# GRAVITATIONAL BACK-REACTION OF MATTER INHOMOGENEITIES

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AT SCALES $L \gg 10$ MPC

$$\overline{\delta\rho / \bar{\rho}} \ll 1$$

AND THE UNIVERSE IS WELL DESCRIBED BY THE HOMOGENEOUS F.R.W. SOLUTION

$$H^2 = \frac{8\pi G}{3}(\bar{\rho} + \rho_V) \quad \left(-\frac{K}{R^2}\right)$$

$$\dot{H} = -4\pi G(\bar{\rho} + \underset{\substack{\text{IS}\\ 0}}{\bar{p}}) \quad \left(+\frac{K}{R^2}\right)$$

.... OR IS IT?

AT $L \sim 10\ \mathrm{MP}$

$$\delta\rho / \bar{\rho} \sim 1$$

EINSTEIN'S EQS. ARE NON LINEAR

$\overline{\delta\rho / \bar{\rho}} = 0$ BUT:

$$\frac{1}{L^6} \int_{L^3} d^3x \int_{L^3} d^3y\ \delta\rho(x)\, L(x,y)\, \delta\rho(y) \neq 0$$

E.G. $\dfrac{1}{|x-y|}$

CAN (RELATIVELY) SMALL SCALE INHOMOGENEITIES "FEED UP" TO LARGER, COSMOLOGICAL SCALES?

# A PARADOX?

FRW :

$$ds^2 = -dt^2 + a^2(t)\,\gamma_{ij}\,dx^i dx^j$$

AVERAGE MATTER DISTRIBUTION

$$\bar{T}_0^{\;0} = \bar{\rho}(t) \qquad \bar{T}_i^{\;j} = \delta_i^{\;j}\,\bar{p}(t)$$

ADD INHOMOGENEOUS DUST

$$t_0^{\;0}(\vec{x},t) = \delta\rho(\vec{x},t)$$

$$t_i^{\;j} \approx 0$$

PERTURBED METRIC (LINEAR)

$$ds^2 = -(1+2\phi)\,dt^2 + a^2(t)(1-2\phi)\,\gamma_{ij}\,dx^i dx^j$$

$$a^{-2} D_i D^i \phi = 4\pi G\,\delta\rho$$

WHAT IS THE LEADING NON LINEAR CORRECTION?

GRAVITY COUPLES TO ALL FORMS OF ENERGY, INCLUDING GRAVITATIONAL ENERGY

$$a^{-2} D_i D^i \phi = 4\pi G\, \delta\rho - \frac{1}{2} a^{-2} D_i \phi\, D^i \phi$$

NON-RELATIVISTIC GRAVITATIONAL ENERGY

HEURISTIC BUT GOOD APPROX. FOR NON-RELATIVISTIC INHOMOGENEITIES

HOMOGENEOUS COMPONENT ARBITRARY

THE PARADOX :

$$\gamma_{ij} = \delta_{ij}$$

"DUST" MADE OF COMPACT OBJECTS OF RADIUS $r$ UNIFORMLY DISTRIBUTED AT AVERAGE MUTUAL DISTANCE $\ell$

E.G. ON A CUBIC LATTICE

$$\delta\rho(\vec{x}) = \sum_{\vec{n}\in\mathbb{Z}^3} \rho_r(a\vec{x} - \ell\vec{n}) - \bar{\rho}$$

NORMALIZATION :

$$\int d^3x \, \rho_r(\vec{x}) = m$$

AVERAGE DENSITY :

$$\bar{\rho} = m/\ell^3$$

AVERAGE OVER A CUBE OF SIZE $V \gg \ell^3$

$$\Delta \overline{\phi} = 4\pi G \underbrace{\overline{\delta\rho}}_{= 0} - \frac{1}{2} \overline{\nabla\phi\nabla\phi}$$

$$\overline{\nabla\phi\nabla\phi} = \frac{1}{V}\int d^3x \, \nabla\phi \, \nabla\phi \neq 0 \; !$$

