Quantizzazione dei campi su spazio-tempo di Moyal-Weyl: (dove) si manifesta la noncommutativita'?

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Introduction

The idea of spacetime NC is rather old: goes back to Heisenberg.

Simplest NC: constant commutators

Moyal-Weyl space: $$[\hat{x}^\mu, \hat{x}^\nu] = i\mathbf{1}\theta^{\mu\nu} \tag{1}$$

Algebra $\widehat{\mathcal{A}}$ of functions on Moyal-Weyl space: generated by $\mathbf{1}, \hat{x}^\mu$ fulfilling (1). With $\mu = 0, 1, 2, 3$ and $\eta_{\mu\nu}$: deformed Minkowski space. $\theta^{\mu\nu} = 0$: $\mathcal{A}$ generated by commuting x^μ.

(1) are translation invariant, not Lorentz-covariant.

Contributions to the construction of QFT on it start in 1994-95. I would divide them into 3 groups, according to the used approaches. By no means are they equivalent!

1. DFR Approach

(Doplicher, Fredenhagen, Roberts 1994-95; Bahns, Piacitelli,..a): field quantization in (rigorous) operator formalism on def. Minkowski space. (1) motivated by the interplay of QM and GR in what they call the **Principle of gravitational stability against localization of events:**

The gravitational field generated by the concentration of energy required by the Heisenberg Uncertainty Principle to localise an event in spacetime should not be so strong to hide the event itself to any distant observer - distant compared to the Planck scale.

(Goes back to Wheeler?)

In the first, simplest version $\theta^{\mu\nu}$ are not fixed constants, but central operators (obeying additional conditions) which on each irrep become fixed constants $\sigma^{\mu\nu}$, the joint spectrum of $\theta^{\mu\nu}$.

In more recent versions $\theta^{\mu\nu}$ is no more central, but commutation relations remain of Lie-algebra type.

But the wished Lorentz covariance is sooner or later lost.

Speculations by Doplicher: $\theta^{\mu\nu}$ should be finally related to v.e.v. of $R^{\mu\nu}$, which in turn should be influenced by the presence of matter quantum fields in spacetime (through quantum equations of motions).

2. (Naive) path-integral quantization on $\mathbb{R}^4_\theta$

(Filk 1996,...). Main initial motivation: effective theory from string theory in a constant background B-field (A lot of string theorists: N. Seiberg, E. Witten, M. R. Douglas, A.S. Schwarz, S. Minwalla, M. Van Raamsdonk, N. Seiberg, J. Gomis, T. Mehen, L. Alvarez-Gaume, M.A. Vazquez-Mozo, M. R. Douglas, N. A. Nekrasov, R.J. Szabo,....).

(Wick-rotated) Lorentz covariance is lost, but this is expected in effective string theory because of the B-field.

Many pathologies: violation of causality, non-unitarity (for $\theta^{0i} \neq 0$), UV-IR mixing of divergences, subsequent non-renormalizability, claimed changes of statistics, etc.

UV-IR MIXING:

Planar Feynman diagrams remain as the undeformed, apart from a phase factor, in particular have the same UV divergences. Nonplanar Feynman diagrams which were UV divergent become finite for generic non-zero external momentum, but diverge as the latter go to zero, even with massive fields: IR divergences!

3. Twisted Poincaré covariant approaches

This is the framework of our work, subject of this talk.
It recovers Poincaré covariance in a deformed version.
Field quantization either in an operator or in a path-integral approach (on the Euclidean).

Chaichian *et al*, Wess, Koch *et al*, Oeckl:
(1) are twisted Poincaré group covariant.

How to implement twisted Poincaré covariance in QFT?

Different proposals, [Chaichian *et al* 04,05,06], [Tureanu06], [Balachandran *et al* 05,06] [Lizzi *et al* 06], [Bu *et al* 06], [Zahn 06], [Abe 06]...:

a) do coordinates x, y of different spacetime points commute?
b) deform the CCR of $a_p, a_p^\dagger$ for free fields?

Abstract of our contribution

We note that a proper enforcement of the "twisted Poincaré" covariance of [Chaichian *et al*], [Wess], [Koch *et al*], [Oeckl] requires nontrivial ("braided") commutation relations between any pair of coordinates x, y generating two different copies of the Grönewold-Moyal-Weyl space, or equivalently a $\star$-tensor product $f(x) \star g(y)$ (in the parlance of [Aschieri *et al*]).

