From Quantum Field Theory to Quantum Geometry

Harald Grosse

Institute for Theoretical Physics, University of Vienna

Introduction

1. Introduction
   QFT: Scalar fields, QED, Standard Model, Gravity
   (electro-weak, strong, gravitational)
2. Space-Time Concepts
   quantize, use noncommutative geometry
3. Regularization of QFT
   Fuzzy $S^2, CP^N, T^2, ...$
4. Noncommutative Quantum field Theory
   renormalization, scalar fields, ϕ^3, gauge models?
5. Induced Gauge Models
6. Conclusions

History

- 1900-1926 Quantum Mechanics:
  (Planck, Bohr, Heisenberg, Schrödinger,...)
- 1905, 1915 Special and General Relativity
  Einstein
- ...1950 perturbative Quantum Field Theory
  renormalization, scalar fields, fermions
- UV, IR, convergence problems
- ...1970 renormalized gauge models
  Standard Model, W, Z, Higgs?
- add Gravity or deform Space-Time

Project

merge general relativity with quantum physics through noncommutative geometry

Requirements

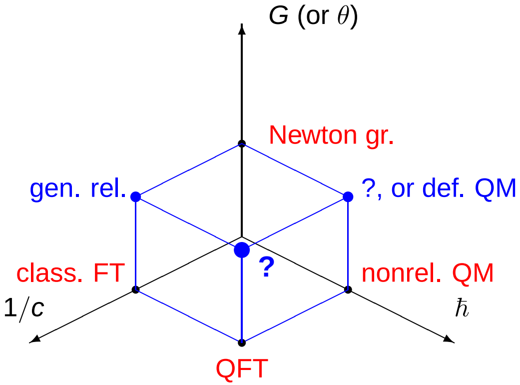
QFT: Wightman, Symanzik,.... Minkowski-Euclidean, reflection positivity
Require: Covariance, spectrum condition, locality, vacuum
representation of Lorentz group, scalar, spinor; vector; tensor fields
renormalization gives a calculus (Connes, Kreimer)
IR, UV, convergence problems
nonperturbative approches, renormalization group, $\beta-$ function
Models: Dimension: 1 , 2 , 3 , 4 ?, ...10, 11

Attempts

Supersymmetry, Superstrings,..........Loop Quantum Gravity
Noncommutative Quantum Field Theory!!! (there are relations among them)

Space-Time Concepts

in Newton gr., QM, ED, GR, QFT and QG ?



History

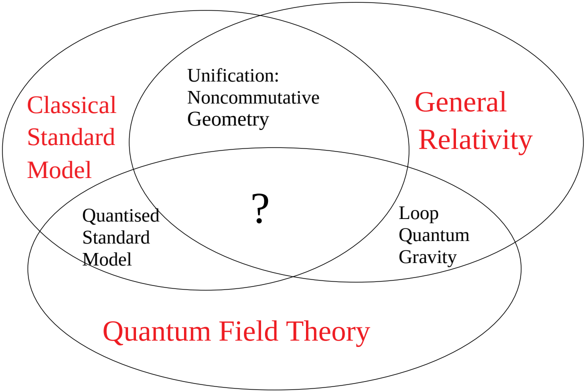
- Limited localisation of events in space-time

$$D \geq R_{ss} = G/c^4 hc/\lambda \geq G/c^4 hc/D \tag{1}$$

  gives Planck lenght as a lower bound
  Riemann, Schrödinger, Heisenberg, Peierls, Pauli, Oppenheimer, Snyder, ...

- 1986 Connes: Noncommutative Geometry
  1992 H. G. and Madore: Regularization using nc manifolds
  1995 Filk: Feynman rules, Doplicher et al: Free fields
  1995.. H.G. and Klimcik, Presnajder
  1999 Schomerus: obtains nc models from strings
  2000..H. G. and Schweda, Wulkenhaar, nc Gauge models

Strategy



Approach

1. renormalisation of quantum field theories on noncommutative geometries

Main ideas

- Replace manifold by algebra deform it
- keep differential calculus derivations....
- Replace fields by projective modules
- Replace integrals by traces
- use renormalized perturbation expansion

Connes, Majid, S.-Jabbari, Landi, Lizzi, Vitale, Szabo, Dabrowski, Piacitelli, Filk, Doplicher, Fredenhagen, Roberts, Bahns, Liao, Sibold, Schweda, O´Connor, Madore, Steinacker, Dolan, Klimcik, Presnajder, Wess, Schupp, Jurco, Aschieri, Chaichian, Balachandran, Jochum, Gayral, Wohlgenannt, Gracia-Bondia, Ruiz-Ruiz, Lukierski,Rivasseau, Magnen, Seiberg, Vignes-Tourneret, Witten, Wulkenhaar, Varilly ......