AVERAGE GRAVITATIONAL ENERGY COMPUTED IN TERMS OF FOURIER TRANSFORM OF DENSITY

$$\tilde{\rho}_r(\vec{k}) = \int d^3x \; e^{i\vec{k}\cdot\vec{x}} \rho_r(x)$$

( HERE AND AFTERWARD $a = 1$ )

AVERAGE GRAVITATIONAL ENERGY:

$$-\frac{1}{8\pi G}\overline{\nabla\phi\,\nabla\phi} = -\frac{G}{\ell^6}\sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ \vec{m}\neq 0}}\frac{\ell^2}{2\pi\vec{m}^2}\left|\tilde{\rho}_r\left(\frac{2\pi\vec{m}}{\ell}\right)\right|$$

THE PARADOX:

POINT-LIKE DUST IMPLIES

$$\lim_{K\to\infty}\left|\tilde{\rho}_r(\vec{K})\right|^2 = m \neq 0$$

SO THE LATTICE SUM DIVERGES.

THIS IS THE USUAL SELF-ENERGY DIVERGENCE

REGULATE BY GIVING A FINITE SIZE TO THE "DUST"

$$\rho_r(x) = \begin{cases} 3m/4\pi r^3 & |\vec{x}| \leq r \\ 0 & |\vec{x}| > r \end{cases}$$

THIS IS NOT GOOD ENOUGH:

$$r = 2Gm \quad (= r_{SCHWARZSCHILD})$$

THEN

$$\mathcal{E}_{GRAV} = -\frac{1}{8\pi G}\overline{\nabla\phi\nabla\phi} \approx \bar{\rho}$$

SO A CRITICAL $\left(\frac{8\pi}{3}G\bar{\rho} = H^2\right)$ UNIVERSE MADE OF EQUAL SIZE BLACK HOLES, UNIFORMLY SPACED WOULD RECEIVE $O(1)$ CORRECTIONS TO THE AVERAGE, HOMOGENEOUS F.R.W. EQUATIONS

EVEN WHEN $R_{HUBBLE} \equiv \frac{1}{H} \gg r_{SCHW.}$

$$R_{HUBBLE} \gg \ell$$

EVEN WHEN $r \gg r_{SCHW.}$

THE SELF-ENERGY EXHIBITS A STRONG DEPENDENCE ON THE SOURCE'S SIZE

THIS IS ODD FOR AN AVERAGED QUANTITY!

THE TRUE SOLUTION IS DIFFERENT: WE MUST PERFORM A CLASSICAL RENORMALIZATION OF THE MASS $m$

# AN ESTIMATE OF THE AVERAGE GRAVITATIONAL ENERGY

AN ISOLATED BODY OF MASS $m$ AND SIZE $r$ HAS GRAVITATIONAL ENERGY

$$E = -\frac{Gm^2}{r} C_1$$

$C_1$ = NUMERICAL CONSTANT, INDEPENDENT OF $m, r$

THE ENERGY DENSITY IN A CELL OF SIDE $\ell$ IS

$$\mathcal{E} = -\frac{Gm^2}{\ell^3 r} C_1$$

A REGULAR LATTICE OF MASSES $m$ GIVES INSTEAD

$$\mathcal{E}_G = -Gm^2\left[\frac{C_1}{r\ell^3} + \frac{C_2}{\ell^4} + O\left(\frac{r}{\ell^5}\right)\right].$$

CORRECTIONS MUST BE THERE BECAUSE

$$\mathcal{E}_G = 0$$

FOR A BODY OF UNIFORM DENSITY

$$\bar{\rho} = \frac{m}{\ell^3}$$

FILLING THE CELLS.