Then all $(x-y)^\mu$ behave like undeformed coordinates.

Consequently, one can formulate QFT in a way physically equivalent to the undeformed counterpart, as observables involve only coordinate differences. (Similarly for n-particle QM)

We briefly comment on what we can learn from these results

Plan

1. Introduction
2. Twisted Poincaré Hopf algebra, several spacetime variables, $\star$-products
3. Revisiting Wightman axioms for QFT and their consequences
4. Free fields
5. Interacting fields
6. (Some) Conclusions?

2. The Hopf algebra $H \equiv U_\theta \mathcal{P}$

This is $U\mathcal{P}$ ($\mathcal{P} =$ Poincaré Lie algebra) "twisted" [Drinfel'd 83] with $\mathcal{F}$: $U\mathcal{P}, H$ have

1. same $*$-algebra and counit ε
2. coproducts $\Delta, \hat{\Delta}$ related by

$$\Delta(g) \equiv \sum_I g^I_{(1)} \otimes g^I_{(2)} \longrightarrow \hat{\Delta}(g) = \mathcal{F}\Delta(g)\mathcal{F}^{-1} \equiv \sum_I g^I_{(\hat{1})} \otimes g$$

The *twist* $\mathcal{F}$ is not uniquely determined. The simplest choice is

$$\mathcal{F} \equiv \sum_I \mathcal{F}_I^{(1)} \otimes \mathcal{F}_I^{(2)} := \exp\left(\tfrac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu\right).$$

$$\hat{\Delta}(P_\mu) = \Delta(P_\mu) = P_\mu \otimes \mathbf{1} + \mathbf{1} \otimes P_\mu = \Delta(P_\mu),$$
$$\hat{\Delta}(M_\omega) = M_\omega \otimes \mathbf{1} + \mathbf{1} \otimes M_\omega + P[\omega, \theta] \otimes P \neq \Delta(M_\omega).$$

where $M_\omega = \omega^{\mu\nu} M_{\mu\nu}$. Translations undeformed!

Let $\triangleright, \hat{\triangleright}$ be the (left) actions of $U\mathcal{P}, H$ on $\mathcal{A}, \widehat{\mathcal{A}}$ ($g \in U\mathcal{P}$ acts on $\mathcal{A}$ as the corresponding differential operators, e.g. $P_\mu \sim i\partial_\mu$).

- $\triangleright, \hat{\triangleright}$ act in the same way on 1st degree polynomials in $x^\nu, \hat{x}^\nu$

$$P_\mu \triangleright x^\rho = i\delta^\rho_\mu = P_\mu \hat{\triangleright} \hat{x}^\rho, \qquad M_\omega \triangleright x^\rho = 2i(x\omega)^\rho, \qquad M_\omega \hat{\triangleright} \hat{x}^\rho = 2i(\hat{x}\omega$$

and more generally on irreps (irreducible representations); $\Rightarrow$
Same classification of elementary particles as unitary irreps of $\mathcal{P}$!

- $\triangleright, \hat{\triangleright}$ differ on higher degree polynomials in $x^\nu, \hat{x}^\nu$, and more generally on tensor products of representations, after the rules

$$g \triangleright (ab) = \sum_I \big(g_{(1)} \triangleright a\big)\big(g_{(2)} \triangleright b\big)$$
$$g \hat{\triangleright} (\hat{a}\hat{b}) = \sum_I \big(g^I_{(\hat{1})} \hat{\triangleright} \hat{a}\big)\big(g^I_{(\hat{2})} \hat{\triangleright} \hat{b}\big) \quad \Leftrightarrow \quad g \hat{\triangleright} (a \star b) = \sum_I \big(g^I_{(\hat{1})} \hat{\triangleright} a\big) \star$$

(resp. reducing to usual or *deformed* Leibniz rule if $g = P_\mu, M_{\mu\nu}$).

Summarizing: (1) are H-covariant, or $\widehat{\mathcal{A}}$ is a H-module algebra.

