)^2 - 1 + \frac{2\mathcal{M}}{R}} + RR'\log\left(\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 - 1 + \frac{2\mathcal{M}}{R}}\right) + \tag{17}$$

$$-R'\ f'(R)$$

has the property of generating of the conjugate momenta as follows

$$\pi_L = \frac{\partial F}{\partial L} \tag{18}$$

$$\pi_R = \frac{\delta F}{\delta R} = \frac{\partial F}{\partial R} - \frac{\partial}{\partial r}\frac{\partial F}{\partial R'} \tag{19}$$

The total derivative $\frac{\partial f(R)}{\partial r} = R' f'(R)$ **of the arbitrary function** $f(R)$ **does not contribute to the momenta. Choose the arbitrary function** $f(r)$ **such that** $F = 0$ **for** $R' = 1$**. Such a requirement fixed** $f'(R)$ **uniquely.**

Large gauge freedom. With regard to L **we shall adopt the usual gauge** $L = 1$**. Then we have**

$$F(R, R', \mathcal{M}) = R\sqrt{(R')^2 - 1 + \frac{2\mathcal{M}}{R}} + RR' \left[\log(R' - \sqrt{(R')^2 - 1 + \frac{2\mathcal{M}}{R}}) \right.$$

$$\left. -\sqrt{\frac{2\mathcal{M}}{R}} - \log\left(1 - \sqrt{\frac{2\mathcal{M}}{R}}\right)\right] \tag{20}$$

For R **we shall choose** $R = r$ **for** $r > \hat{r}$ **and also** $R = r$ **for** $r < \hat{r} - l$**. We shall call this class of gauges "outer gauges". Alternatively one can consider the gauge** $R = r$ **for** $r < \hat{r}$ **and also** $R = r$ **for** $r > \hat{r} + l$ **which we shall call "inner gauges".**

Contrary to what is done in [?, ?] we will not take any limit $l \to 0$ and prove that the results are independent of the deformation of R.
Variation of S by keeping fixed the metric and in particular R and L fixed at the boundaries. Such a variation is given by

$$-N^r(r_m)\delta\pi_L(r_m) + N^r(r_0)\delta\pi_L(r_0) \tag{21}$$

The π_L can be obtained from the two equations of motion for the gravitational field

$$0 = N[\frac{\pi_L}{R^2} - \frac{\pi_R}{R}] + (N^r)'; \qquad \dot{R} = -N\frac{\pi_L}{R} + N^r R' \tag{22}$$

Outside the deformation region $(\hat{r}-l, \hat{r})$ we have $N^r = N\sqrt{\frac{2H}{r}}$ for $r > \hat{r}$, $N = \text{const}$ and $N^r = N\sqrt{\frac{2M}{r}}$ for $r < \hat{r}$, $N = \text{const}$ where the two constants differ. Thus the variation of the boundary term is

$$-N(r_m)\delta H + N(r_0)\delta M. \tag{23}$$

In the next section we shall also connect $N(r_m)$ with $N(r_0)$ being $N(r)$ not constant in the deformation region.

The one shell effective action in the outer gauge

As outlined in the previous section we shall choose

$$R(r,t) = r + \frac{V(t)}{\hat{r}(t)} \int_0^r \rho(r' - \hat{r}) dr' = r + \frac{V(t)}{\hat{r}} g(r - \hat{r}) \qquad (24)$$

with ρ **with support** $(-l, 0)$**, and** $\rho(0) = 1$ **and**

$$\int_{-l}^{0} \rho(r) dr = 0 \qquad (25)$$

As a consequence also the deformation $g(r)$ **has support in** $(-l, 0)$ **and** $g'(0 - \varepsilon) = 1$**. Such** R **satisfies the discontinuity requirements at** $r = \hat{r}$ **which are imposed by the constraints. The constraints impose the following discontinuity relations at** $\hat{r}$ **(we recall that** $L \equiv 1$**)**

$$\Delta R' = -\frac{V}{R}; \qquad V = \sqrt{\hat{p}^2 + m^2} \qquad (26)$$

and

$$\Delta \pi_L = -\hat{p} \qquad (27)$$

In the outer gauge the bulk gravitational action becomes ($L \equiv 1$**)**

$$S_g = \int_{t_i}^{t_f} I_g \; dt \qquad (28)$$

$$I_g = \int_{r_0}^{\infty}(\pi_L\dot{L}+\pi_R\dot{R})dr = \int_{r_0}^{\infty}\pi_R\dot{R}dr = \int_{r_0}^{\infty}\left((\frac{\partial F}{\partial R} - \frac{\partial}{\partial r}\frac{\partial F}{\partial R'})\dot{R}\right)dr$$

