Black-holes, topological strings and q-deformed two-dimensional Yang-Mills theory

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Summary of the talk

- Introduction: black-hole entropy
- BPS black-holes in $\mathcal{N}=2$ supergravity
- From black-holes to topological strings: Ooguri-Strominger-Vafa (OSV) conjecture
- Checking the conjecture: instanton counting in $\mathcal{N}=4$ topological SYM_4 and q-deformed YM_2
- Results in collaborations with N. Caporaso (MIT), M. Cirafici (Heriot-Watt), M. Marino (CERN), S. Pasquetti (Parma), D. Seminara (Firenze), R. J. Szabo (Heriot-Watt), A. Tanzini (SISSA)

Black-hole's Thermodynamics

Laws	Thermodynamics	Black-Hole
Zeroth Law	T constant throughout body in thermal equilibrium	κ constant over horizon of stationary black hole
First Law	$dU = TdS+$work terms	$dM = \frac{\kappa dA}{4\pi} + \Omega_H dJ$
Second Law	$\delta S \geq 0$ in any process	$\delta A \geq 0$ in any process

Attempt of a Dictionary

Possible Dictionary:

$$A \text{ (area)} \mapsto S(\text{entropy}) \qquad \kappa \begin{pmatrix}\text{surface}\\ \text{gravity}\end{pmatrix} \mapsto k_B T \text{ (temperature)}$$

Notice that the dimensions are wrong

$$L^2 \longleftrightarrow \text{adimensional} \qquad \text{acceleration} \longleftrightarrow \text{energy}$$

There is no classical constant or combination of classical constants to restore the right dimensions. But if we borrow $\hbar$ from Q.M., the following combinations possess the right dimensions

$$\underbrace{\frac{c^3 A}{G\hbar}}_{S} \qquad \underbrace{\frac{\hbar\kappa}{c}}_{k_B T}$$

We need Q.M. to complete the dictionary

Hawking Temperature (with a little trick)

Consider the Schwarzschild black-hole

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega$$

Let us look at the near-horizon geometry. Setting $x = 2\sqrt{(r - r_g) r_g}$ and $r_g = 2GM/c^2$, we can write for small x

$$ds^2 = \underbrace{-\frac{c^2 x^2}{4 r_g} dt^2 + dx^2}_{Rindler\ space} \quad + \quad \underbrace{r_g^2 d\Omega}_{S^2} + \text{sublead.}$$

To investigate a thermal field theory in this background, we exploit the usual trick to rotate in Euclidean space-time $t \mapsto i\tau_E$

$$ds^2 = \underbrace{\frac{c^2 x^2}{4 r_g^2} d\tau_E^2 + dx^2}_{cone} \quad + \quad \underbrace{r_g^2 d\Omega}_{S^2} + \text{sublead.}$$

This geometry has generically conical singularity for $x = 0$. However there is no reason for a conical singularity at $x = 0$ because the Minkowskian geometry is regular there.

We can avoid the apex of the cone requiring that

$$\frac{c}{2r_g} \times \tau_E \quad \text{is periodic of} \quad 2\pi$$

Namely

$$\tau_E \text{ is periodic of } \frac{4\pi r_g}{c} = \frac{2\pi c}{\kappa}$$

But in the path-integral approach a system with a periodic time means a system at finite temperature. The period of the time is identified with $\beta\hbar$, thus

$$k_B T = \frac{\hbar\kappa}{2\pi c} \qquad \Rightarrow \qquad S = \frac{c^3 A}{4G\hbar}$$

But

In statistical thermodynamics $S = \log(N_{microstates})$

$$\Downarrow$$

What are black-hole's microstates?

Extremal black-holes

Consider the Reissner-Nordstrom black-hole

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega$$

This BH possesses an inner and an outer horizon, we shall consider the limit when these two horizons coincide:

$$M = |Q| \quad \text{(Electric force = Gravitational force)} \quad \Rightarrow \quad T_H = 0$$

In isotropic coordinates, where the horizon is located at $r = 0$, the metric (for $M = |Q|$) takes the form

$$ds^2 = -\left(1 + \frac{Q}{r}\right)^{-2} dt^2 + \left(1 + \frac{Q}{r}\right)^2 (dr^2 + r^2 d\Omega)$$

This BH already displays some of the general properties we are interested in. For large r, the metric is asymptotically flat

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega \qquad (\text{Minkowski space})$$

For small r (near horizon metric)

$$ds^2 = \underbrace{-\frac{Q^2}{r^2}dt^2 + \frac{r^2}{Q^2}dr^2}_{AdS_2} + \underbrace{Q^2 d\Omega}_{S_2} \qquad (AdS_2 \otimes S_2 : \text{ Bertotti space})$$

(Solution of Eins.-Maxw. system with covariantly constant e.m. field strength)

Both the metrics possess two Killing spinors, namely they can be considered as a vacuum with $\mathcal{N} = 2$ supersymmetry

$$\Downarrow$$

Extremal RS BH as a soliton interpolating between 2 SUSY vacua

This BH preserve 1/2 of the supersymmetry. The complete supersymmetry is restored only at $r = 0$ (Bertotti) and at $r = \infty$ (Minkowski).