Several spacetime variables

Let $\mathcal{A}^n$ be the n-fold tensor product algebra of $\mathcal{A}$,
$x_1^\mu \equiv x^\mu \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}, \quad x_2^\mu \equiv \mathbf{1} \otimes x^\mu \otimes \ldots \otimes \mathbf{1}, \ldots$
$\mathcal{A}^n$ is $U\mathcal{P}$-covariant, i.e. $[x_i^\mu, x_j^\nu] = 0$ are compatible with $\triangleright$.

$[\hat{x}_i^\mu, \hat{x}_j^\nu] = 0$ is not compatible with $\hat{\triangleright}$ (apply e.g. $M_\omega \hat{\triangleright}$) .

The H**-covariant** NC generalization of $\mathcal{A}^n$ is the unital $*$-algebra $\widehat{\mathcal{A}}^n$ generated by real variables $\hat{x}_i^\mu$ fulfilling

$$[\hat{x}_i^\mu, \hat{x}_j^\nu] = \mathbf{1} i \theta^{\mu\nu}, \tag{2}$$

dictated by the braiding associated to the quasitriangular structure $\mathcal{R} = \mathcal{F}_{21} \mathcal{F}^{-1}$ of H.

$\star$-Products

Equivalent formulation of $\widehat{\mathcal{A}}^n$: For $n \geq 1$ let $\mathcal{A}_\theta^n$ be the algebra coinciding with $\mathcal{A}^n$ as a vector space, but with the new product

$$a \star b := \sum_I (\overline{\mathcal{F}}_I^{(1)} \triangleright a)(\overline{\mathcal{F}}_I^{(2)} \triangleright b), \tag{3}$$

with $\overline{\mathcal{F}} \equiv \mathcal{F}^{-1}$. This encodes both the $\star$-product within each copy of $\mathcal{A}$, and the "$\star$−tensor product" algebra [Aschieri *et al*]. $\mathcal{A}_\theta^n$ has $\star$-commutation relations isomorphic to (**??**), $\Rightarrow \widehat{\mathcal{A}}^n, \mathcal{A}_\theta^n$ are isomorphic H-module $*$-algebras: choosing (**??**) $\mathcal{F}$ as in (**??**,

$$x_i^\mu \star x_j^\nu = x_i^\mu x_j^\nu + i\theta^{\mu\nu}/2 \qquad \Rightarrow \qquad [x_i^\mu \stackrel{\star}{,} x_j^\nu] = \mathbf{1} i\theta^{\mu\nu}.$$

In general, (**??** gives $f(x_i) \star g(x_j) = \exp[\frac{i}{2}\partial_{x_i}\theta\partial_{x_j}] f(x_i) g(x_j)$, *after* which we must set $x_i = x_j$ if $i = j$.

In the sequel we express NC only by $\star$-products.

Alternative generators of $\mathcal{A}_\theta^n$

$$\xi_i^\mu = x_{i+1}^\mu - x_i^\mu, \qquad X^\mu = \sum_{j=1}^n a_j x_j^\mu \qquad (i<n, \quad \sum_{j=1}^n a_j = 1$$

1. $[X^\mu \stackrel{\star}{,} X^\nu] = \mathbf{1} i\theta^{\mu\nu}$, so X^μ generate a $\mathcal{A}_{\theta,X}$, whereas $\forall b \in \mathcal{A}_\theta^n$

$$\xi_i^\mu \star b = \xi_i^\mu b = b \star \xi_i^\mu \qquad \Rightarrow \qquad [\xi_i^\mu \stackrel{\star}{,} b] = 0, \qquad (4)$$

ξ_i^μ generate a $\star$-central subalgebra $\mathcal{A}_\xi^{n-1}$, and $\mathcal{A}_\theta^n \sim \mathcal{A}_\xi^{n-1} \otimes \mathcal{A}_{\theta,X}$.

2. $\mathcal{A}_\xi^{n-1}, \mathcal{A}_{\theta,X}$ are actually H-module subalgebras, with

$$\begin{aligned} & g \hat{\triangleright} a = g \triangleright a & & a \in \mathcal{A}_\xi^{n-1}, \quad g \in H \\ & g \hat{\triangleright} (a \star b) = \left(g_{(1)} \triangleright a\right) \star \left(g_{(2)} \hat{\triangleright} b\right), & & b \in \mathcal{A}_\theta^n, \end{aligned} \qquad (5)$$

i.e. *on* $\mathcal{A}_\xi^{n-1}$ *the* H*-action is undeformed*, including the related part of the Leibniz rule. [By (10) $\star$ can be also dropped.] All ξ_i^μ are translation invariant.