$$\int_{r_0}^{\hat{r}(t)}\left(\frac{dF}{dt} - \frac{\partial}{\partial r}\left(\frac{\partial F}{\partial R'}\dot{R}\right)\right)dr =$$

$$= \frac{d}{dt}\int_{r_0}^{\hat{r}(t)} F dr - \dot{M}(t)\int_{r_0}^{\hat{r}(t)}\frac{\partial F}{\partial M}dr - \left[\dot{\hat{r}}(t)F - \frac{\partial F}{\partial R'}\dot{R}\right]_{\hat{r}(t)-\varepsilon} \quad (29)$$

where we used the fact that F vanishes at $r = r_0$.
Adding $I_{shell} = \hat{p}\dot{\hat{r}}$ we obtain for the reduced action in the outer gauge, neglecting the total time derivative which does not contribute to the equations of motion
Reduced action

$$\int_{t_i}^{t_f}\left(p_c\,\dot{\hat{r}} - \dot{M}(t)\int_{r_0}^{\hat{r}(t)}\frac{\partial F}{\partial M}dr\right)dt + (-N^r\pi_L + NRR')\Big|_{r_0}^{r_m} \quad (30)$$

where

$$p_c = -F(\hat{r}(t)-\varepsilon) - \frac{1}{\dot{\hat{r}}(t)}\,\frac{\partial F}{\partial R'}\dot{R}\Big|_{\hat{r}(t)-\varepsilon} + \hat{p} = \quad (31)$$

$$= \sqrt{2M\,\hat{r}} - \sqrt{2H\,\hat{r}} - \hat{r}\log\left(\frac{\hat{r} + \sqrt{\hat{p}^2+m^2} - \hat{p} - \sqrt{2H\,\hat{r}}}{\hat{r} - \sqrt{2M\,\hat{r}}}\right) \quad (32)$$

Comments:

1) No limit $l \to 0$ (FLW) is necessary for obtaining p_c of eq.(??) which holds for any deformation g.

2) The $\dot{M}(t)$ term is important as we shall see if we consider the variational problem in which M is varied. On the other hand if we consider M as a datum of the problem and vary H the contribution $\dot{M}(t)$ is absent.

3) $\hat{p}$ in eq.(??) is a function of $\hat{r}$, H and M as given by the discontinuity relation which is equivalent to the implicit equation

$$H - M = \frac{V}{\hat{r}} + \frac{m^2}{2\hat{r}} - \hat{p}\sqrt{\frac{2H}{\hat{r}}} \tag{33}$$

4) Derivation greatly simplified.

Let us consider first M as a *datum* of the problem ($M = \text{const}$ and vary $H(t)$. This is the situation considered in (KW1, FLW) where the expression (??) for p_c was derived by a limit process in which $l \to 0$. From eq.(??) we see that the equation of motion for $\dot{\hat{r}}$ is given by

$$\dot{\hat{r}}\frac{\partial p_c}{\partial H} - N(r_m) = 0 \tag{34}$$

where $N(r_m)$ can be replaced by $N(\hat{r})$ being $N(r)$ in the outer gauge constant for $r > \hat{r}$. The computation of eq.(??) keeping in mind the implicit definition of $\hat{p}$ eq.(??) gives the correct equations of motion for the massive shell

$$\dot{\hat{r}} = \frac{\hat{p}}{V} N(\hat{r}) - N^r(\hat{r}) = (\frac{\hat{p}}{V} - \sqrt{\frac{2H}{\hat{r}}}) N(\hat{r}) \tag{35}$$

Alternatively one can consider $H = \text{const}$ as a datum of the problem and vary $M(t)$.