Regularization

FUZZY SPHERE

expand smooth functions on the sphere:

$$f(x) = f_0 + f_i x^i + f_{ij} x^i x^j + \dots$$

use generators of $su(2)$

$$\mathcal{A}_N = [0] \oplus \cdots \oplus [j]$$

use differential calculus on matrices, integration is trace

$$\sum_{i=1,2,3} (X_N^i)^2 = R^2, \; [X_N^i, X_N^j] = \frac{i\epsilon_{ijk} R X_N^k}{\sqrt{N^2-1}}$$

embed sequence of algebras....

Regularization

Scalar Field becomes regularized

$$S_N[\phi] = Tr_N \, [X_N^i, \phi^\dagger][X_N^i, \phi] + m^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2$$
$$< \phi \ldots \phi >_N = \frac{1}{Z_N} \int [d\phi]_N e^{-S_N[\phi]} \phi \ldots \phi$$

Quantization of sections of line bundles through sequence of embedded *NxM* matrices
of functions on superspace through sequence of embedded graded matrices
of representations of $SU_q(2)$ gives q-deformed sphere
Regularization preserves symmetries

Renormalization

Formulation

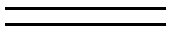
ϕ^4 on nc $\mathbb{R}^4$, $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ or equivalently

$$(a * b)(x) = \int dy \int dk a(x + \frac{\theta k}{2}) b(x+y) e^{iky}$$

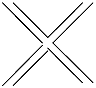
ϕ^4 action

$$S = \int dp (p^2 + m^2) \phi_p \phi_{-p} + \lambda \int \prod_{j=1}^{4} (dp_j \phi_{p_j}) \, \delta(\sum_{j=1}^{4} p_j) e^{-\frac{i}{2} \sum_{i<j} p_i^\mu p_j^\nu \theta_{\mu\nu}}$$

Feynman rules



$\frac{1}{p^2+m^2}$

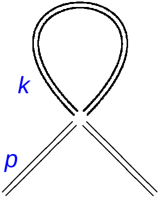


$\frac{\lambda}{4!} e^{-\frac{i}{2} \sum_{i<j} p_i^\mu p_j^\nu \theta_{\mu\nu}}$

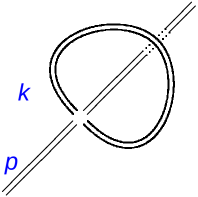
cyclic order of momenta leads to ribbon graphs

IR/UV mixing

One-loop two-point function planar and nonplanar contributions:



$$= \frac{\lambda}{6} \int dk \frac{1}{k^2 + m^2}$$



$$= \frac{\lambda}{12} \int dk \frac{e^{ip^\mu k^\nu \theta_{\mu\nu}}}{k^2 + m^2} \quad p \overset{\sim}{\to} 0 \ \frac{1}{\tilde{p}^2}$$

planar graphs: renormalize BPHZ ....

IR/UV mixing

$\Rightarrow$ non-planar graph finite (noncommutativity serves as regulator),
but behaves $\sim \tilde{p}^{-2}$ for small momenta (renormalisation not possible)
$\Rightarrow$ this leads to non-integrable integrals when inserted as subgraph into bigger graphs: IR/UV-mixing Minwalla, van Raamsdonk & Seiberg, 1999
– more rigorous treatment: power-counting theorem for ribbon graphs Chepelev & Roiban 1999/2000
– proposals: resummation, supersymmetry (all integrals exist, but unbounded as momenta $\to 0$), ...
satisfactory solution: modify action

Main result

H. G. and R. Wulkenhaar ϕ^4 model modified
IR/UV mixing: short and long distances related
Theorem: Action

$$S = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{\mu_0^2}{2}\phi\star\phi + \lambda\phi\star\phi\star\phi\star\phi\Big)(x)$$

for $\tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu}x^\nu$
is perturbativly renormalizable to all orders in λ
$p_\mu \leftrightarrow \tilde{x}_\mu$, $\hat{\phi}(p) \leftrightarrow \pi^2\sqrt{|\det\theta|}\,\phi(x)$

Fourier transformation $\hat{\phi}(p_a) = \int d^4x\; e^{(-1)^a i p_{a,\mu} x_a^\mu}\phi(x_a)$,
leads to Langmann-Szabo duality
$S\left[\phi; \mu_0, \lambda, \Omega\right] \mapsto \Omega^2 S\left[\phi; \frac{\mu_0}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}\right]$