THE RESOLUTION OF THE PARADOX IS NOW CLEAR:

THE PHYSICAL, OBSERVABLE MASS OF THE "DUST" PARTICLE IS

$$m_{PHYS} = \underbrace{m}_{\text{"BARE" MASS}} - \underbrace{C_1 \frac{Gm^2}{r}}_{\text{GRAVITATIONAL SELF ENERGY}}$$

BY EXPRESSING $\delta\rho$ IN TERMS OF $m_{PHYS}$ THE $1/r$ DIVERGENCE IS CANCELED THROUGH $\mathcal{O}(Gm^2)$

$$\Delta\bar{\phi} = -4\pi G^2 m^2 \left[ C_2/\rho^4 + \mathcal{O}\left(\frac{r}{\rho^5}\right) \right]$$

CORRECTIONS DUE TO INHOMOGENEITIES EXIST:

- THEY SCALE AS $\frac{1}{\ell^4}$ $\left(= \frac{1}{a^4}\right)$

I.E. AS RADIATION

- THEY ARE FINITE AND SMALL:

THEY INDUCE ANOTHER EFFECTIVE SOURCE OF ENERGY WITH DENSITY $(r \ll \ell)$

$$\rho_{eff} = -Gm^2 \frac{C_2}{\ell^4} = O\left(G\ell^2 \bar{\rho}^2\right)$$

I.E.

$$H^2 = \frac{8\pi}{3} G \bar{\rho} \left(1 + C \ell^2 \bar{H}^2\right)$$

O(1) CONSTANT

UNPERTURBED HUBBLE PARAMETER

FOR VIRIALIZED MATTER

$$H^2 \ell^2 \sim v^2$$

AVERAGE PECULIAR VELOCITY$^2$
INSIDE A CLUSTER

$$O(10^{-5})$$

IN OUR UNIVERSE

OPEN QUESTIONS:

- COMPUTE ALL CORRECTIONS TO EINSTEIN'S EQS. THROUGH $\mathcal{O}(Gm^2)$

- IS THE EFFECT REALLY THERE? ($\equiv$ IS THERE A GAUGE WHERE IT VANISHES?)

- WHAT IS THE EFFECT TO $\mathcal{O}(G^{N-1} m^{2N})$? (DO LARGE EFFECTS ARISE BY RESUMMATION OF EXPANSION)

- HOW TO TREAT A MULTI-SCALE (CONTINUOUS) DISTRIBUTION OF MATTER?

WE TAKE ANOTHER APPROACH:

- LOOK AT SIMPLE, EXACTLY SOLUBLE MODELS OF INHOMOGENEITY

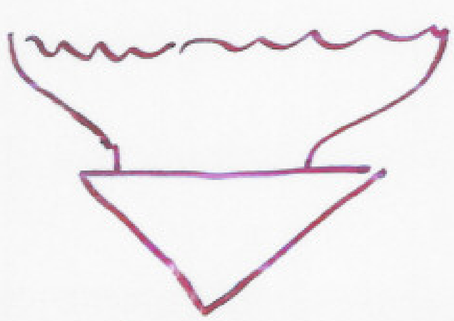
- TRY TO EXTRACT A GENERAL LESSON FROM THERE

MODELS:

