*Espansione a grandi **N** per la gravità e 'softening' ultravioletto*

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Outline of the talk

- Reasons behind the Large N expansion in Gauge Theories.
- Large N in gravity and "pictorial" interpretation.
- Diagrammatic "manifestation" of the Holographic Principle
- UV-softening at large N in gravity, similarities with other models and physical interpretation.

Motivations

- In QCD, the 't Hooft and Veneziano limits are one of the main tools to investigate non perturbative phenomena which standard perturbation theory is not able to disclose (such as confinement, baryons and mesons physics, chiral symmetry breaking and so on).
- Many models are not renormalizable in the standard perturbative expansion and, in fact, are renormalizable in the large N expansion.
- Thus, some sort of large N limit(s) would be very useful in General Relativity which is not perturbatively renormalizable and whose quantistic features are still far from being fully understood.

But…

- In Gauge Theories the main fields are 1-forms taking values in the algebra of the gauge group: the large N limit is the limit in which the internal gauge group is larger and larger while the background is kept fixed.
- In Gravity the main field is the space-time metric, a rank-two covariant tensor field: there is not a clear separation between space-time and internal symmetries because General Relativity is the theory of space-time itself.
- Nevetheless, it is possible to formulate General Relativity in a way which is very close to a Gauge Theory. This will help in formulating the large N limit as well as in comparing the Gravitational and Yang-Mills cases.

Review of large N QCD.

The Lagrangian and the basic fields are:

$$L = -\frac{1}{2e^2} tr F_{\mu\nu} F^{\mu\nu} + \left(\overline{\psi}\right)^I \gamma^\mu \left(D_\mu\right)_{IJ} \psi^J$$

$$\left(D_\mu\right)_{IJ} = \partial_\mu \delta_{IJ} - e\left(A_\mu\right)_{IJ},$$

$$\left(F_{\mu\nu}\right)_{IJ} = \left[D_\mu, D_\nu\right]_{IJ}$$

Propagators and Vertices

$$\left\langle A^{IJ}{}_{\mu} A^{KL}{}_{\nu} \right\rangle (k) = \frac{1}{2}\left(\delta^{IL}\delta^{KJ} - \frac{\delta^{IJ}\delta^{KL}}{N_c} \right) P_{\mu\nu}(k)$$

$$P_{\mu\nu}(k) \approx -\frac{\delta_{\mu\nu}}{k^2 - i\varepsilon}$$

$$\left\langle \psi_{(j)}{}^{I} \psi_{(j)}{}^{L} \right\rangle (k) \approx -\frac{\delta_{\mu\nu}}{m_{(j)} + i\gamma^{\mu} k_{\mu} - i\varepsilon}\delta^{IL}$$

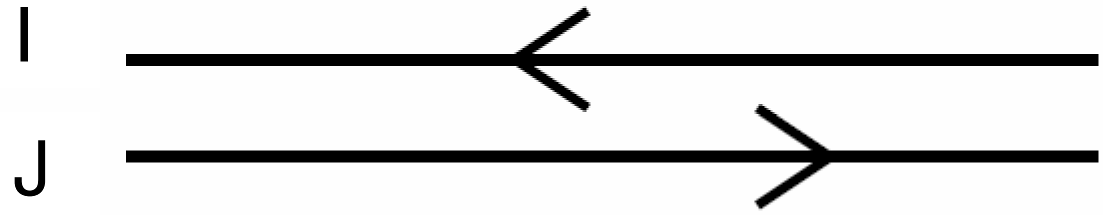
$$V^{(3)} \approx ie\left(\delta_{\rho\nu}(k-q)_{\mu} + \delta_{\rho\mu}(p-k)_{\nu} + \delta_{\mu\nu}(q-p)_{\rho} \right)$$