Embedding in $\mathcal{N}=2$ supergravity

In order to investigate this family of supersymmetric BHs, we need to recall some basics fact on $\mathcal{N}=2$ supergravities, whose bosonic Lagrangian is

$$S=\int\sqrt{-g}d^4x(2R+\mathrm{Im}\mathcal{N}_{LM}F^L_{\mu\nu}F^{M|\mu\nu}+\mathrm{Re}\mathcal{N}_{LM}\epsilon^{\mu\nu\rho\sigma}F^L_{\mu\nu}F^M_{\rho\sigma}+\frac{1}{6}g_{IJ}\partial_\mu\phi^I\partial\phi^J)$$

Field Content:

- *Gravity:* $g_{\mu\nu}$ *(graviton)* ψ^I_μ *(2 gravitinos)* A_μ *graviphoton*
- n_v *vector multiplets* X^L *:* $A^I_\mu \quad \lambda^I \quad \phi^I;$
- n_H *hypermultiplets (Luckily they are spectators, we shall forget about them!).*

The nice feature of $\mathcal{N}=2$ supersymmetry is that the interaction of the vector multiplets with gravity can be encoded just in one holomorphic function the *prepotential* $F(X^I)$ (homogeneous function of degree 2). X^I can be essentially identified with the scalars in the vector multiplets. We shall also define the *potential* $F_L=\partial_L F(X)$.

In this setting our BPS black-hole solutions are those saturating the bound

$$|Z|_\infty = M$$

where $|Z|_\infty$ is the central charge appearing in the SUSY algebra and M the mass of the BH. Here

$$Z_\infty = Z|_{r\to\infty} \quad \text{with} \quad Z = (q_L X^L - F_L p^L) e^{K/2(X,\bar{X})}$$

(q_L, p^L) are the electric and the magnetic charges carried by the BH. The function K is defined as

$$K = \log[i(\bar{X}^L F_L - X^L \bar{F}_L)]$$

K is the Kahler potential.

Notice that Z might depend on the value of the scalars at infinity!

Computing the entropy for this family of $\mathcal{N}=2$ black-holes: Attractor Mechanism

Recall the ansatz for the metric

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}(dr^2 + r^2 d\Omega).$$

The anti-selfdual part of the gauge field strength is

$$F^{I-} = \frac{1}{2}[p^I - i[\mathrm{Im}\mathcal{N}]^{IJ}(q_J - [\mathrm{Re}\mathcal{N}]_{JL}p^L)]\left[\sin\theta d\theta \wedge d\phi - i\frac{e^{2U}}{r^2}dt \wedge dr\right]$$

The function $U(r)$ and the scalars X^I obey to the following equations

$$r^2\frac{dU(r)}{dr} = -|Z|e^U \qquad \Rightarrow \qquad \frac{d\mu}{dr} = \frac{|Z|}{r^2}$$

$$r^2\frac{dX^I}{dr} = -2e^U g^{i\bar{j}}\partial_{\bar{j}}|Z| \qquad \Rightarrow \qquad \mu\frac{dX^I}{d\mu} = -g^{I\bar{J}}\partial_J \log|Z|^2$$

where $\mu = e^{-U}$.

- The second equation looks like an equation for the RG. When we approach the horizon, $g_{tt} = e^U \to 0$ and consequently $\mu = e^{-U} \to \infty$. This suggests that X^I are attracted to the minima of the potential $\log|Z|^2$. In other words the scalar field on the horizon are fixed by the equation

$$\partial_I |Z| = 0 \qquad \text{(Attractor Equations)}$$

  only in terms of the charges.

- The first equation teaches us that $e^{-U(r)}$ has the form

$$\mu = e^{-U(r)} = \frac{|Z_0|}{r} \quad \text{where} \quad Z_0 = Z|_{r=0}.$$

For the whole family of these black-holes, the near horizon metric is

$$ds^2 = -\frac{|Z_0|^2}{r^2} dt^2 + \frac{r^2}{|Z_0|^2} dr^2 + |Z_0|^2 d\Omega$$

We are ready to compute the entropy for this family of black-holes

$$S_{BH} = \frac{A}{4} = \frac{1}{4} 4\pi \lim_{r \to 0} r^2 e^{-2U} = \pi |Z_0|^2$$

Since the attractor equation determines the value of scalars at the horizon just in terms of the charges, the entropy will be a function only of the charges as well: attractor mechanism

This property is fundamental if we want to find a microscopical interpretation of the entropy for $\mathcal{N} = 2$ BPS black-holes.

First hint to OSV conjecture

Actually with some manipulations, we can rewrite the Attractor Equations and the entropy in a form that is more natural for the future developments:

$$q_I = \mathrm{Re}[F_I] \qquad p^I = \mathrm{Re}[X^I] \qquad \text{(Attractor Equations)}$$

$$S_{BH} = \frac{i\pi}{4}(\bar{X}^I F_I - X^I \bar{F}_I) \qquad \text{(Entropy)}$$

The second equation states that $X^I = p^I + \frac{i}{\pi}\phi^I$, while the first one states that $q_I = (F_I + \bar{F}_I)/2$. Then using the homogeneity property $X^I F_I = 2F$

$$S_{BH} = \frac{\pi i}{4}(X^I F_I - \bar{X}^I \bar{F}_I) + \frac{1}{2}\phi^I(F_I + \bar{F}_I) = \mathcal{F} + \phi^I q_I$$

$$\text{with} \quad q^I = -\frac{\partial \mathcal{F}}{\partial \phi_I} \qquad \text{and} \quad \mathcal{F} = \frac{\pi i}{2}[F(X) - \bar{F}(\bar{X})]$$

The entropy is the Legendre Transform of the $\mathcal{F}$ with respect to ϕ^I.

This observation opens an interesting connection between the BH entropy and the imaginary part of the prepotential F. In fact F has a natural and simple interpretation at level of compactification of type II superstrings on Calabi-Yau.