Remark 1. $(x-y)^\mu\star = (x-y)^\mu\cdot$, same spectral decomposition on all $\mathbb{R}$ (including 0). On each irrep of $\mathcal{A}_\theta^n$ this amounts to multiplication by either a space-like, or a null, or a time-like 4-vector, in the usual sense.

Summing up, coordinate differences can be treated as classical variables; any x_i^μ is a combination of $\star$-commutative ξ_i^μ and the $\star$-noncommutative X^μ, e.g. if $X := x_1$

$$x_i = \sum_{j=1}^{i-1} \xi_j + X.$$

X =Global "noncommutative translation".

1.,2. can be reformulated in terms of $\hat{x}_i, \widehat{\mathcal{A}}^n$, etc. $\hat{X}$ is like the "quantum shift operator" of [Chaichian *et al*].

The **differential calculus is not deformed** with $\theta \neq 0$, also on $\mathcal{A}_\theta^n$ (or the isomorphic $\widehat{\mathcal{A}}^n$), since $P_\mu \triangleright \partial_{x_i^\nu} = 0$:

$$\partial_{x_i^\mu} \star x_j^\nu = \delta_\mu^\nu \delta_j^i + x_j^\nu \star \partial_{x_i^\mu} \qquad \left[\partial_{x_i^\mu} \stackrel{\star}{,} \partial_{x_j^\nu}\right] = 0 \tag{6}$$

In the sequel we shall drop the symbol $\star$ beside a derivative.

Also **integration over the space is not deformed** with $\theta \neq 0$:

$$\int d^4x \, a \star b = \int d^4x \, ab \tag{7}$$

Stoke's theorem still applies.

Consequences for QFT

Wightman axioms (grouped into subsets **QM**, **R**, [Strocchi]):

QM1. The states are described by vectors of a (separable) Hilbert space $\mathcal{H}$.

QM2. The group of space-time translations $\mathbb{R}^4$ is represented on $\mathcal{H}$ by strongly continuous unitary operators $U(a)$. The spectrum of the generators P_μ is contained in $\bar{V}_+ = \{p_\mu : p^2 \geq 0,\, p_0 \geq 0\}$. There is a unique Poincaré invariant state Ψ_0, the *vacuum state*.

QM3. The fields (in the Heisenberg representation) $\varphi^\alpha(x)$ [α enumerates field species and/or $SL(2,\mathbb{C})$-tensor components] are operator (on $\mathcal{H}$) valued tempered distributions on Minkowski space, with Ψ_0 a *cyclic* vector for the fields, i.e. polynomials of the (smeared) fields applied to Ψ_0 give a set $\mathcal{D}_0$ dense in $\mathcal{H}$.

Taking v.e.v.'s we define Wightman functions (distributions):

$$\mathcal{W}^{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = \left(\Psi_0, \varphi^{\alpha_1}(x_1)\star\ldots\star\varphi^{\alpha_n}(x_n)\Psi_0\right), \quad (8)$$

or (their combinations) Green's functions

$$G^{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = \left(\Psi_0, T[\varphi^{\alpha_1}(x_1)\star\ldots\star\varphi^{\alpha_n}(x_n)]\Psi_0\right); \quad (9)$$

no problem in defining time-ordering T as on commutative Minkowski space, even if $\theta^{0i} \neq 0$,

$$T[\varphi^{\alpha_1}(x)\star\varphi^{\alpha_2}(y)] = \varphi^{\alpha_1}(x)\star\varphi^{\alpha_2}(y)\vartheta(x^0-y^0) + \varphi^{\alpha_2}(y)\star\varphi^{\alpha_1}(x)\vartheta(y^0-x^0$$

as $\vartheta(x^0 - y^0)$ are $\star$-central ($\vartheta \equiv$Heavyside function). [The $\star$'s preceding all ϑ can be and have been dropped, by (10).]

Argue as in [Streater & Wightman 1964] for ordinary QFT.