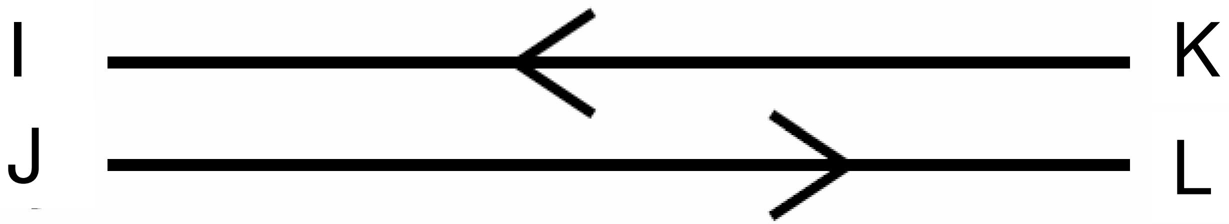
$$V^{(4)} \approx e^2\left(2\delta_{\rho\mu}\delta_{\beta\nu} - \delta_{\rho\beta}\delta_{\mu\nu} - \delta_{\beta\mu}\delta_{\rho\nu} \right)$$

$$V^{(M)} \approx -e\gamma_{\mu}$$

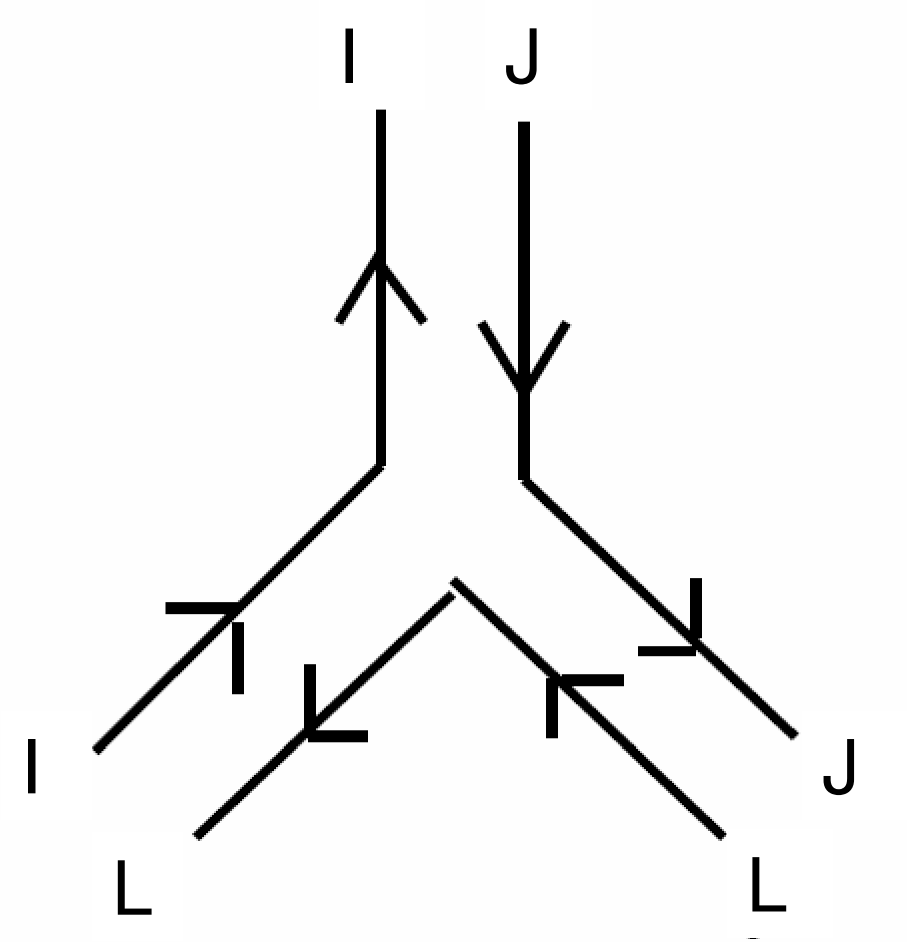
- A very clever way to take into account the internal index structures of the fields disentangling it from the space-time-momentum dependence in the path integral has been introduced by ‘t Hooft.
- To each gauge boson one has to associate two internal lines carrying internal indices with arrows pointing in opposite direction (in order to distinguish the fundamental and anti-fundamental representation of the gauge group SU(N)).
- To each quark one has to associate an internal line carrying an internal index and an arrow, to each anti-quark one has to associate an internal line carrying an internal index and an arrow pointing in the direction opposite to the quark arrow.