The variation of $M(t)$ in the outer gauge is a far more complicated procedure in the outer gauge, due to the presence of $\dot{M}(t)$, but gives rise to the same result obtained by varying H and keeping M fixed.
In this case the $\dot{M}$ term plays a major role; in fact the equation of motion now takes the form

$$\dot{\hat{r}} \frac{\partial p_c}{\partial M} + \frac{d}{dt} \int_{r_0}^{\hat{r}(t)} \frac{\partial F}{\partial M} dr + N^r(r_0) \frac{\partial \pi_L}{\partial M}(r_0) = 0 \tag{36}$$

where due to the vanishing of $g(x)$ for $x < -l$ we have $\pi_L(r_0) = \sqrt{2Mr_0}$. $N^r(r)$ on the other hand is obtained by solving the two coupled equations (??) with the condition

that for $r > \hat{r}$ $N^r(r)$ equals $N(r_m)$. One easily finds that for $r < \hat{r}$

$$N^r(r) = W\left[\int_{\hat{r}}^{r} \dot{R}\frac{\partial \pi_R}{\partial M}dr + \frac{\sqrt{2H\hat{r}}}{\sqrt{2H\hat{r}} + \hat{p}}\right] \tag{37}$$

Taking into account that

$$\frac{d}{dt}\int_{r_0}^{\hat{r}(t)} \frac{\partial F}{\partial M}dr = \int_{r_0}^{\hat{r}} \dot{R}\frac{\partial \pi_R}{\partial M}dr + \left[\dot{\hat{r}}\frac{\partial F}{\partial M} + \frac{\partial^2 F}{\partial M \partial R'}\dot{R}\right]_{\hat{r}-\varepsilon} \tag{38}$$

we obtain

$$\dot{\hat{r}}[\frac{\partial \hat{p}}{\partial M} - \frac{v}{W}\frac{\partial R'}{\partial M}|_{\hat{r}-\varepsilon}] + \frac{\sqrt{2H\hat{r}}}{\sqrt{2H\hat{r}} + \hat{p}} = 0 \tag{39}$$

where $W = \sqrt{R'^2 - 1 + 2M/R}$. Using

$$\frac{\partial \hat{p}}{\partial M} = -\frac{1}{\frac{\hat{p}}{V} - \sqrt{\frac{2H}{\hat{r}}}} \tag{40}$$

we obtain again the correct equation of motion.
Again we remark that no limit process $l \to 0$ is necessary for all these developments.
To summarize, in this section we have derived the reduced action for the one shell problem in the outer gauge with an arbitrary deformation.
1. One can vary H (the exterior ADM mass) considering the interior mass as given,

or

2. One can very the interior mass M considering the exterior mass H as given

or

3. One can vary both M and H always obtaining the correct equations of motion. Whenever M is varied the $\dot{M}$ term in eq.(??) plays a crucial role.

4. All the results do not depend on the deformation g.

The one shell problem in the inner gauge

It is of interest in order to analyze the gauge dependence of the treatment to outline the result in the inner gauge i.e. when $R = r$ for $r < \hat{r}$ and for $r > \hat{r} + l$.
The p_c's are different but the imaginary part of the action, which intervenes in the tunneling process are exactly the same, thus proving the independence of the physics form these gauge choices.
Due to the similarity with the treatment of Sect.(3) we shall go through rather quickly. This time the bulk action takes the form

$$S = \int_{t_i}^{t_f} dt \left(p_c^i \dot{\hat{r}} - \dot{H} \int_{r_0}^{\hat{r}(t)} \frac{\partial F}{\partial H} dr \right) + (-N^r \pi_L + NRR')|_{r_0}^{r_m} \quad (41)$$

where now p_c^i is given by

$$p_c^i = F + \frac{\dot{R}}{\dot{\hat{r}}} \frac{\partial F}{\partial R'}|_{\hat{r}+\varepsilon} + \hat{p} \quad (42)$$

whose explicit value is

$$p_c^i = \sqrt{2M\,\hat{r}} - \sqrt{2H\,\hat{r}} - \hat{r} \log \left(\frac{\hat{r} - \sqrt{2H\,\hat{r}}}{\hat{r} - V + \hat{p} + \sqrt{2M\hat{r}}} \right) \quad (43)$$

and now $\hat{p}$ which is given again by the discontinuity equation (??) is given by the implicit equation

$$H - M = V - \frac{m^2}{2\hat{r}} - \hat{p}\sqrt{\frac{2M}{\hat{r}}} \tag{44}$$

which is different from eq.(??) showing the gauge dependence of $\hat{p}$. Now the simple procedure is the one in which one varies M keeping H as a fixed datum of the problem. This time the solution of the system eq.(??) gives simply $N = \text{const}$ and $N^r = N\sqrt{\frac{2M}{r}}$ for $r_0 < r < \hat{r}$.