F is the genus zero partition function F_0 of the topological string theory on the Calabi-Yau manifold on which we compactify the superstrings to get the $\mathcal{N} = 2$ theory in 4 dimensions (to be precise $F = -iF_0/\pi$).

At the microscopic level F_0 counts the leading contribution to the number of bound states of branes that we need to build the black-hole.

Notice that the topological string is not directly related to the partition function of a black-hole with fixed magnetic and electric charges. It is rather related to an ensamble of BHs with fixed magnetic charges, but free electric charges. What is fixed is the chemical potential associated to the electric charges.

Higher derivative contributions

Actually the above analysis can be improved to include string corrections to the entropy.

At the level of supergravity, this means that we can include higher derivative terms in the action and try to compute the new BH solutions and their entropy.

String loops corrections are not obviously easy to compute. There is however a nice class of amplitudes that generates at supergravity level a peculiar set of terms: **the F-terms**

$$\int d^4x d^4\theta F(X, W^2) = \sum_{g=0}^{\infty} \int d^4x d^4\theta (W_{ab}W^{ab})^g F_g(X),$$

where W_{ab} is the Weyl superfield. This gives origin to terms such R^2T^{2g-2} where T is the graviphoton field strength.

These terms originate, at the level of type II superstring compactified on CY, from amplitudes of the type two gravitons into graviphotons. The index g here denotes the genus expansion.

These amplitudes are special. They are "topological" and depend only on the moduli space of the CY we have compactified on. They can be computed in a simpler string theory called "Topological String Theory": $N = (2,2)$ twisted sigma models coupled to topological gravity with CY_3 target space.

Here

$$F(X, W^2) = \sum_{g=0}^{\infty} (W_{ab} W^{ab})^g F_g(X)$$

is a generalization of the prepotential, which is the first term of its expansion. Each coefficient F_g has an interpretation at the level of topological strings!

F_g is the genus g free energy of the topological string theory whose target space is the CY under consideration

On the other hand we expect the entropy of a BPS black-hole to be of topological nature and thus to receive contributions only from terms that respect this property.

Using the new lagrangian with the insertion of the F-terms, we can repeat the analysis for the BPS black-holes step by step. However we miss an ingredient

How do we compute the entropy for a theory of gravity whose lagrangian is an arbitrary polynomial of the Riemann?

The answer was provided by Wald who has reinterpreted the Hawking entropy as a generalized Noether charge. In this formulation, the entropy can be computed for any gravity theory admitting BH solutions.

We can write a corrected set of attractor equations

$$C^2 W^2 = 256$$

$$p^L = \mathrm{Re}[C X^L] \qquad q_L = \mathrm{Re}[C \partial_L F \left(X^L, 256/C^2\right)]$$

Here C is a scaling field, that in principle can be gauged away with a Kahler transformation. Using the Wald's formula the corrected entropy is

$$S_{BH} = \underbrace{\frac{\pi i}{2}(q_L \bar{C} \bar{X}^L - p^L \bar{C} F_L)}_{OLD\ \ ENTROPY} + \underbrace{\frac{\pi}{2}\mathrm{Im}[C^3\ \partial_C F]}_{CORRECTION}$$

If we define, as before, the function

$$\mathcal{F}(\phi, p) = -\pi \mathrm{Im}[C^2 F(X^L, 256/C^2)]$$

the entropy is given by

$$S_{BH} = \mathcal{F}(\phi, p) - \phi^L \frac{\partial}{\partial \phi^L} \mathcal{F}(\phi, p) \quad \text{with } \ q^L = -\frac{\partial}{\partial \phi^L} \mathcal{F}(\phi, p)$$

In other words, $\mathcal{F}(\phi, p)$ is the Legendre-Transform of S_{BH} with respect to the chemical electric potential ϕ^L. Posing

$$Z_{BH} = \exp(\mathcal{F}(\phi, p)),$$

the partition function defined in this way can be interpreted as follows

$$Z_{BH} = \sum_{q_L} \underbrace{\Omega(p^L, q_L)}_{\mathrm{BH's\ with\ (p^L, q_L)}} e^{-\phi^L q_L}.$$

It is a mixed partition function: microcanonical for the magnetic charge and grand canonical for the electric charges.

By expanding $\mathcal{F}(\phi, p)$ we can find the following relation with perturbative topological string free energy

$$\mathcal{F}(\phi, p) = F_{top} + \bar{F}_{top} = 2\mathrm{Re}F_{top},$$

namely

$$Z_{BH} = \exp(F_{top} + \bar{F}_{top}) = |Z_{top}|^2 \qquad \boxed{\text{OSV Conjecture}}$$

Topological string free energy on a Calabi-Yau threefold X has the following structure (in IIA case): it depends on the Kahler parameters t^L, describing the Kahler moduli of X, and on the string coupling g_s

$$\mathcal{F}_X = \sum_{g=0}^{\infty} g_s^{2g-2}\ \mathcal{F}_g(\boldsymbol{t})$$

with

$$\mathcal{F}_g(\boldsymbol{t}) = \mathcal{F}_g^{\boldsymbol{C}lass.}(\boldsymbol{t}) + \sum_{\boldsymbol{n}\in H_2(X,\mathbb{Z})_+} \mathrm{e}^{-\boldsymbol{n}\cdot\boldsymbol{t}}\ \mathrm{N}^g{}_{\boldsymbol{n}}\ ,$$

$\mathrm{N}^g{}_{\boldsymbol{n}} \in \mathbb{Q}$ are the infamous Gromov-Witten invariants of X, at genus g.