A_{IJ}

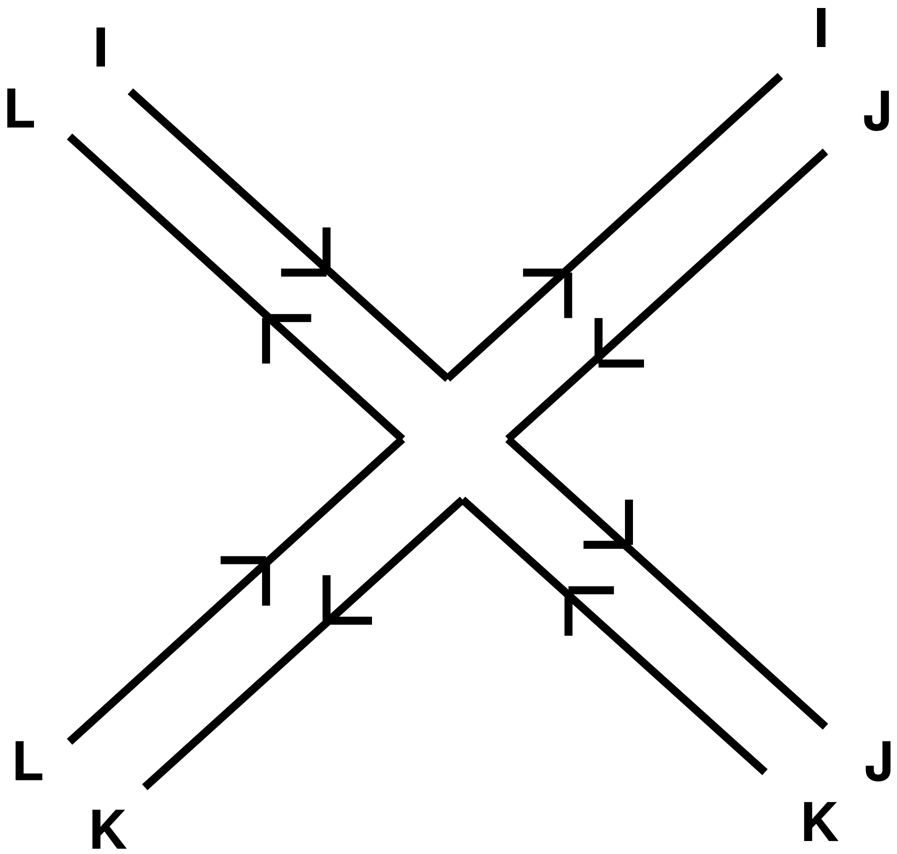




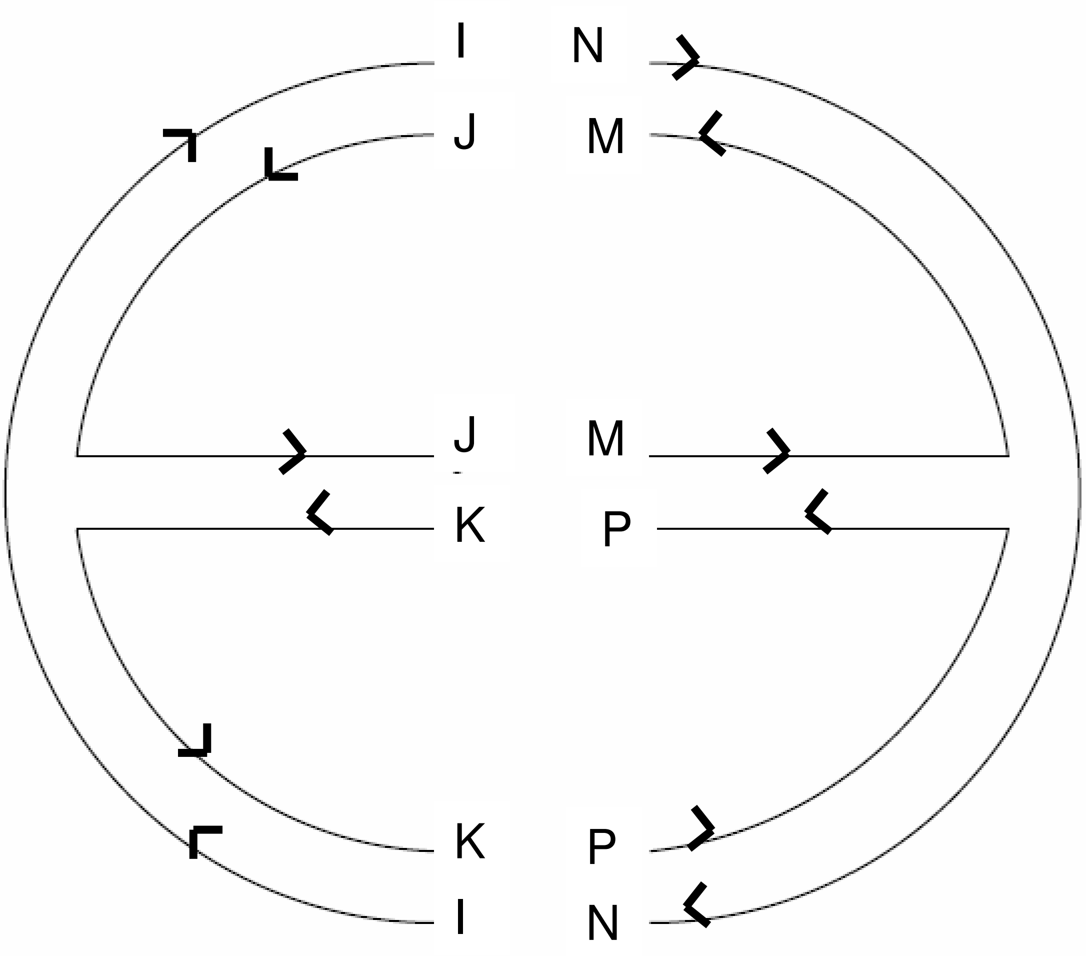
$\delta_{IK}\delta_{JL}$



~e

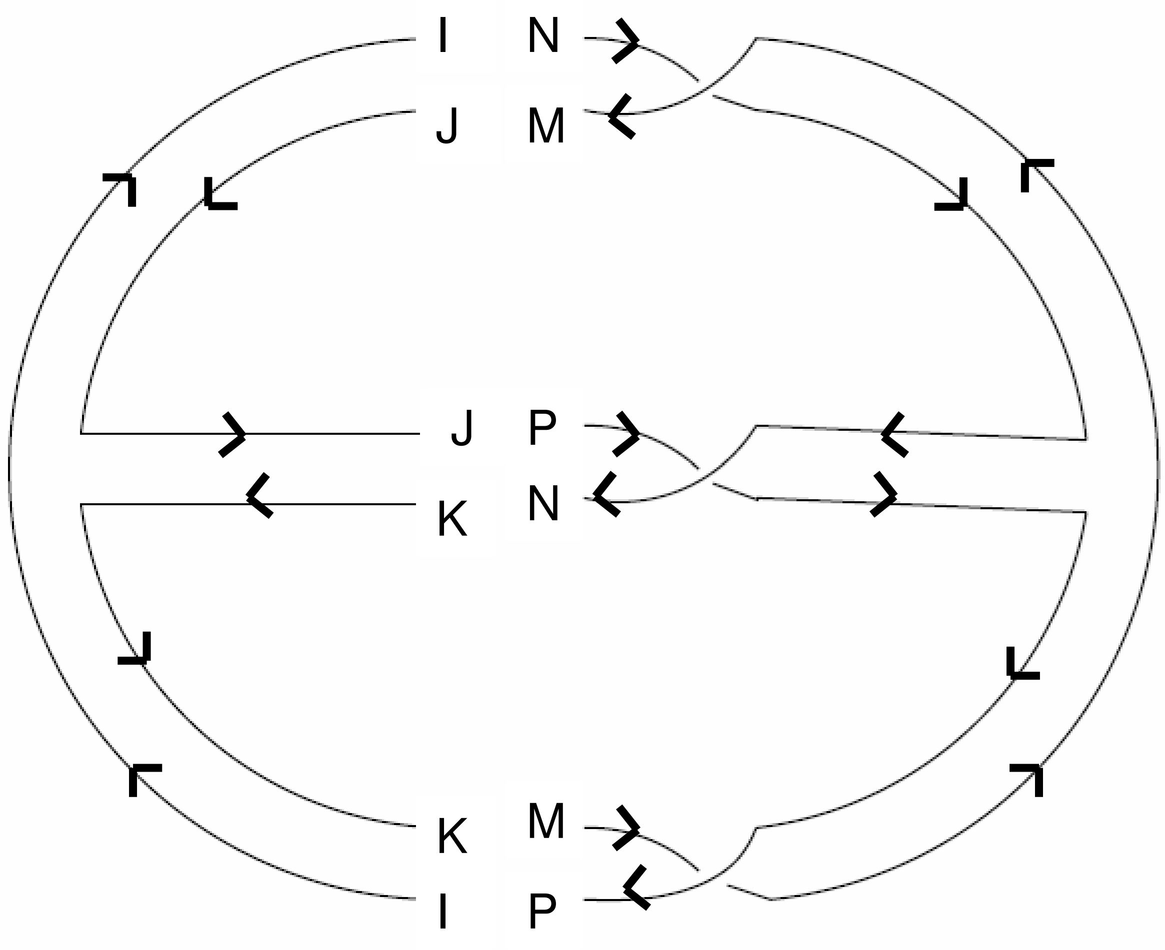


$\sim \mathbf{e}^2$



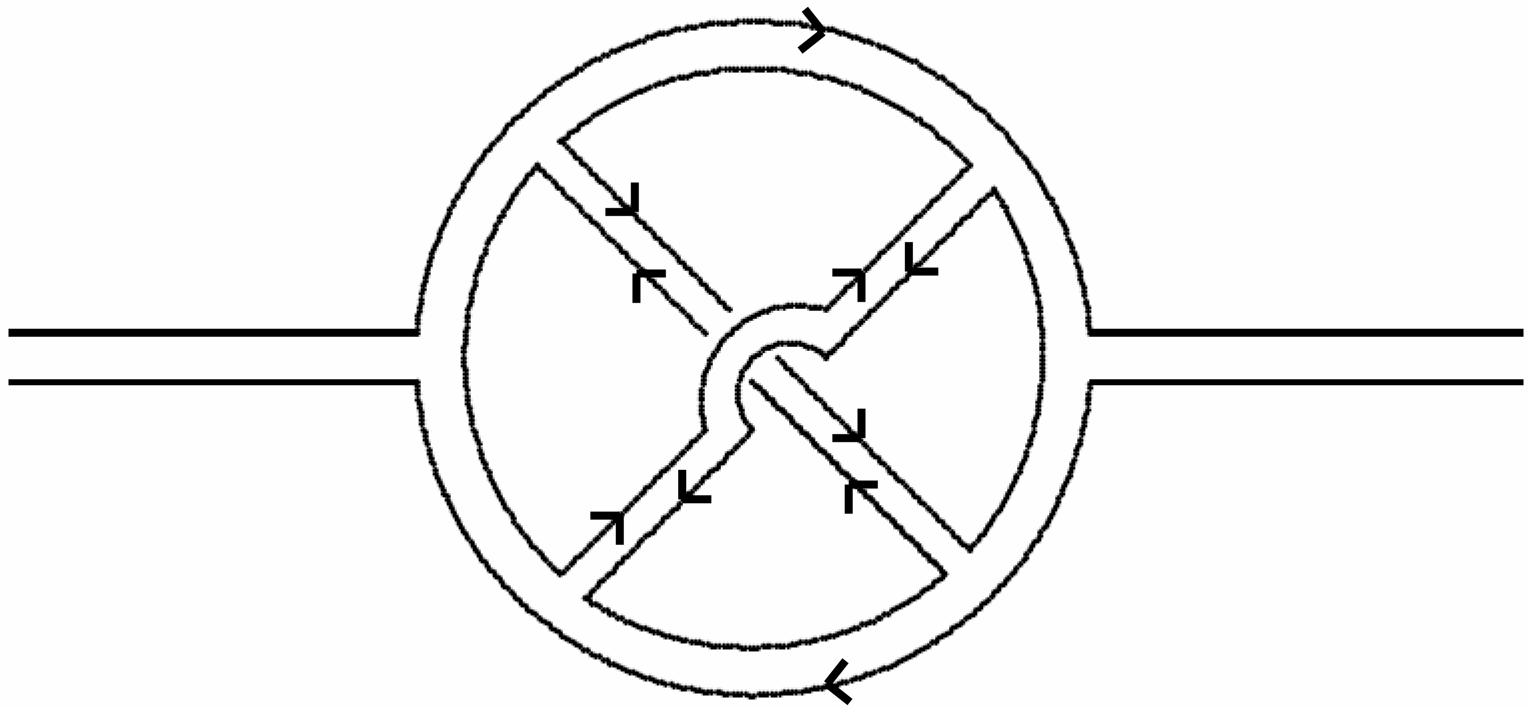
$\sim N^3$

$\sim e^2$

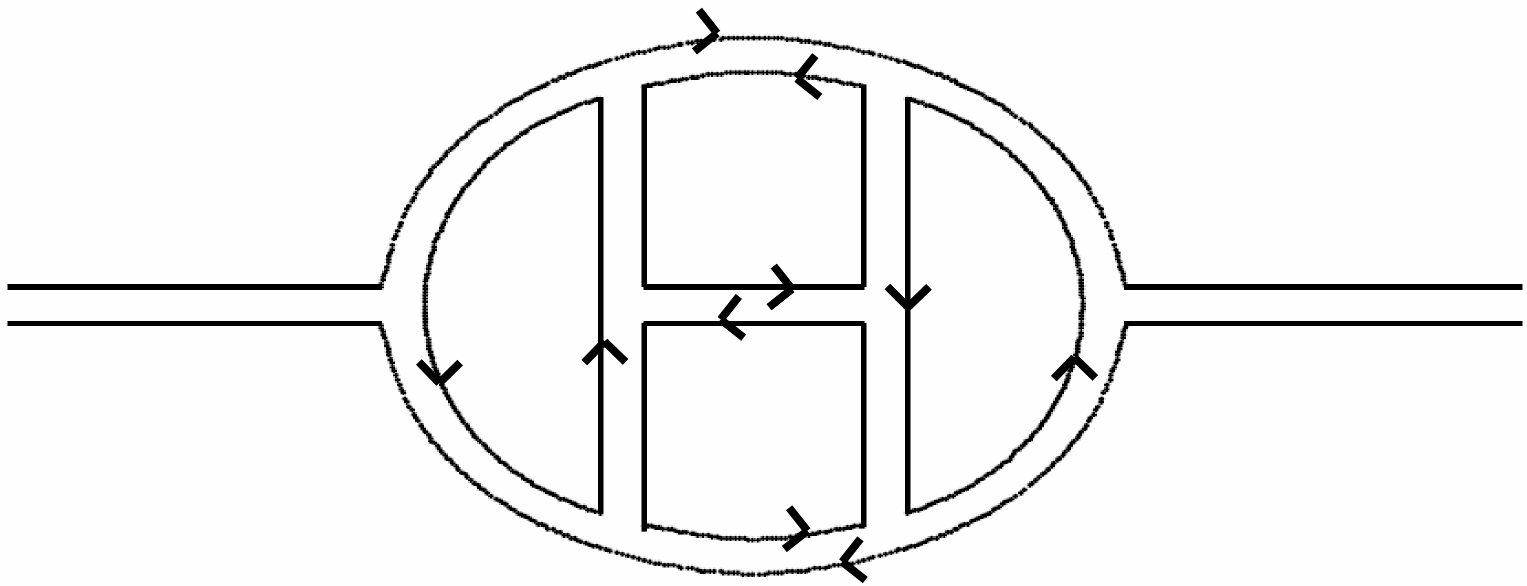