QM2 $\Rightarrow$ **Wightman and Green's functions are translation invariant** and therefore may depend only on the ξ_i^μ.

$$\begin{aligned} \mathcal{W}^{\alpha_1,\dots,\alpha_n}(x_1,\dots,x_n) &= W^{\alpha_1,\dots,\alpha_n}(\xi_1,\dots,\xi_{n-1}), \\ \mathcal{G}^{\alpha_1,\dots,\alpha_n}(x_1,\dots,x_n) &= G^{\alpha_1,\dots,\alpha_n}(\xi_1,\dots,\xi_{n-1}). \end{aligned} \tag{10}$$

From QM3, QM2, QM1 it follows

W1. $\mathcal{W}^{\{\alpha\}}(x_1,\dots,x_n) = W^{\{\alpha\}}(\xi_1,\dots,\xi_{n-1})$ are tempered distributions.

W2. (**Spectral condition**) The support of the Fourier transform $\tilde{W}$ of W is contained in the product of forward cones, i.e.

$$\tilde{W}^{\{\alpha\}}(q_1,\dots q_{n-1}) = 0, \qquad \text{if } \exists j: \quad q_j \notin \overline{V}_+. \tag{11}$$

W3. $\mathcal{W}^{\{\alpha\}}$ fulfill the **Hermiticity and Positivity** properties following from those of the scalar product in $\mathcal{H}$.

Ordinary relativistic conditions on QFT:

R1. (**Lorentz Covariance**) $SL(2,\mathbb{C})$ is represented on $\mathcal{H}$ by strongly continuous unitary operators $U(A)$, and under the Poincaré transformations $U(a,\ A) = U(a)\,U(A)$

$$U(a,A)\,\varphi^\alpha(x)\,U(a,A)^{-1} = S^\alpha_\beta(A^{-1})\,\varphi^\beta\big(\Lambda(A)x+a\big), \qquad (12)$$

with S a finite dimensional representation of $SL(2,\mathbb{C})$.

R2. (**Microcausality or locality**) The fields either commute or anticommute at spacelike separated points

$$[\,\varphi^\alpha(x),\varphi^\beta(y)\,]_\mp = 0, \qquad \text{for} \;\; (x-y)^2 < 0. \qquad (13)$$

As a consequence of QM2,R1 in ordinary QFT one finds

W4. (Lorentz Covariance of Wightman functions)

$$\mathcal{W}^{\alpha_1..\alpha_n}(\Lambda(A)x_1,...,\Lambda(A)x_n) = S^{\alpha_1}_{\beta_1}(A)..S^{\alpha_n}_{\beta_n}(A)\mathcal{W}^{\beta_1..\beta_n}(x_1,...,x_n). \tag{14}$$

R1 needs a "twisted" reformulation R1$_\star$, which we defer. R1$_\star$ should imply that $W^{\{\alpha\}}$ are $SL_\theta(2,\mathbb{C})$ tensors, anyway. But, as the $W^{\{\alpha\}}$ should be built only in terms of ξ_i^μ and other $SL(2,\mathbb{C})$ tensors (like $\partial_{x_i^\mu}$, $\eta_{\mu\nu}$, γ^μ, polarization vectors, spinors, etc.), which are all annihilated by $P_\mu\triangleright$, $\mathcal{F}$ should act as id and $W^{\{\alpha\}}$ should transform under $M^{\rho\sigma}$ as for $\theta=0$. Therefore **we shall require W4 also if** $\theta\neq 0$ as a temporary substitute of R1.

R2$_\star$? Simplest: with a $\star$-commutator; makes sense, as space-like separation is well-defined. Alternatively, $\exists$ some reasonable weakening? In fact, an open question also on commutative space; the same restrictions should apply.

R2$_\star$. $$[\,\varphi^\alpha(x) \stackrel{\star}{,} \varphi^\beta(y)\,]_\mp = 0, \qquad \text{for } (x-y)^2 < 0.$$

Argue as [S. & W. 1964] to prove QM1-3, W4, R2$_\star$ are (independent and) compatible: they can be fulfilled by free fields (see below)! So in particular **the noncommutativity structure of a Moyal-Weyl space is compatible with R2$_\star$!**
As consequences of R2 one again finds

W5. (**Locality**) if $(x_j - x_{j+1})^2 < 0$

$$\mathcal{W}(x_1, ...x_j, x_{j+1}, ...x_n) = \pm\mathcal{W}(x_1, ...x_{j+1}, x_j, ...x_n). \qquad (15)$$

W6. (**Cluster property**) For any spacelike a and for $\lambda \to \infty$

$$\mathcal{W}(x_1, ...x_j, x_{j+1}+\lambda a, ..., x_n+\lambda a) \to \mathcal{W}(x_1, ..., x_j)\,\mathcal{W}(x_{j+1}, ..., x \qquad (16)$$

(convergence as distributions); true also with permuted x_i's.