On the other hand Z_{BH} depends on the charges p^L and the chemical potential ϕ^L: the dictionary is

$$t^L = 2\pi i \frac{p^L + i\phi^\Lambda \pi}{p^0 + i\phi^0/\pi}, \qquad g_s = \pm \frac{4\pi i}{p^0 + i\phi^0/\pi}$$

Just a formulation a bit more precise

Consider Type IIA superstring theory on $X \times \mathbb{R}^{3,1}$, X is a Calabi-Yau threefold.

N=2 BPS Black-Hole	$\Leftrightarrow$	D6, D4, D2, D0 branes wrapping over the cycles of X
Charges of the B.H.	$\Leftrightarrow$	D6,D4 magnetic charges D2, D0 electric charges

One can define a partition function for a mixed ensemble of $\mathcal{N}=2$ BPS black hole states by fixing the magnetic charges Q_6 and Q_4 and summing over the D2 and D0 charges with fixed chemical potentials ϕ_2 and ϕ_0 to get

$$Z_{\rm BH}(Q_6,Q_4,\phi_2,\phi_0)=\sum_{Q_2,Q_0}\Omega(Q_6,Q_4,Q_2,Q_0)\; e^{-Q_2\phi_2-Q_0\phi_0}$$

$\Omega(Q_6, Q_4, Q_2, Q_0)$ is the contribution from BPS states with fixed D-brane charges. The conjecture states that

$$Z_{\mathrm{BH}}(Q_6, Q_4, \phi_2, \phi_0) = \sum_{Q_2, Q_0} \Omega(Q_6, Q_4, Q_2, Q_0)\, e^{-Q_2\phi_2 - Q_0\phi_0} = \left|Z_{\mathrm{top}}(g_s, t_s)\right|^2 ,$$

or conversely

$$\Omega(Q_6, Q_4, Q_2, Q_0) = \int d\phi_I e^{Q_I \phi^I} |Z_{\mathrm{top}}|^2$$

$$g_s = \frac{4\pi\,\mathrm{i}}{\frac{\mathrm{i}}{\pi}\,\phi_0 + Q_6}\,, \qquad t_s = \tfrac{1}{2}\, g_s \big(-\tfrac{\mathrm{i}}{\pi}\,\phi_2 + Q_4\big)$$

$Z_{\mathrm{top}}(g_s, t_s)$ is the (A-model) topological string partition function.

- Although magnetic and electric charges are treated differently, the formulae must be invariant under electromagnetic dualities!
- Actually the sum defining the partition function of the BH is divergent (instability of the mixed ensemble). Integration contour!
- Z_{BH} is formally periodic in $i\phi_I$ due to charge quantization. Z_{top} is not!

Instanton Counting N=4 (twisted) YM_4

Checking the OSV conjecture requires to find situation where

- Z_{BH} is computable from the microscopic theory of branes
- Z_{top} is known at any order in the genus expansion.

For BPS black-holes, the first step is sometimes equivalent to evaluating an observable in the twisted SYM theory leaving on a brane. Here, following Aganagic, Ooguri, Saulina and Vafa (AOSV), we shall consider the simplified situation where D6 branes are absent, but we have D4, D2 and D0-branes. We shall take a non-compact Calabi-Yau containing a 4-cyle of the type

$$C_4 = \mathcal{O}(-p) \longrightarrow \Sigma_g$$

where D4 branes wrap. D2-branes wrap the Riemann surface Σ_g.

The number of D4-branes is fixed to be N and one should count the ensemble of bound states on it. The complete CY threefold is then taken

$$X = \mathcal{O}(2g-2+p) \oplus \mathcal{O}(-p) \longrightarrow \Sigma_g$$

The relevant gauge theory on the N D4-branes is the $\mathcal{N}=4$ topologically twisted $U(N)$ Yang-Mills theory on C_4 in the presence of chemical potentials for D2 and D0-branes. This is simulated by turning on the observables

$$S_c = \frac{1}{2g_s}\int_{C_4} \mathrm{Tr}\big(F \wedge F\big) + \frac{\theta}{g_s}\int_{C_4} \mathrm{Tr}\big(F \wedge K\big)$$

where F is the Yang-Mills field strength and K is the unit volume form of Σ_g. The chemical potentials ϕ_0, ϕ_2 and gauge parameters g_s, θ are related by

$$\phi_0 = \frac{4\pi^2}{g_s}\ , \quad \phi_2 = \frac{2\pi\,\theta}{g_s}.$$

This means that the charges q_0, q_2 of the D0 and D2 branes are

$$q_0 = \frac{1}{8\pi^2}\int_{C_4} \mathrm{Tr}\big(F\wedge F\big)\ , \qquad q_2 = \frac{1}{2\pi}\int_{C_4} \mathrm{Tr}\big(F\wedge K\big)\ .$$

Obtaining $Z_{\rm BH}$ is therefore equivalent to computing

$$Z_{\rm BH} = \Big\langle \exp\Big[-\frac{1}{2g_s}\int_{C_4}\mathrm{Tr}\big(F\wedge F\big) - \frac{\theta}{g_s}\int_{C_4}\mathrm{Tr}\big(F\wedge K\big)\Big]\Big\rangle = Z_{\mathcal{N}=4}\ .$$

The partition function $Z_{\mathcal{N}=4}$ has an expansion of the form

$$Z_{\mathcal{N}=4} = \sum_{q_0,q_2}\Omega(q_0,q_2;N)\ \exp\Big(-\frac{4\pi^2}{g_s}\,q_0 - \frac{2\pi\,\theta}{g_s}\,q_2\Big)$$

where $\Omega(q_0,q_2;N)$ is under suitable assumptions the Euler characteristic of the moduli space of $U(N)$ instantons on C_4 in the topological sector labelled by the Chern numbers q_0 and q_2.