$\sim N$

$\sim e^2$



Double-line representation of a three-loop non-planar diagram for the gluon propagator. The diagram has six three-gluon vertices but only one closed index line (while three loops!). The order of this diagram is e^6N.



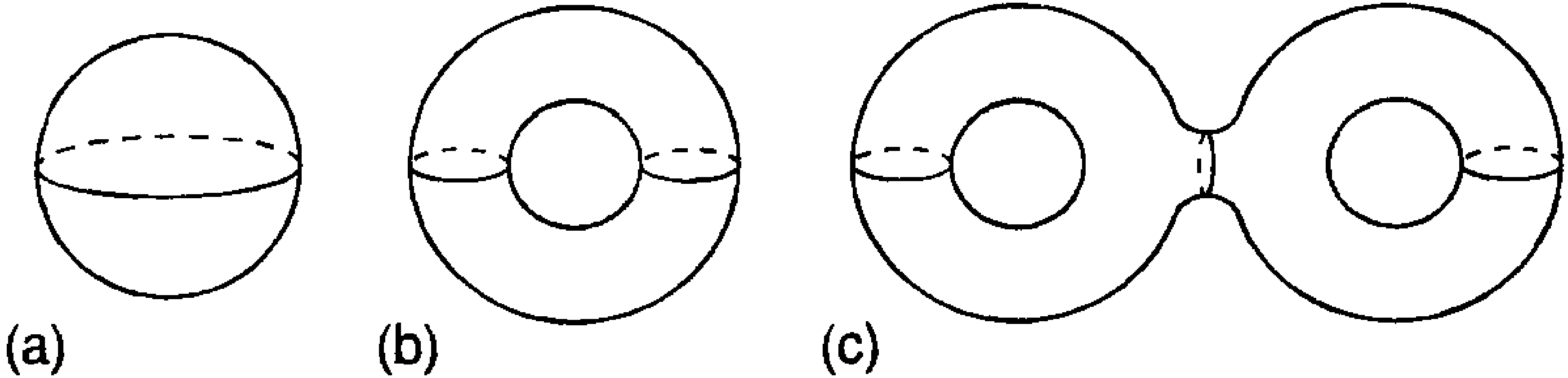
Double-line representation of a four-loop diagram for the gluon propagator. The sum over the Nc indices is associated with each of the four closed index lines whose number is equal to the number of loops. The contribution of this diagram is e^8N^4.

Large *N* Counting

$$W_{\Gamma}(k, x, g, L) \approx \lambda^{V_4 + V_3/2}\left(N^{2-2g-L}\right)$$

$$\lambda = e^2 N$$

Thus, ‘t Hooft classified the large N diagrams according to their topological properties.
From the ‘t Hooft notation it is also clear that, in the topological expansion, only orientable surfaces enter since for SU(N) the fundamental representation is not real and the adjoint is the tensor product of the fundamental and the anti-fundamental representations. To derive this formula one only needs to use the Euler formula 2g-2=E-V-F.