Summarizing: QFT framework with **QM1-3, W4, R2$_\star$** or alternatively with constraints **W1-6** on $\mathcal{W}^{\{\alpha\}}$ exactly as in QFT on Minkowski space.

We stress that these results should hold for all $\theta^{\mu\nu}$, and not only if $\theta^{0i} = 0$, as in other approaches.

Free fields

Free field e.o.m. remain undeformed (as $\Box$, $\not\partial$, etc), hence also their constraints on $\mathcal{W}^{\{\alpha\}}, \mathcal{G}^{\{\alpha\}}$ and on the field comm. relations. For simplicity Hermitean scalar field $\varphi_0(x)$ of mass m. One finds

$$\begin{aligned} &(\Box_x + m^2)\varphi_0 = 0, \qquad\qquad \Rightarrow \\ &\varphi_0(x) = \varphi_0^+(x)+\varphi_0^-(x) = \int d\mu(p)\,[e^{-ip\cdot x}a^p + a_p^\dagger e^{ip\cdot x}], \end{aligned} \tag{17}$$

where $d\mu(p) := \delta(p^2-m^2)\vartheta(p^0)d^4p$, and

$$\begin{aligned} W(x-y) &= \int \frac{d\mu(p)}{(2\pi)^3} e^{-ip\cdot(x-y)} = -iF^+(x-y) \\ G(x-y) &= \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}, \end{aligned} \tag{18}$$

(22), (23), *are independent of* **R2**$_\star$ *or any other assumption about field commutation relations*, which are not used in the proof.

Adding **R2**$_\star$ and reasoning as in the proof of the Jost-Schroer Thm. (4-15 in [S. & W. 1964]) one proves (up to a factor> 0) the **free field commutation relation**

$$[\varphi_0(x) \stackrel{\star}{,} \varphi_0(y)] = iF(x-y), \qquad F(\xi) := F^+(-\xi) - F^+(\xi) \quad (19)$$

(F undeformed!). Applying ∂_{y^0} and then setting $y^0 = x^0$ [this is compatible with (7)] one even finds **the c.c.r.**

$$[\varphi_0(x^0, \mathbf{x}) \stackrel{\star}{,} \dot{\varphi}_0(x^0, \mathbf{y})] = i\,\delta^3(\mathbf{x} - \mathbf{y}). \quad (20)$$

As a consequence of (24), also the n-point Wightman functions coincide with the undeformed ones, i.e. vanish if n is odd and are sum of products of two point functions if n is even (factorization). This agrees with the cluster property, as expected.

A φ_0 fulfilling (24) can be obtained from (22) plugging $a^p, a_p^\dagger$ satisfying the commutation relations

$$a_p^\dagger a_q^\dagger = e^{ip\theta' q}\, a_q^\dagger a_p^\dagger, \qquad a^p a^q = e^{ip\theta' q}\, a^q a^p,$$

$$a^p a_q^\dagger = e^{-ip\theta' q}\, a_q^\dagger a^p + 2\omega_{\mathbf{p}}\delta^3(\mathbf{p}-\mathbf{q}), \qquad \theta' = \theta \qquad (21)$$

$$[a^p, f(x)] = [a_p^\dagger, f(x)] = 0,$$

($p\theta q := p_\mu \theta^{\mu\nu} q_\nu$), as adopted in [Balachandran *et al* 05,06] first, then [Lizzi *et al* 06, Abe 06]. The choice $\theta' = 0$ gives the CCR, assumed in most of the literature, explicitly in [Doplicher *et al* 95], apparently in [Chaichian *et al* 04,05,06], [Tureanu06], or implicitly in path-integral approach to quantization. Correspondingly, one finds non-local $\star$-commutation relations

$$\varphi_0(x) \star \varphi_0(y) = e^{i\partial_x(\theta-\theta')\partial_y} \varphi_0(x) \star \varphi_0(y) + i\,F(x-y), \quad (22)$$

unless $\theta' = \theta$. [But taking $\theta' = \theta$ and using $\varphi_0(x)\varphi_0(y)$ instead of $\varphi_0(x) \star \varphi_0(y)$ one also finds non-local relations.]