The analytic properties of p_c

We saw that in the outer gauge

$$p_c = \sqrt{2M\,\hat{r}} - \sqrt{2H\,\hat{r}} - \hat{r}\log\left(\frac{\hat{r} + V - \hat{p} - \sqrt{2H\,\hat{r}}}{\hat{r} - \sqrt{2M\,\hat{r}}}\right) \tag{45}$$

The solution of eq. for $\hat{p}$ is

$$\hat{p} = \frac{\sqrt{2H\hat{r}}\; b \pm \sqrt{\hat{r}^2(b^2 - m^2\hat{r}^2) + 2H\hat{r}^3 m^2}}{\hat{r}^2 - 2H\hat{r}} \tag{46}$$

where

$$b = \hat{r}(H - M) - \frac{m^2}{2} \tag{47}$$

If we want $\hat{p}$ to describe an outgoing shell we must chose the plus sign in front of the square root. Moreover the shell reaches $r = +\infty$ only if $H - M > m$ as expected.
For $\hat{r} \to 2H + 0$ we have $\hat{p} \to +\infty$
When $\hat{r} \to 2M{+}0$ the argument of the log reverts to positive values.
In order to compute the tunneling amplitude, below $\hat{r} = 2H$ we have to use the prescription $\hat{r} - 2H \to \hat{r} - 2H - i\varepsilon$ and as a consequence the p_c below $2H$ acquires the imaginary part $i\pi\hat{r}$. Below $\hat{r} = 2M$ the denominator of the argument of the logarithm in eq.(??) becomes negative so that the argument of the logarithm reverts to real values
and the argument of the logarithm reaches zero for

$$\hat{r} = 2M \tag{48}$$

Below $\hat{r} = 2M$ the argument of the logarithm reverts to real values. Thus the classically forbidden region is $2M < \hat{r} < 2H$ independent of m and for any deformation g and the integral of the imaginary part of p_c for any deformation g is

$$i \int \mathrm{Im} p_c dr = i\ \pi \int_{2M}^{2H} r dr = 2i\pi(H^2 - M^2) \tag{49}$$

which is Parikh-Wilczek result [?] but now valid also for massive emission. The semiclassical emission probability is given by

$$\exp(-2\int \mathrm{Im} p_c dr) = \exp(-4\pi(H^2 - M^2)) =$$

$$= \exp(-8\pi\omega(M + \omega/2))$$

In the inner gauge the analytic properties of the conjugate momentum p_c^i can be similarly discussed. We have

$$p_c^i = \sqrt{2M\hat{r}} - \sqrt{2H\hat{r}} - \hat{r}\log\left(\frac{\hat{r} - \sqrt{2H\hat{r}}}{\hat{r} - V - \sqrt{2M\hat{r}} + \hat{p}}\right) \tag{50}$$

and we obtain the same results.

The two shell reduced action

$\hat{r}_1$ and $\hat{r}_2$ $\hat{r}_1 < \hat{r}_2$.
For $r < \hat{r}_1$ M, for $\hat{r}_1 < r < \hat{r}_2$ by M_0 for $r > \hat{r}_2$ H.

$$R(r) = r + v_2 g(r - \hat{r}_2) + v_1 h(r - \hat{r}_1) \tag{51}$$

The action is given by

$$S = \int dt \ \left[\hat{p}_2 \dot{\hat{r}}_2 + \hat{p}_1 \dot{\hat{r}}_1 + \int \pi_R \dot{R} \ dr \right] + \text{b.t.} \tag{52}$$

Breaking the integration range from r_0 to $\hat{r}_1$ and from $\hat{r}_1$ to $\hat{r}_2$ and using eq.(??) for π_R and the same technique as used for the one shell case we reach the expression

$$\hat{p}_2 \dot{\hat{r}}_2 + \hat{p}_1 \dot{\hat{r}}_1 + \frac{d}{dt} \int_{r_0}^{\hat{r}_2} F dr - \dot{M}_0 \int_{\hat{r}_1}^{\hat{r}_2} \frac{\partial F}{\partial M_0} dr - \dot{\hat{r}}_2 F(\hat{r}_2 - \varepsilon) - (\dot{R} \frac{\partial F}{\partial R'})(\hat{r}_2 - \varepsilon) +$$

$$\dot{\hat{r}}_1 \Delta F(\hat{r}_1) + \Delta(\dot{R} \frac{\partial F}{\partial R'})(\hat{r}_1)$$