- DUST-WALL UNIVERSE

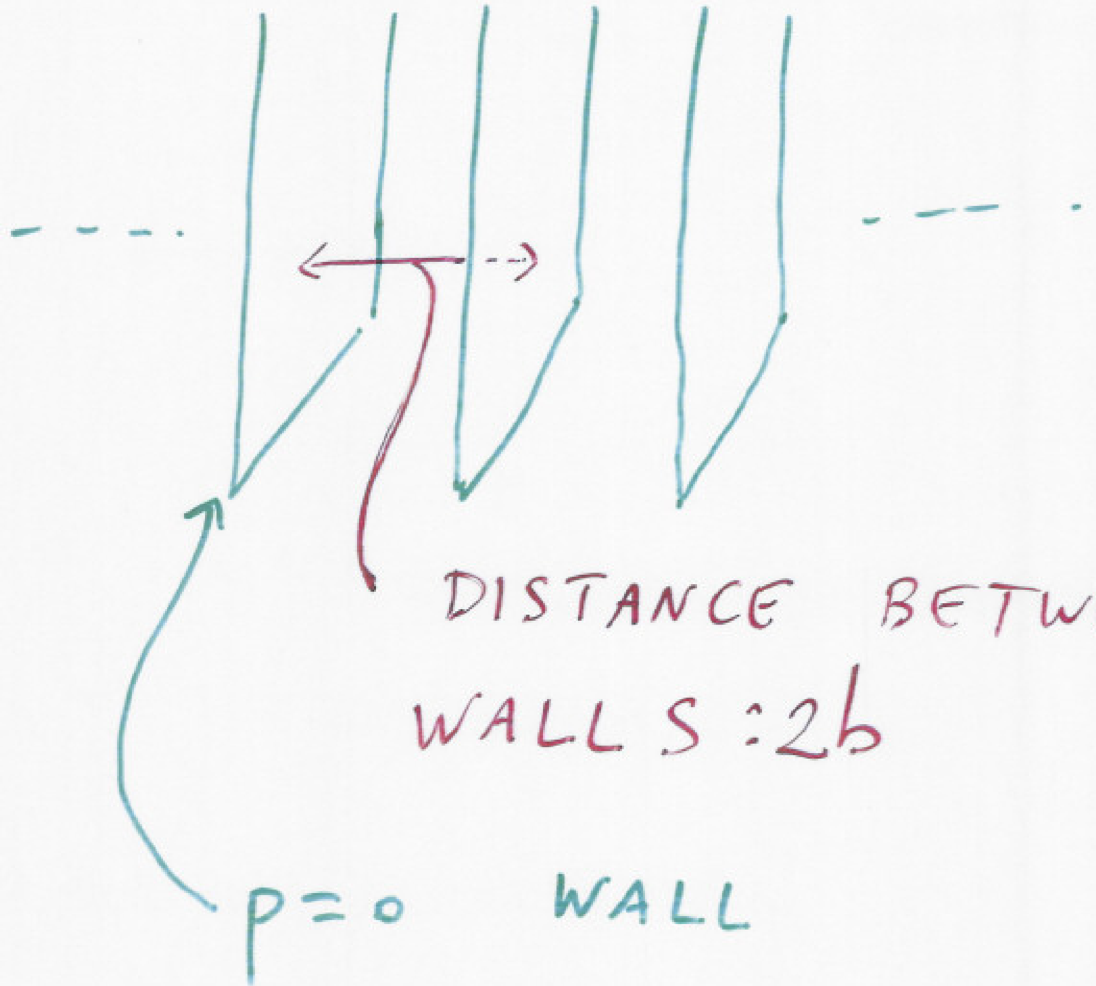
- 3D GRAVITY WITH POINTLIKE MATTER SOURCES

- SWISS-CHEESE UNIVERSE



CORRECTIONS ALWAYS $O(H^2 \ell^2)$

# DUST-WALL UNIVERSE




PARALLEL PRESSURELESS WALLS OF SURFACE DENSITY $\sigma(t)$

AT $\quad x = (2n+1)\,b \quad n \in \mathbb{Z}$

AVERAGE DENSITY : $\sigma/2b$

IT IS A ONE-DIM. PROBLEM

$$ds^2 = e^{2u(t,x)}(-dt^2 + dx^2) + e^{2v(t,x)}(dy^2 + dz^2)$$

THIS ONE-DIM. PROBLEM CAN BE SOLVED EXACTLY

TO LOWEST ORDER IN $b$ THE METRIC IS HOMOGENEOUS

$$e^{2u} \propto e^{2v} \propto t^4$$

IT DESCRIBES A HOMOGENEOUS MATTER-DOMINATED UNIVERSE

$t$ = CONFORMAL TIME

"STANDARD" SYNCRONOUS TIME $\propto t^3$

NEXT TO LEADING ORDER IN $b$

$$e^{2u} \propto t^4 + t^2(20b^2 - 6x^2)$$

$$e^{2v} \propto t^4 + t^2(40b^2 + 6x^2)$$

$$-b \leq x \leq b$$

CORRECTIONS TO $H$

Q: WHICH $H$?