$$\mathcal{W}(x_1,x_2,x_3,x_4) = W(x_1-x_2)W(x_3-x_4) + e^{i\partial_{x_2}(\theta-\theta')\partial_{x_3}}W(x_1-x_3)W(x_2-x_4) + \ldots \qquad (23)$$

The first term at the rhs comes from the v.e.v.'s of $\varphi_0(x_1)\star\varphi_0(x_2)$ and $\varphi_0(x_3)\star\varphi_0(x_4)$; it is Lorentz invariant and factorized. The second, nonlocal term comes from the v.e.v.'s of $\varphi_0(x_1)\star\varphi_0(x_3)$ and $\varphi_0(x_2)\star\varphi_0(x_4)$, after commuting $\varphi_0(x_2),\varphi_0(x_3)$. Only if $\theta'=\theta$ is Lorentz invariant and factorizes to $W(x_1-x_3)W(x_2-x_4)$. As it depends only on x_1-x_3, x_2-x_4, it is invariant under $(x_1,x_2,x_3,x_4)\to(x_1,x_2+\lambda a,x_3,x_4+\lambda a)$. By taking a space-like and $\lambda\to\infty$, we conclude that if $\theta'\neq\theta$ $\mathcal{W}$ violates W4 and W6, as expected.

There is also an “exotic” way to realize the free com. rel. (22). Assume $P_\mu \triangleright a^\dagger_p = p_\mu a^\dagger_p$, $P_\mu \triangleright a^p = -p_\mu a^p$ and extend the $\star$-product law also to $a^p, a^\dagger_p$. It amounts to $\theta' = -\theta$ (inserting $\star$'s) and nontrivial com. rel. between the $a^p, a^\dagger_p$ and functions:

$$a^\dagger_p \star a^\dagger_q = e^{-ip\theta q}\, a^\dagger_q \star a^\dagger_p, \qquad a^p \star a^q = e^{-ip\theta q}\, a^q \star a^p,$$

$$a^p \star a^\dagger_q = e^{ip\theta q}\, a^\dagger_q \star a^p + 2\omega_{\mathbf{p}} \delta^3(\mathbf{p}-\mathbf{q}), \tag{24}$$

$$a^p \star e^{iq\cdot x} = e^{-ip\theta q}\, e^{iq\cdot x} \star a^p, \qquad a^\dagger_p \star e^{iq\cdot x} = e^{ip\theta q}\, e^{iq\cdot x} \star a^\dagger_p.$$

Whence $[\varphi_0(x) \stackrel{\star}{,} f(y)] = 0$. The first three relations define an example of a general deformed Heisenberg algebra [G. F. 95]

$$a^q \star a^p = R^{qp}_{rs}\, a^s \star a^r \qquad a^\dagger_p \star a^\dagger_q = R^{sr}_{pq}\, a^\dagger_r \star a^\dagger_s$$

$$a^p \star a^\dagger_q = \delta^p_q + R^{rp}_{qs}\, a^\dagger_r \star a^s$$

covariant under a triangular Hopf algebra H. Here the R-matrix

Interacting QFT (T-ordered perturb. th.)

Def. Normal ordering: $\mathcal{A}_\theta^n$-bilinear map of field algebra into itself such that $(\Psi_0, :M:\Psi_0) = 0$, in particular $:\mathbf{1}: = 0$. Applying it to (26) we find that it is consistent to define

$$:a^p a^q: = a^p a^q, \quad :a_p^\dagger a^q: = a_p^\dagger a^q, \quad :a_p^\dagger a_q^\dagger: = a_p^\dagger a_q^\dagger, \quad :a^p a_q^\dagger: = a_q^\dagger a^p e^{-ip\theta' q}$$