$$= \hat{p}_1 \dot{\hat{r}}_1 + \frac{d}{dt} \int_{r_0}^{\hat{r}_2} F dr - \dot{M}_0 \int_{\hat{r}_1}^{\hat{r}_2} \frac{\partial F}{\partial M_0} dr + p^0_{c2} \, \dot{\hat{r}}_2 + \dot{\hat{r}}_1 \, \Delta F(\hat{r}_1) + \Delta(\dot{R} \frac{\partial F}{\partial R'})(\hat{r}_1)$$

where p^0_c is given by eq.(??) and Δ stays for the jump across the discontinuity at $\hat{r}_1$. It will be useful to write

$$F = \pi_L + R' R(\mathcal{L} - \mathcal{B}) \tag{53}$$

where

$$\mathcal{L} = \log(R' - \sqrt{R'^2 - 1 + \frac{2\mathcal{M}}{R}})$$

and

$$\mathcal{B} = \sqrt{\frac{2\mathcal{M}}{R}} + \log(1 - \sqrt{\frac{2\mathcal{M}}{R}})$$

and notice that

$$\frac{\partial F}{\partial R'} = R(\mathcal{L} - \mathcal{B}) \tag{54}$$

With

$$T = \log v_2; \qquad \mathcal{D} = R(\Delta\mathcal{L} - \Delta\mathcal{B}); \qquad p_{c1} = R'(\hat{r}_1 + \varepsilon)\mathcal{D}; \tag{55}$$

$$\tilde{p}_{c2} = -(R'(\hat{r}_1 + \varepsilon) - 1)\mathcal{D} + (R_1 - \hat{r}_1)\frac{\partial T}{\partial \hat{r}_2}\mathcal{D} \tag{56}$$

Reduced action for the two shell system in the outer gauge

$$\dot{\hat{r}}_1 p_{c1} + \dot{\hat{r}}_2 p_{c2} + \dot{H}(R(\hat{r}_1) - \hat{r}_1)\frac{\partial T}{\partial H}\mathcal{D} + \dot{M}_0(R(\hat{r}_1) - \hat{r}_1)\frac{\partial T}{\partial M_0}\mathcal{D} +$$

$$-\dot{M}_0 \int_{\hat{r}_1}^{\hat{r}_2} \frac{\partial F}{\partial M_0} dr + \frac{d}{dt}\int_{r_0}^{\hat{r}_2} F dr + (-N^r \pi_L + NRR')|_{r_0}^{r_m} \tag{57}$$

where $p_{c2} = p_c^0 + \tilde{p}_c$

Equations of motion

Can be derived from the above reduced action. Variation w.r.t. H gives non contribution in $\dot{\hat{r}}_1$ (these contributions cancels). One is left with $\dot{\hat{r}}_2$ which are exactly the equations of motion of the outer shell in the Schwarzschild mass M_0, irrespective to the inner dynamics (as expected). Variation w.r.t. M_0 gives a combination of $\dot{\hat{r}}_2$ and $\dot{\hat{r}}_1$ equations of motion.

Exchange relations

Some hypothesis has to be done on the dynamics of the collision.

General treatment: Just before the collision i.e. at $t = t_0 - \varepsilon$ **assuming that** $\hat{r}_2(t_0 - \varepsilon) > \hat{r}_1(t_0 - \varepsilon)$ **we have for the momentum** π_L

$$\pi_L(\hat{r}_1 - \varepsilon, t_0 - \varepsilon) = \pi_L(\hat{r}_1 + \varepsilon, t_0 - \varepsilon) + \hat{p}_1 = \tag{58}$$

$$\pi_L(\hat{r}_2 - \varepsilon, t_0 - \varepsilon) + \hat{p}_1 = \pi_L(\hat{r}_2 + \varepsilon, t_0 - \varepsilon) + \hat{p}_1 + \hat{p}_2$$

and after the collision i.e $t = t_0 + \varepsilon)$ **we have**

$$\pi_L(\hat{r}_2 - \varepsilon, t_0 + \varepsilon) = \pi_L(\hat{r}_2 + \varepsilon, t_0 + \varepsilon) + \hat{p}'_1 = \tag{59}$$

$$\pi_L(\hat{r}_1 - \varepsilon, t_0 + \varepsilon) + \hat{p}'_1 = \pi_L(\hat{r}_1 + \varepsilon, t_0 + \varepsilon) + \hat{p}'_1 + \hat{p}'_2$$

the collision point $\hat{r}_2 = \hat{r}_1 = \hat{r}_0$**.**

The sum $V_1 + V_2$ **has to be continuous in time**

$$V_1 + V_2 = V'_1 + V'_2$$

due to eq.(??).