DEFINITION AMBIGUOUS

E.G.

$$H_{11} = \frac{d \log a_{11}}{d\tau}$$

$a_{11}(t) \propto e^{v(t,b)}$ IS THE SCALE FACTOR ALONG $y, z$

$\tau$ = PROPER TIME ON THE WALL

$$H_\perp = \frac{d \log a_\perp}{d\tau}$$

$$a_\perp = e^{u(t,b)}$$

SCALE FACTOR ALONG X

BOTH DEFINITIONS ARE EQUALLY PHYSICAL

IN TERMS OF THE PROPER DISTANCE BETWEEN WALLS

$$\ell \equiv \int_{-b}^{b} dx\, e^{u(x,t)}$$

$$H_{\parallel}^2 a_{\parallel}^3 \propto 1 + \frac{9}{16} H^2 \ell^2$$

$$H_\perp^2 a_\perp^3 \propto 1 - \frac{7}{16} H^2 \ell^2$$

$$\left( H \approx H_\perp \approx H_\parallel \approx \frac{2}{3\tau} \right)$$

MORAL OF THE STORY:

- INHOMOGENEITY EFFECTS ARE INDEED

$$O(H^2 \ell^2)$$

AND

⋮

CAVEAT EMPTOR

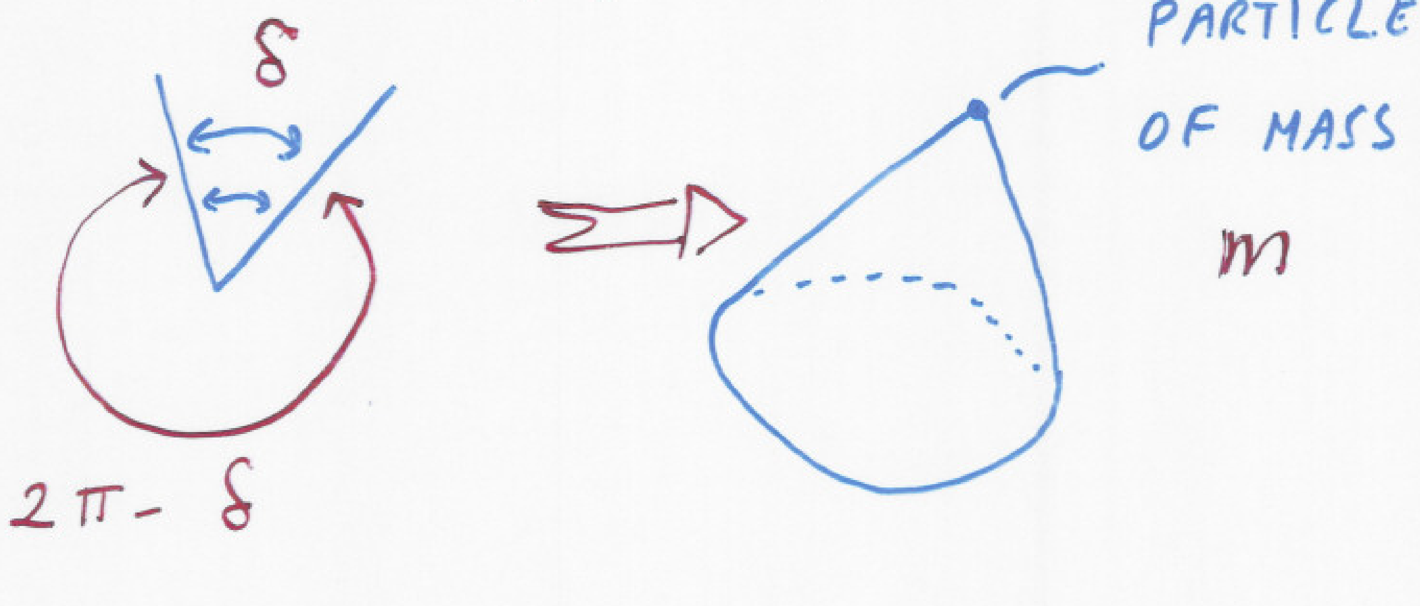
DEFINITION OF $H$ AMBIGUOUS.