Counting of BH microstates is equivalent to an instanton counting in the $\mathcal{N}=4$ topological gauge theory!

q-Deformed YM_2

A fundamental observation due to Vafa is that the $\mathcal{N} = 4$ instanton counting can be effectively reduced in this case to the partition function of a two-dimensional field theory, under suitable assumptions.

This is achieved by introducing certain massive perturbations that localize the theory to $U(1)$-invariant modes and reduce the model to an effective gauge theory on Σ_g. The topological nature of the theory makes the result actually independent of the massive deformations

$$W = mUV + \omega T^2$$

and one can send the masses to infinity obtaining the localization.

Remark: this is a clever trick! But recently new results from brute force instanton counting on C_4 have been derived...

The localization is not simply a dimensional reduction because the non trivial nature of the line-bundle $\mathcal{O}(-p)$ generates an extra term in the 2D effective action

$$S_p = -\frac{p}{2g_s}\int_{\Sigma_g} \mathrm{Tr}\,\Phi^2\; K\;,$$

where

$$\Phi(z) = \oint_{S^1_{z,|u|=\infty}} A$$

is the holonomy of the gauge field A around a circle at infinity in the fiber over the point $z \in \Sigma_g$. The relevant two-dimensional action becomes

$$S_{\mathrm{YM}_2} = \frac{1}{g_s}\int_{\Sigma_g} \mathrm{Tr}\big(\Phi\, F\big) + \frac{\theta}{g_s}\int_{\Sigma_g} \mathrm{Tr}\,\Phi\; K - \frac{p}{2g_s}\int_{\Sigma_g} \mathrm{Tr}\,\Phi^2 K.$$

This is just YM_2 theory on the Riemann surface Σ_g..but not exactly!

The new ingredient is that the scalar field Φ is periodic

It parameterizes the holonomy of the gauge field at infinity. The periodicity affects the path integral measure and consequently the quantum theory has an interpretation as a q-deformation of two-dimensional Yang-Mills theory. The partition function is

$$Z_{\mathcal{N}=4} = Z^q_{\mathrm{YM}} = \sum_R \dim_q(R)^{2-2g}\; q^{\frac{p}{2}\,C_2(R)}\; e^{\,\mathrm{i}\,\theta\, C_1(R)}$$

Let us compare it with the usual YM_2

$$Z_{\mathrm{YM}} = \sum_R \dim(R)^{2-2g}\; e^{\frac{g^2 A}{2}\,C_2(R)}\; e^{\,\mathrm{i}\,\theta\, C_1(R)}.$$

In both cases, R runs through the unitary irreducible representations of the gauge group $U(N)$, $C_1(R)$ and $C_2(R)$ are respectively its first and second Casimir invariants.

The difference is in the dimensions: in the usual YM_2 we have the standard dimension

$$\dim(R) = \prod_{1\leq i<j\leq N} \frac{R_i - R_j + j - i}{j - i}$$

while in the other case we have *quantum* dimension

$$\begin{aligned}\dim_q(R) &= \prod_{1\leq i<j\leq N} \frac{\left[R_i - R_j + j - i\right]_q}{\left[j - i\right]_q} = \\ &= \prod_{1\leq i<j\leq N} \frac{\left[q^{(R_i - R_j + j - i)/2} - q^{-(R_i - R_j + j - i)/2}\right]}{\left[q^{(j-i)/2} - q^{-(j-i)/2}\right]}\end{aligned}$$

where $q = e^{-g_s}$. Clearly as $g_s \to 0$ the quantum dimension goes smoothly into the classical one.

We have therefore:

$$Z_{\mathrm{BH}} = Z_{\mathcal{N}=4} = Z^q_{\mathrm{YM}}$$

Topological strings should emerge at large charges (namely large N)!

Nicely at large N, $Z^q_{\rm YM}$ undergoes a Gross-Taylor like factorization into

$$\sum_{l=-\infty}^{\infty}\ \sum_{R_1,\dots,R_{2g-2}} Z^{{\rm YM}^q,+}_{R_1,\dots,R_{2g-2}}(t_s+p\,g_s l)\, Z^{{\rm YM}^q,-}_{R_1,\dots,R_{2g-2}}(\bar{t}_s-p\,g_s l)$$