The relation between $\pi_L(\hat{r}_0 - \varepsilon, t_0 - \varepsilon)$ **and** $\pi_L(\hat{r}_0 + \varepsilon, t_0 - \varepsilon)$

$$\hat{r}_0\sqrt{(1 + \frac{\hat{V}_1 + \hat{V}_2}{\hat{r}_0})^2 - 1 + \frac{2M}{\hat{r}_0}} = \sqrt{2H\hat{r}_0} + \hat{p}_1 + \hat{p}_2 \tag{60}$$

and the same written at $t_0+\varepsilon$

$$\hat{r}_0\sqrt{(1+\frac{\hat{V}_1'+\hat{V}_2'}{\hat{r}_0})^2-1+\frac{2M}{\hat{r}_0}}=\sqrt{2H\hat{r}_0}+\hat{p}_1'+\hat{p}_2' \qquad (61)$$

gives the relation

$$\hat{p}_1'+\hat{p}_2'=\hat{p}_1+\hat{p}_2$$

$\hat{p}_2$ **is given by the implicit equation**

$$H-M_0=V_2+\frac{m_2^2}{2\hat{r}_0}-\hat{p}_2\sqrt{\frac{2H}{\hat{r}_0}} \qquad (62)$$

and a similar equation gives $\hat{p}_1$. M_0' is given by

$$H-M_0'=V_1'+\frac{{m'}_1^2}{2\hat{r}_0}-\hat{p}_1'\sqrt{\frac{2H}{\hat{r}_0}} \qquad (63)$$

E.g. for "transparent crossing"

$$m_1^2+m_2^2+2(V_1'V_2-\hat{p}_1'\hat{p}_2)+2(V_1'+V_2)\hat{r}_0+2M\hat{r}_0=2Hr_0+2(\hat{p}_1'+\hat{p}_2)\sqrt{2H\hat{r}_0}$$

where $\hat{p}_2$ is given by eq.(??) and $\hat{p}_1'=\hat{p}_1$ by eq.(??).
The massless case is most easily treated.

$$\hat{p}_2=-\frac{H-M_0}{1+\sqrt{\frac{2H}{\hat{r}_0}}} \qquad (64)$$

and

$$\hat{p}_1' = -\frac{H - M_0'}{1 + \sqrt{\frac{2H}{\hat{r}_0}}} \tag{65}$$

gives which gives

$$H\hat{r}_0 + \sqrt{2H\hat{r}_0}(\hat{p}_1 + \hat{p}_2) = \hat{r}_0(\hat{p}_1 - \hat{p}_2) + M\hat{r}_0 - 2\hat{p}_1\hat{p}_2 \tag{66}$$

Substituting

$$H\hat{r}_0 + M\hat{r}_0 - 2HM = M_0\hat{r}_0 - 2M_0M_0' + M_0'\hat{r}_0 \tag{67}$$

Dray- 't Hooft and Redmount relation

Integrability of the form $p_{c1}d\hat{r}_1 + p_{c2}d\hat{r}_2$

We are interested in computing the action for the two shell system i.e. the time integral of eq.(??) on the equations of motion i.e. $\dot{M}_0 = \dot{H}_0 = 0$. This apart the boundary term reduces to

$$\int_{t_i}^{t_f} dt(p_{c1}\dot{r}_1 + p_{c2}\dot{r}_2) = \int(p_{c1}d\hat{r}_1 + p_{c2}d\hat{r}_2) \tag{68}$$

This is of interest in the computation of the semiclassical wave function and for the tunneling problem in the two shell case. We shall prove that the form $p_{c1}d\hat{r}_1 + p_{c2}d\hat{r}_2$ is integrable. Such a result will be very useful in the actual computation. We recall that

$$p_{c1} = R'(\hat{r}_1 + \varepsilon)D \tag{69}$$

$$p_{c2} = p^0_{c2} + (-R'(\hat{r}_1 + \varepsilon) + 1 + (R(\hat{r}_1) - \hat{r}_1)\frac{\partial T}{\partial \hat{r}_2})D \tag{70}$$