IF WE WANT TO DO PRECISION COSMOLOGY WE MUST DECIDE EXACTLY WHAT WE CALL $H$

# 3D GRAVITY WITH POINT PARTICLES

IN 2+1 D A POINT PARTICLE IN G.R. DOES NOT CURVE SPACE. INSTEAD IT CREATES A CONICAL SINGULARITY WITH DEFICIT ANGLE

$$\tilde{\delta} = +8\pi G m$$



CONSIDER A UNIVERSE WITH AN ARBITRARY DISTRIBUTION OF PARTICLES WITH MASS $m_1 \ldots m_N$

WITH $$\sum_{i=1}^{N} m_i > \frac{1}{2G}$$

AN EXACT SOLUTION EXISTS:

$$ds^2 = -dt^2 + a^2(t)\, e^{\phi(z,\bar{z})} dz\, d\bar{z}$$

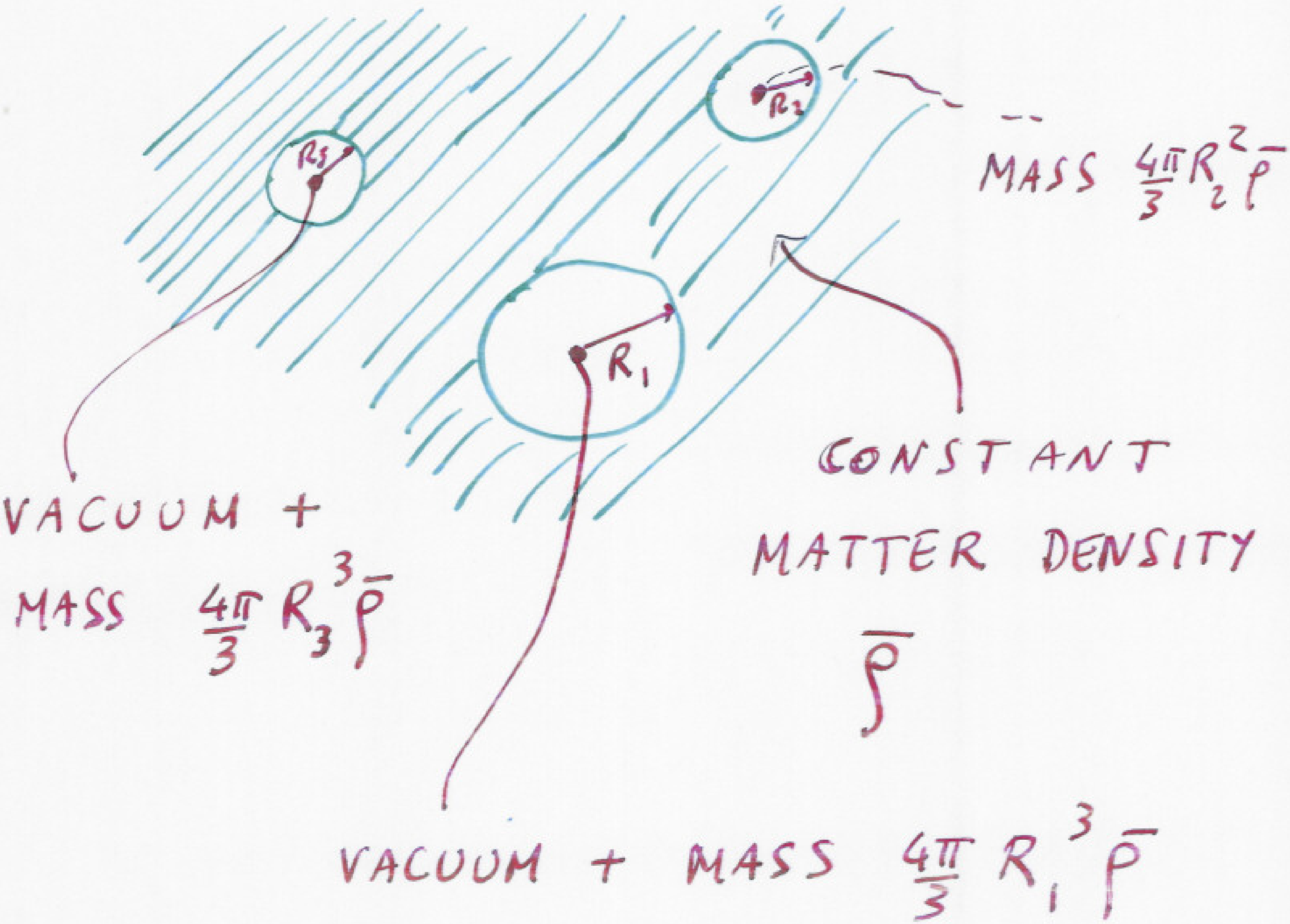
$$\partial_z \partial_{\bar{z}} \phi = \frac{1}{2} e^{\phi} - \sum_{i=1}^{N} 8\pi G m_i\, \delta^2(z - z_i)$$