Unitary Transform

operator base

$$\phi(x) = \sum_{m_1, m_2, \in \mathbb{N}} \phi_{m_1 m_2} b_{m_1 m_2}(x)$$

where

$$b_{m_1 n_1}(x_1, x_2) = \frac{(x_1 + \mathrm{i} x_2)^{\star m_1}}{\sqrt{m_1!(2\theta)^{m_1}}} \star \left(2 \mathrm{e}^{-\frac{1}{\theta}(x_1^2 + x_2^2)}\right) \star \frac{(x_1 - \mathrm{i} x_2)^{\star n_1}}{\sqrt{n_1!(2\theta)^{n_1}}}$$

$$(b_{mn} \star b_{kl})(x) = \delta_{nk} b_{ml}(x), \qquad \int d^4 x \, b_{mn}(x) = (2\pi\theta)^2 \, \delta_{mn}$$

interaction becomes matrix product no oscillations

$$S = (2\pi\theta)^2 \sum_{m,n,k,l \in \mathbb{N}} \left(\frac{1}{2} \phi_{mn} G_{mn;kl} \phi_{kl} + \lambda \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \right)$$

Decay Properties

Propagator complicated!

Decay exponents decide on renormalization!

$\Delta_{\substack{mm\ mm \\ 00;00}}(0) \sim \frac{\theta/8}{\sqrt{\frac{4}{\pi}(m+1) + \Omega^2(m+1)^2}}$

$\Delta_{\substack{m_1\ m_1\ \ 0\,0 \\ m_2 m_2;00}}(0) = \frac{\theta}{2(1+\Omega)^2(m_1+m_2+1)} \left(\frac{1-\Omega}{1+\Omega}\right)^{m_1+m_2}$ exact results for special values

Δ has equidistant spectrum, study Jacobi matrix using Meixner polynomials

closed formula (finite sum) due to identity for Meixner polynomials

Propagator has asymmetric decays, is quasilocal

proof power counting rule for ribbon graphs

renormalization by flow equation

Renormalization Group

Wilson-Polchinski approach to nonlocal matrix models
Define QFT by cutoff partition function

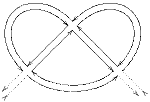
$$S[\phi,\Lambda] = (2\pi\theta)^2 \Big(\sum_{m,n,k,l} \frac{1}{2} \phi_{mn} G^K_{mn;kl}(\Lambda)\, \phi_{kl} + L[\phi,\Lambda] \Big)$$

startfrom interaction
$L[\phi,\infty] = \lambda \sum_{m,n,k,l} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}$
require cutoff independence, add power series of interactions
graphs drawn on Riemann surface of genus g
$1 - 2g = L - I + V$ with B holes
L....single line loops for closed external lines
I....double line propagators
B....loops which cary external legs

Polchinski

$$\Lambda \frac{\partial L[\phi,\Lambda]}{\partial \Lambda} = \sum_{m,n,k,l} \frac{1}{2} \Lambda \frac{\partial \Delta^K_{nm;lk}(\Lambda)}{\partial \Lambda} \Big(\frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} - \frac{1}{(2\pi\theta)^2} \frac{\partial^2 L[\phi,\Lambda]}{\partial \phi_{mn}\, \partial \phi_{kl}} \Big)$$

Modified nc ϕ^4 model is renormalizable
adjust four parameters: coupling, mass, frequency, field amplitude
Proof Power counting rule
use "quasilocality" of propagator to estimate ribbon graphs
example



3 loop diagram, four independent sums!
proof: all nonplanar graphs irrelevant
planar graphs with more than 4 legs irrelevant
4leg graph log. divergent, 2leg graph quadr. divergent
need 4 rel/marginal parameters!

RG functions

evaluate β function

$$\lim_{\mathcal{N}\to\infty}\Big(\mathcal{N}\frac{\partial}{\partial\mathcal{N}}+N\gamma+\mu_0^2\beta_{\mu_0}\frac{\partial}{\partial\mu_0^2}+\beta_\lambda\frac{\partial}{\partial\lambda}+\beta_\Omega\frac{\partial}{\partial\Omega}\Big)\Gamma[\mu_0,\lambda,\Omega,\mathcal{N}]=0$$

$$\beta_\lambda=\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\lambda[\mathcal{N}]\Big)=\frac{\lambda_{\text{phys}}^2}{48\pi^2}\frac{(1-\Omega_{\text{phys}}^2)}{(1+\Omega_{\text{phys}}^2)^3}+\mathcal{O}(\lambda_{\text{phys}}^3)$$

$$\beta_\Omega=\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\Omega[\mathcal{N}]\Big)=\frac{\lambda_{\text{phys}}\Omega_{\text{phys}}}{96\pi^2}\frac{(1-\Omega_{\text{phys}}^2)}{(1+\Omega_{\text{phys}}^2)^3}+\mathcal{O}(\lambda_{\text{phys}}^2)$$

$\Omega=1$ special.....integrable ?