where p^0_{c2} depends on $\hat{r}_2$ while p_{c1} and $\tilde{p}_{c2}$ depend both on $\hat{r}_1$ and $\hat{r}_2$ as $R(\hat{r}_1)$, $R'(\hat{r}_1 + \varepsilon))$, D depend on $\hat{r}_1$ and $\hat{r}_2$. By looking at the structure of R we can rewrite the integrability condition

$$\frac{\partial p_{c1}}{\partial \hat{r}_2} = \frac{\partial p_{c2}}{\partial \hat{r}_1} \tag{71}$$

as

$$\frac{\partial}{\partial \hat{r}_2}\left[\frac{\partial R(\hat{r}_1)D}{\partial \hat{r}_1}\right] = \frac{\partial}{\partial \hat{r}_1}\left[\frac{\partial (R(\hat{r}_1)-\hat{r}_1)D}{\partial \hat{r}_2}\right] \tag{72}$$

In the massless case this relation is simply satisfied because $D = D(R)$. In the massive case the relation (??) takes the more complicated form

$$R'(+)\left[-\frac{1}{W(+)}(-R''(+)+\frac{\partial T}{\partial \hat{r}_2}(R'(+)-1)+ \right. \tag{73}$$

$$\left. \frac{1}{W(-)}(-R''(+)+\frac{\partial T}{\partial \hat{r}_2}(R'(+)-1)+\frac{\partial v_1}{\partial \hat{r}_2}\right] =$$

$$\left[-(R'(+)-1)+\frac{\partial T}{\partial \hat{r}_2}(R-\hat{r}_1)\right]\left[-\frac{1}{W''(+)}R''(+)+\frac{1}{W(-)}(R''(+)+\frac{\partial v_1}{\partial \hat{r}_1})\right]$$

where $\hat{p}_1$, $\hat{p}_2$ are given by implicit equations and $+$ stays for $\hat{r}_1+\varepsilon$. In absence of an algebraic proof we verified them numerically down to machine precision (10^{-16}) for random values of $H, M_0, M, \hat{r}_1, \hat{r}_2$.

If the crossing occurs at $\hat{r}_0$ $2M < \hat{r}_0 < 2H$, with e.g. $\hat{r}_1 < \hat{r}_2$ before the crossing, we choose the path $\hat{r}_1 = \hat{r}_2 - \varepsilon$ before the crossing and $\hat{r}_2 = \hat{r}_1 - \varepsilon$ after the crossing.

In this way we prove that for the crossing of a null shell and a massive shell, into a null shell and another massive shell even with change of mass, provided that at the crossing

the crossing relations are satisfied, the imaginary part of the integral

$$S = \int (p_{c1}\ d\hat{r}_1 + p_{c2}\ d\hat{r}_2) + \text{b.t.}$$

is still given by

$$i\ \text{Im} \int (p_{c1}\ d\hat{r}_1 + p_{c2}\ d\hat{r}_2) = 2\pi i(H^2 - M^2)$$

Before

$$H - M_0 = V_2 + \frac{m_2^2}{2\hat{r}2} - \hat{p}\sqrt{\frac{2H}{\hat{r}_2}}$$

After

$$\frac{\hat{p}_2'}{\hat{R}} = [(\frac{M_0' - M}{R} - \frac{{m_2'}^2}{2R})W(+) + \frac{R'(+)}{R^2}(\frac{{m'}_2^4}{4} +$$

$${m'}_2^2(M_0' + M)R[(M_0' - M)^2 - m_2^2]R^2)^{1/2}]/(1 - \frac{2M_0'}{R})$$

where

$$W(+) = \sqrt{\frac{2H}{\hat{r}_2}} + \frac{\hat{p}_1'}{\hat{r}_2}$$

and

$$\hat{p}_1' = \frac{H - M_0'}{1 - \sqrt{\frac{2H}{\hat{r}_2}}}$$

$$R'(+) = 1 + \frac{\hat{p'}_1}{\hat{r}_1}$$

CONCLUSIONS

1. Notable simplification of the Kraus-Wilczek Friedman-Louko-Winters Hilts derivation of the reduced action.

2. No limiting procedure necessary

3. Outer and inner gauges are equivalent

4. More general gauge allows consistent formulation with two or more shells which can intersect.

5. Consistency of the equations of motion.

6. Re-derivation of the Dray-'t Hooft, Redmount exchange relations for light- like shells.

7. Integrability relation and factorization of the emission probability for one massless and one massive shell independently of the exchange interaction.

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