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{a^2}$$

HUBBLE PARAMETER INSENSITIVE TO INHOMOGENEITIE

OTHER PHYSICAL QUANTITIES DO CHANGE

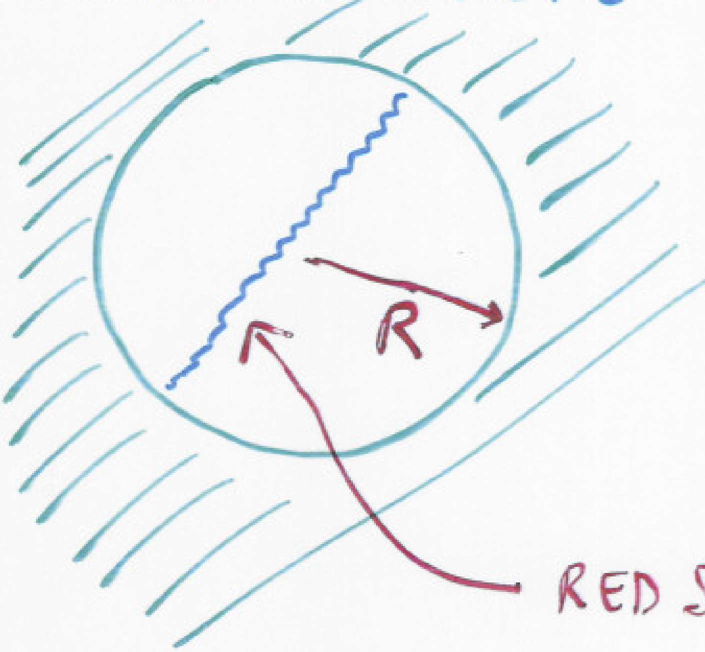
# SWISS CHEESE UNIVERSE




IN THIS EXACT SOLUTION THE HUBBLE PARAMETER OUTSIDE THE HOLES IS UNCHANGED

# SACHS-WOLFE EFFECT






REDSHIFT $\propto$

$$\left.\frac{d}{dr}\phi(r)\right|_{r=R} \cdot \dot{R} \cdot \Delta T$$

LIGHT FALLS INTO POTENTIAL WELL OF HOLE AT TIME $T$ AND COMES OUT AT TIME $T + \Delta T$ $\quad \Delta T \sim R$

$$\phi(r) = -G\frac{4\pi}{3}\bar{\rho}R^3/r$$

REDSHIFT $\sim$ $\quad H^3R^3 + \ldots$

STANDARD REDSHIFT OF HOMOGENEOUS MATTER:

$\propto HR$

MORAL IS THE SAME AS IN PREVIOUS EXAMPLES

INHOMOGENEITIES AT SCALE

$R$ CORRECT COSMOLOGICAL OBSERVABLES BY TERMS SUPPRESSED BY

$$O(H^2 R^2)$$

THIS IS THE SAME ORDER OF MAGNITUDE AS THE AMBIGUITIES PRESENT IN THE VERY DEFINITION OF PHYSICAL QUANTITIES THAT CAN BE IDENTIFIED WITH THE HUBBLE PARAMETER

$$H$$