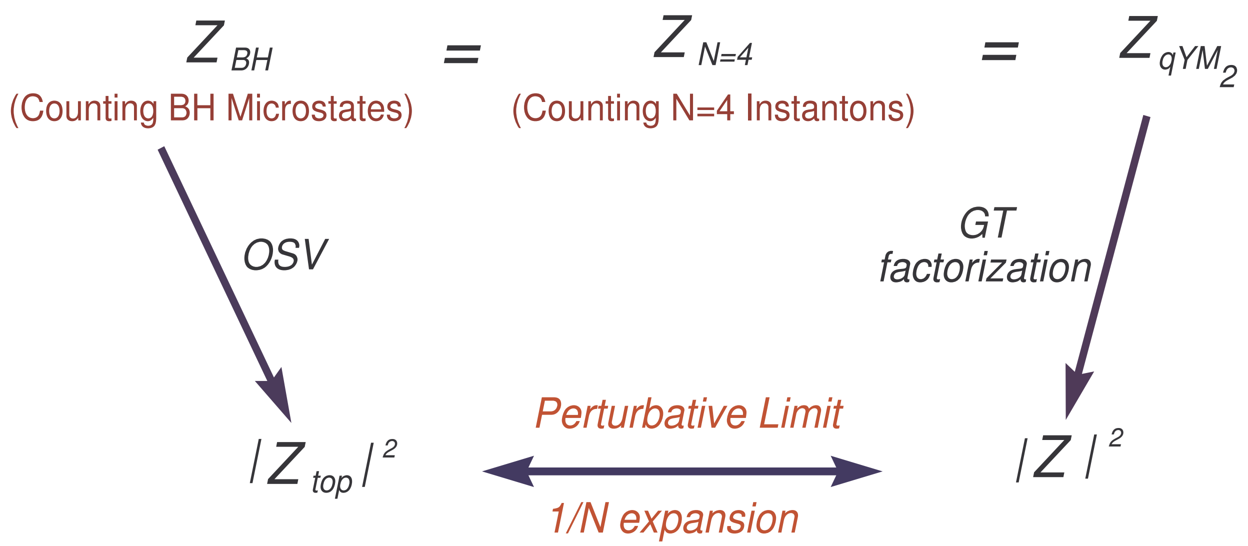
- $Z^{{\rm YM}^q,+}_{R_1,\dots,R_{2g-2}}(t_s+p\,g_s l)$ is the perturbative A-model topological string amplitude on X_p with $2g-2$ stack of D-branes inserted into the fibers: it is an open string amplitude. This partition function has been computed recently by Bryan and Pandharipande.
- $Z^{{\rm YM}^q,+}, Z^{{\rm YM}^q,-}$ are then glued together to give a closed string amplitude.
- The extra sum over the integer l originates from the $U(1)$ degrees of freedom contained in the original gauge group $U(N)$.
- Since the factorization as well as the perturbative topological string amplitude appear in the large N expansion while $Z^q_{\rm YM}$ is non-perturbative in N, $Z_{\rm BH}$ has been proposed to be the non-perturbative completion of $Z_{\rm top}$.

Summary of the different relations

On the Calabi-Yau X_p

$$X_p = \mathcal{O}(2g-2+p) \oplus \mathcal{O}(-p) \longrightarrow \Sigma_g$$

we have the following chain of relations



Other relations: $\underset{D=2}{qYM_2} \quad \Leftrightarrow \quad \underset{D=3}{\text{Chern} - \text{Simons on Siefert manifolds}}$

Results

- We have explored the phase structure of the q-deformed YM_2 in the large N-limit. On S^2 for $p > 2$ we have found that the theory has a phase transition at a critical value of the Kahler parameter t

$$t_c = p\log(\sec^2(\pi/p))$$

separating a weak from a strong coupling region.

  – for $t < t_c$ the theory describes a topological string on the resolved conifold, but no sign of factorization

  – for $t > t_c$ we have found a double-cut solution of the relevant matrix model describing this regime. The solution organizes in terms of the correct modular parameters and the desired factorization appears: it exactly parallels the DK transition of usual YM_2

- The recovery of the correct Calabi-Yau geometry can be tested in a simpler manner by investigating a more fundamental block: the chiral version of our theory. In this case one can prove that the chiral version of q-deformed YM_2 reproduces, in the strong coupling phase, the correct topological string on

$$X_p = \mathcal{O}(p-2) \oplus \mathcal{O}(-p) \longrightarrow \mathbb{P}^1$$

Our matrix-model technique leads to new exact results for topological string amplitudes on X_p at any genus

  – We derived in closed form the Gromov-Witten invariants generating functional and the mirror map

  – We established the critical behavior of topological strings on X_p around its transition point: it is the same universality class of 2D gravity

- We clarified the relationship between instanton counting in $\mathcal{N}$=4 topological Yang-Mills theory (extending the analysis on a generic four-dimensional toric orbifold) and q-deformed Yang-Mills theory (on the blowups of the minimal resolution of the orbifold singularity)

$$Z_{\mathcal{N}=4} \simeq Z^q_{\rm YM}$$

  We described explicitly the instanton contributions to the counting of D-brane bound states which are captured by the two-dimensional gauge theory.: the fractional instantons. We derived an intimate relationship between qYM_2 and Chern-Simons theory on generic Lens spaces, and use it to show that the correct instanton counting is only reproduced when the Chern-Simons contributions are treated as non-dynamical boundary conditions in the D4-brane gauge theory

- We extended these analysis to $\Sigma_g = \mathbb{T}^2$ studying the circle of relations on the CY threefold

$$X_p = \mathcal{O}(p) \oplus \mathcal{O}(-p) \quad \longrightarrow \quad \mathbb{T}^2$$

q-deformed YM_2 on S^2

In the case the CY threefold is $X = \mathcal{O}(-2+p) \oplus \mathcal{O}(-p) \;\to\; \mathbb{P}^1$ the q-deformed YM_2 takes a particular simple form

$$Z^q_{\mathrm{YM}} = \sum_R \dim_q(R)^{2-2g}\; q^{\frac{p}{2}\,C_2(R)}\;\, \mathrm{e}^{\,\mathrm{i}\,\theta\, C_1(R)} =$$

$$= \frac{1}{N!} \sum_{n_i \in \mathbb{Z}} \mathrm{e}^{-\frac{g_s p}{2} \sum_{i=1}^N n_i^2 + \mathrm{i}\,\theta \sum_{i=1}^N n_i} \prod_{1\leq i<j\leq N} \sinh^2\left(\frac{g_s}{2}\,(n_i - n_j)\right)$$