Note the phase. More generally, in any monomial reorders all a^p to the right of all $a_q^\dagger$ introducing a $e^{-iq\theta' p}$ for each flip $a^p \leftrightarrow a_q^\dagger$. Assuming (26) or (30), (i.e. free field com. rel.) one finds

$$:\varphi_0(x): \;=\; \varphi_0(x)$$

$$:\varphi_0(x)\star\varphi_0(y): \;=\; \varphi_0(x)\star\varphi_0(y) - (\Psi_0, \varphi_0(x)\star\varphi_0(y)\Psi_0)$$

$$:\varphi_0(x)\star\varphi_0(y)\star\varphi_0(z): = \varphi_0(x)\star\varphi_0(y)\star\varphi_0(z) - (\Psi_0, \varphi_0(x)\star\varphi_0(y)\Psi_0)\,\varphi$$

$$-(\Psi_0, \varphi_0(x)\star\varphi_0(z)\Psi_0)\,\varphi_0(y) - \varphi_0(x)\,(\Psi_0, \varphi_0(y)\star\varphi_0(z)\Psi_0$$

Well-defined operators also if coinciding coordinates (e.g. $y \to x$). Moreover, **the same Wick theorem will hold**:

$$T\left[\varphi_0(x)\star\varphi_0(y)\right] = :\varphi_0(x)\star\varphi_0(y): + \Big(\Psi_0, T\left[\varphi_0(x)\star\varphi_0(y)\right]\Psi$$

$$T\left[\varphi_0(x)\star\varphi_0(y)\star\varphi_0(z)\right] =: \varphi_0(x)\star\varphi_0(y)\star\varphi_0(z): + \Big(\Psi_0, T\left[\varphi_0(x)\star\varphi_0(y)\right]\Psi_0\Big) :$$

$$+\Big(\Psi_0, T\left[\varphi_0(x)\star\varphi_0(z)\right]\Psi_0\Big) :\varphi_0(y): + \Big(\Psi_0, T\left[\varphi_0(y)\star\varphi_0(z)\right]\Psi_0\Big) :\varphi_0$$

$$\ldots$$

Interacting theory. Wish to apply the Gell-Mann–Low formula

$$G(x_1,\ldots,x_n) = \frac{\left(\Psi_0, T\left\{\varphi_0(x_1)\star\ldots\star\varphi_0(x_n)\star\exp_\star\left[-i\lambda\int dy^0\; H_I(y^0)\right]\right\}\Psi\right.}{\left(\Psi_0, T\exp_\star\left[-i\int dy^0\; H_I(y^0)\right]\Psi_0\right)} \qquad (26)$$

Here $\varphi_0 \equiv$free "in" field, and $H_I(x^0)$ is the interaction Hamiltonian in the interaction representation.

The derivation of (22) involves unitary evolution operators of

Choose

$$H_I(x^0) = \lambda \int d^3x \ :\varphi_0^{\star m}(x): \qquad\qquad \varphi_0^{\star m}(x) \equiv \underbrace{\varphi_0(x) \star ... \star \varphi_0(x}_{m \text{ times}}$$

This is a well-defined, Hermitean operator, with zero v.e.v. Expanding the $\exp_\star$'s we evaluate the generic $O(\lambda^h)$ term in the numerator or denominator as in the undeformed case: applying Wick Thm to the field monomial and $(\Psi_0, :M: \Psi_0) = 0$ we find exactly the *same* integrals over y-variables of products of free propagators having coordinate differences as arguments. Each term is represented by a Feynman diagram. So the **Green's functions coincide with the undeformed ones** and can be computed by Feynman diagrams with the undef. Feynman rules. So, at least perturbatively, **this QFT is completely equivalent to the undeformed one** (no more pathologies like UV-IR mixing!).

Conclusions. What do we learn?

- Glass: half full or half empty?
- Various approaches to QFT on NC spaces. Operator ones (as by [Doplicher, Fredenhagen, Roberts, Bahns, Piacitelli]) on (deformed) Minkowski spaces look safer starting points, but still no completely satisfactory guiding principle.
- Twisting or not Poincaré group, and doing it properly, makes radical differences.
- A sensible theory with twisted Poincaré seems possible: equivalent to the undeformed one. Avoids all complications (IV-UR, causality/unitarity violation, statistics violation, cluster property violation,...).
- Obtained by matching operator $(a, a^\dagger)$ and spacetime noncommutativities somehow to compensate each other

- No new physics, nor a more satisfactory formulation of old one (e.g. by an inthrinsic UV regularization)...
- ... but might be used as a Lab to look for and test equivalent formulations of QFT: Wick rotation into EQFT, path integral quantization, etc.
  Before one should also investigate: how to formulate $R1_\star$, analyticity properties, asymptotic states, spin-statistics, CPT, etc