A solvable model

Scalar field ϕ^3 H. G. and Harold Steinacker
recall: $\partial_i \phi = -i[\tilde{x}_i, \phi]$

$$S = \int -(\tilde{x}_i \phi \tilde{x}_i \phi - \tilde{x}_i \tilde{x}_i \phi \phi) + \Omega^2 \tilde{x}_i \phi \tilde{x}_i \phi + \frac{\mu^2}{2} \phi^2 + \frac{i \tilde{\lambda}}{3!} \phi^3$$

simplifies for $\Omega = 1$ to

$$S = \int (\tilde{x}_i \tilde{x}_i + \frac{\mu^2}{2}) \phi^2 + \frac{i \tilde{\lambda}}{3!} \phi^3 = Tr \Big(\frac{1}{2} J \phi^2 + \frac{i \lambda}{3!} \phi^3 \Big).$$

where

$$J = 2(2\pi\theta)^2 (\sum_i \tilde{x}_i \tilde{x}_i + \frac{\mu^2}{2}) \quad \dots \quad \text{harmonic oscillator !}$$

choose appropriate basis:
in $d = 2$: $J|n\rangle = 4\pi \, (n + \frac{1}{2} + \frac{\mu^2 \theta}{2})|n\rangle$, $n \in \{0, 1, 2, \dots\}$

Free Energy for genus g

Intersection theory of moduli space of Riemann surfaces of genus g with marked points

$$Z^{Kont}(M) = e^{F^{Kont}(M)} = \int dX \exp\left\{Tr\left(-\frac{MX^2}{2} + i\frac{X^3}{6}\right)\right\}$$
$$= \int d\tilde{\phi} \exp\left\{Tr\left(\frac{1}{2i\lambda}M^2\tilde{\phi} - \frac{i}{3!}\tilde{\phi}^3 - \frac{1}{3}M^3\right)\right\}$$

$$F_0^{Kont} = \frac{1}{3}\sum_i m_i^3 - \frac{1}{3}\sum_i (m_i^2 - 2u_0)^{3/2} - u_0 \sum_i (m_i^2 - 2u_0)^{1/2}$$
$$+ \frac{u_0^3}{6} - \frac{1}{2}\sum_{i,k} \ln\left\{\frac{(m_i^2 - 2u_0)^{1/2} + (m_k^2 - 2u_0)^{1/2}}{m_i + m_k}\right\}$$

$$F_1^{Kont} = -\frac{1}{24}\ln(1 - I_1),$$

Renormalize

treated in dimension $2, 4, 6$

$$I_k = -(2k-1)!! \sum_i \frac{1}{(m_i^2 - 2u_0)^{k+\frac{1}{2}}},$$

$$u_0 = -\sum_i \frac{1}{\sqrt{m_i^2 - 2u_0}} = I_0.$$

all F_g^{Kont} with $g \geq 2$ are given by finite sums of polynomials in $I_k/(1-I_1)^{\frac{2k+1}{3}}$.
model nonperturbativly solvable also in 6 dimensions!
(not superrenormalizable) agrees with pertubation expansion

Induced Gauge Models

Couple gauge field to scalar field, H. G. and Michael Wohlgenannt

$$S = \int d^D x \Big(\frac{1}{2}\phi \star [B_\nu, [B^\nu, \phi]_\star]_\star + \frac{\Omega^2}{2}\phi \star \{B^\nu, \{B_\nu, \phi\}_\star\}_\star + \frac{\mu^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \Big)(x)$$

use covariant coordinates

$$B_\nu = \tilde{x}_\nu + A_\nu$$

gauge transformation

$$A_\mu \mapsto \mathrm{i} u^* \star \partial_\mu u + u^* \star A_\mu \star u$$

expand S calculate effective action

One-Loop

Calculation in four dimensions
quadratic divergent

$$\Gamma^{\epsilon}_{1l}[\phi] = -\frac{1}{2}\int_{\epsilon}^{\infty}\frac{dt}{t}\,\mathrm{Tr}\left(e^{-tH} - e^{-tH^{0}}\right)$$

Use: Duhamel expansion

$$\Gamma^{\epsilon}_{1l} = \frac{1}{16\pi^2}\int d^4x\left(\frac{1}{\epsilon\theta}(B_\nu \star B^\nu - \tilde{x}^2)\right.$$

$$\left.+\left(\frac{\mu^2}{2}(B_\nu \star B^\nu - \tilde{x}^2) + \frac{1}{2}\left((B_\mu \star B^\mu)\star(B_\nu \star B^\nu) - (\tilde{x}^2)^2\right)\right)\ln\epsilon\right)$$

for $\Omega = 1$
(tbp) one loop calculation for general Ω
Quantization ?

Conclusions

- formulation of models on nc spaces possible gives symmetry preserving cutoffs
- removing cutoffs leads to IR/UV mixing not renormalizable
- modified actions for matter fields yields a calculus
  renormalons killed, constructive approach ? Nontrivial ?
- gauge fields ?
  formulated
- gravity ? ? ?