As $g_s \to 0$ the quantum dimension goes smoothly into the ordinary one. To recover the undeformed partition function, one has also to send $p \to \infty$ with

$$g_s p = a = g^2 A = \text{fixed}$$

The q-deformed theory can be seen as a peculiar a/p expansion in the ordinary theory, allowing us to carry some standard results about localizations to $q\mathrm{YM}_2$

Large N-Phase Transition: Casimir Picture

Recall that the partition function of the q-deformed gauge theory on S^2 is given by

$$Z^q_{\rm YM}(g_s,p)=\sum_{\substack{n_1,\dots,n_N\in\mathbb{Z}\\ n_i-n_j\geq i-j\ \ \text{for}\ \ i\geq j}}{\rm e}^{-\frac{g_s p}{2}\sum_{i=1}^N n_i^2}\prod_{1\leq i<j\leq N}\sinh^2\left(\tfrac{g_s}{2}(n_i-n_j)\right)\ .$$

The ordering of the integers n_i keeps track of the their meaning in terms of Young tableaux labels and highest weights. Now to take the large N limit we proceed as usual. We keep fixed

$$t=g_sN\qquad \text{as}\quad N\rightarrow\infty$$

We also introduce also a second parameter which is obviously fixed in the limit

$$a=g_sNp.$$

It plays the role of the area of S_2 in the YM_2 language.

When N is large, $x_i = i/N$ becomes continuous and we can introduce a distribution function $\rho(x)$ for the Young Tableaux. The partition function in the large N-limit is then

$$Z^q_{\rm YM}(t,a) = \exp\big(N^2 S_{eff}(\rho)\big)$$

with

$$S_{eff}(\rho) = -\int dx dy \rho(x)\rho(y) \log\left(\sinh\left(\frac{t}{2}(x-y)\right)\right) + \frac{a}{2}\int dx \rho(x) x^2$$

The distribution function is fixed by the requirement of minimizing the action and satisfies the following saddle-point equation

$$\frac{a}{2}x = \int \rho(y)\coth\frac{t}{2}(x-y)$$

whose solution is

$$\rho(x) = \frac{a}{\pi t}\arctan\sqrt{\frac{e^{t^2/a}}{\cosh^2\left(\frac{tx}{2}\right)} - 1}$$

with $x \in [-\text{arccosh}(e^{-t^2/2a}), \text{arccosh}(e^{-t^2/2a})]$. It is the "quantum" deformation of the Wigner distribution.

However there is an important difference with standard matrix models: the distribution ρ cannot be arbitrary but it must satisfy the constraint

$$\rho(x) \leq 1$$

Breaking this constraint will imply a phase transition!

This bound can be easily checked and we find

- If $p \leq 2$ the bound is never violated
- If $p > 2$ the bound is always violated when $t \geq t_c = p \log \sec^2\left(\frac{\pi}{p}\right)$

Thus $q\mathrm{YM}_2$ undergoes a phase transition on S^2 when $p > 2$. Our ρ provides a description holding just in the weak coupling regime. In this regime no equivalence with $|Z_{top}|^2$ via OSV conjecture. Over the transition the answer will become positive. We need a strong coupling analysis (a two-cut solutions in the matrix model language). But, first, what have we really found in the weak coupling regime?

The resolved conifold

Computing the free energy is a tedious (but elementary) exercise. One finds

$$\mathcal{F}(t,a) = -\frac{t^2}{6a} + \frac{\pi^2 a}{6t^2} - \frac{a^2}{t^4}\,\zeta(3) + \frac{a^2}{t^4}\,\mathrm{Li}_3(\mathrm{e}^{-t^2/a}) + c(t)$$

This is easily identified with the genus 0 free energy topological string on the resolved conifold.

We miss the Calabi-Yau X_p and the modulus square: one strange feature is that the result does not change substantially with p. This factor simply scales the Kahler modulus: why do we loose the original geometrical information encoded in $q\mathrm{YM}_2$?
This question can be easily answered if we look at the dual picture: in terms of instantons (small abuse of language).

Instanton Picture: CS on Lens Spaces

The original expression for the partition function of $q\mathrm{YM}_2$ is arranged as a strong coupling expansion $(e^{-g_s p n^2})$. We want to write it in a way that is more suitable for the weak coupling expansion: namely to perform a modular transformation from $g_s \to 1/g_s$. This can be done and we find

$$Z^q_{\mathrm{YM}}(g_s,p) = \frac{1}{N!} \sum_{s_i \in \mathbb{Z}} \mathrm{e}^{-\frac{2\pi^2}{g_s p} \sum_{i=1}^N (s_i-\theta)^2} \; w_q^{\mathrm{inst}}(s_1,\ldots,s_N),$$

where $w_q^{\mathrm{inst}}(s_1,\ldots,s_N)$ is given by

$$\frac{1}{2}\left(\frac{2\pi}{g_s p}\right)^N \mathrm{e}^{-\frac{g_s\left(N^3-N\right)}{6p}} \int_{-\infty}^{\infty} \mathrm{d}z_1\cdots\mathrm{d}z_N \; \mathrm{e}^{-\frac{2\pi^2}{g_s p}\sum_{i=1}^N z_i^2}$$
$$\times \prod_{1\le i<j\le N}\left[\cos\left(\frac{2\,\pi\,(s_i-s_j)}{p}\right) - \cos\left(\frac{2\,\pi\,(z_i-z_j)}{p}\right)\right].$$

In the weak coupling regime (we shall also consider $\theta = 0$) the theory is dominated by the trivial vacuum $w_q^{\mathrm{inst}}(0,\ldots,0)$. All the others are exponentially suppressed: this is an *instanton expansion* of the exact partition function.

All the non-trivial instanton contributions are nonperturbative in the $1/N$ expansion. To detect a possible phase transition study:

$$R = \frac{w_q^{\text{inst}}(0,\ldots,0)}{e^{-\frac{2\pi^2 N}{A}} w_q^{\text{inst}}(1,\ldots,0)}$$

This can be done again with matrix model techniques.

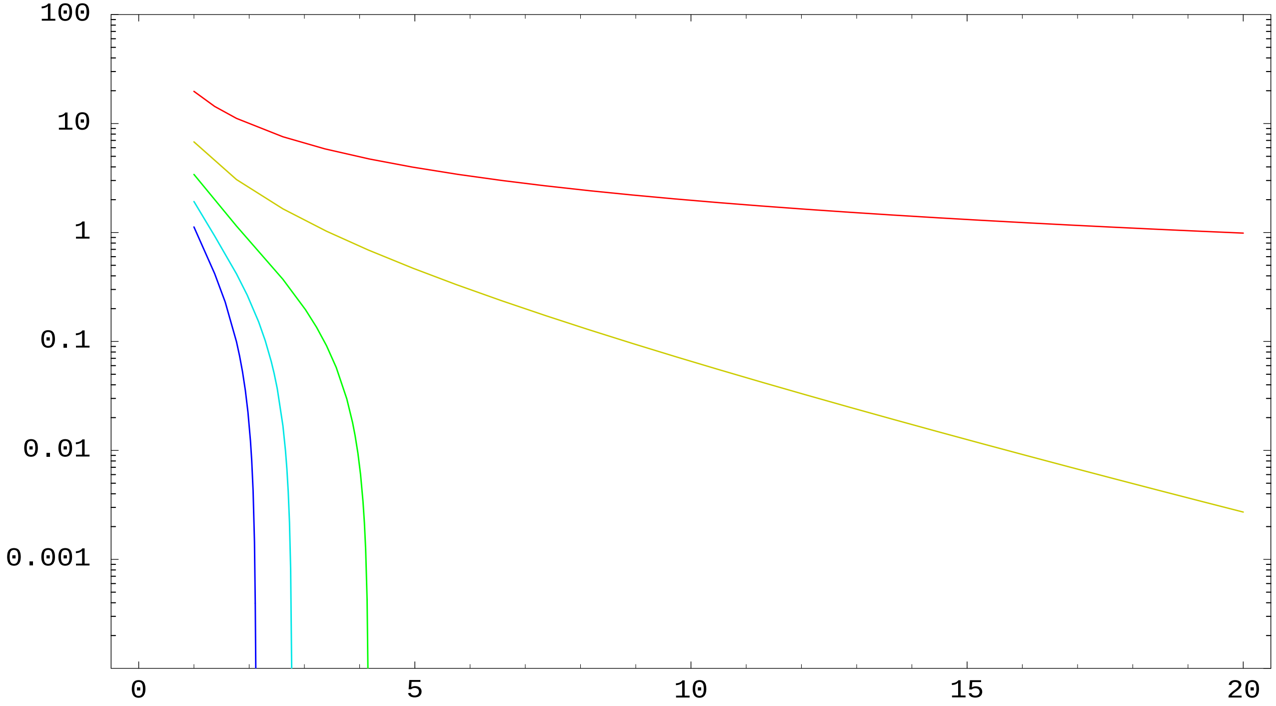
- If $R \geq 1$ the theory is always in the trivial vacuum
- If $R < 1$ all the non trivial sectors contribute

We find that $R = 1$ for:

$$t_c = p \log \sec^2\left(\frac{\pi}{p}\right),$$

which is exactly the transition curve we find in the group theory approach.

One can plot the log of the above ratio for different values of p and the result is impressive



the log of the ratio for $p = 1, 2, 3, 4, 5$ *as a function of* t

This analysis actually explains also why we get always the conifold in the weak-coupling regime. The instanton representation can be also in fact arranged as follows

$$Z^q_{YM} = \sum_{\{N_k\}} \prod_{k=0}^{p-1} \frac{\theta_3\left(\frac{2\pi\,\mathrm{i}\,p}{g_s}\middle|\frac{2\pi\,\mathrm{i}\,k}{g_s}\right)^{N_k}}{N_k!}\; Z^p_{\mathrm{CS}}(\{N_p\})\ .$$

where $Z^p_{\mathrm{CS}}(\{N_p\})$ is the partition function of Chern-Simons on Lens spaces

$$L_p = S^3/\mathbb{Z}_p = \partial(C_4 = \mathcal{O}(-p)\ \rightarrow\ \mathbb{P}^1)$$

.

In the weak coupling regime it dominates Z_{CS} in the vacua with no flat connection wrapping around the cycles of L_p. In this case Z_{CS} on L_p is simply equal to Z_{CS} on S^3. It was proven by Gopakumar and Vafa that Z_{CS} on S^3 in the large limit reproduce the resolved conifold via geometric transition!

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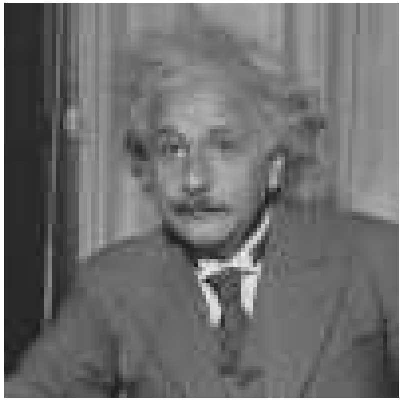

1916

Quantum link?

1906