Here one can see the first three orientable surfaces which enter in the topological expansion: in the case (a) the genus is zero and, therefore, the sphere gives the dominant contribution at large N, in (b) and (c) the genus is equal to one and two so that they are suppressed.

Summarizing the QCD case…

Thus, in gauge theories, the planar (genus zero) contribution is dominant. In the gluonic sector without quarks the non planar contributions are suppressed as $1/N^2$. The quark loops are suppressed as $1/N$. Confinement, baryons and mesons physics are well understood at large N.

BF formulation of Gravity

A useful way to write the Einstein Hilbert action is as a topological action plus a constraint. It is available a seemingly similar formulation for gauge theories which allows interesting comparisons.

The action

$$S_{EH} = \frac{1}{G}\left[S(B,F) - \frac{c_2}{2}\int_M \phi_{IJKL} B^{IJ} \wedge B^{KL} + H_{YM}(\phi) \right]$$

$$S(B,F) = \int_M B^{IJ} \wedge F_{IJ}(A)$$

$$\left(F_{\mu\nu}\right)_{IJ} = \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)_{IJ} + \left(A_\mu\right)^K{}_I \left(A_\nu\right)_{KJ} - \left(A_\nu\right)^K{}_I \left(A_\mu\right)_{KJ}$$

$$B^{IJ} = \frac{1}{2} B^{IJ}{}_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

It is trivial to verify that, when one solves the equations for the Lagrangian multiplyer, the standard Palatini formulation of Einstein-Hilbert action is recovered. The BF theory is exactly solvable, thus gravity appears as an exact term plus a constraint. Yang-Mills theory also can be related to the BF action...

BF-Yang-Mills theory

$$S_{YM} = \frac{1}{G}\left[S(B,F) + e^2 \int_M B \wedge {}^*B \right]$$

$$(F_{\mu\nu})_{IJ} = (\partial_\mu A_\nu - \partial_\nu A_\mu)_{IJ} + (A_\mu)^K{}_I (A_\nu)_{KJ} - (A_\nu)^K{}_I (A_\mu)_{KJ}$$

$$B \wedge {}^*B = trB_{\mu\nu}B^{\mu\nu}$$

Thus, Yang-Mills action is a topological action plus a deformation: this formulation is useful to compare the two theories. It is possible to formulate BF gravity in such a way that it has the same propagators as BF-Yang-Mills theory. This is very useful in order to clearly identify the "guilty" of the perturbative non renormalizability of gravity.

Vertices, Propagators,…

$$V(A^a{}_\mu, A^b{}_\nu, B^c{}_{\alpha\beta}) \approx f^{abc}\varepsilon_{\mu\nu\alpha\beta}$$

$$V(B^a{}_{\mu\nu}, B^b{}_{\alpha\beta}, \phi^{cd}) \approx \delta_{ac}\delta_{bd}\varepsilon^{\mu\nu\alpha\beta}$$

$$V(A^a{}_\mu, \rho^b, \rho^{*d}) \approx -f^{abd}p^\mu, \ldots$$

The first vertex is present both in the Yang-Mills case and in the Gravitational case; the second one only pertains to gravity.
The ghosts vertices are the same as the Yang-Mills case (only one has been explicitly written). Besides the second vertex, all the vertices have a standard connected structure from an "internal index point of view".

The ghosts propagator have similar structures, the only propagator which is bad-behaved in the UV is the BB-propagator. Such propagators are the same as in the BF-Yang-Mills case.

$$\Delta_{AA} = \frac{\delta^{ab}}{p^2}\left(\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right),$$

$$\Delta_{BB} = -\frac{\delta^{ab}}{p^2}\varepsilon_{\mu\nu\alpha\lambda}\varepsilon_{\gamma\rho\beta\lambda}\, p^\alpha p^\beta,$$

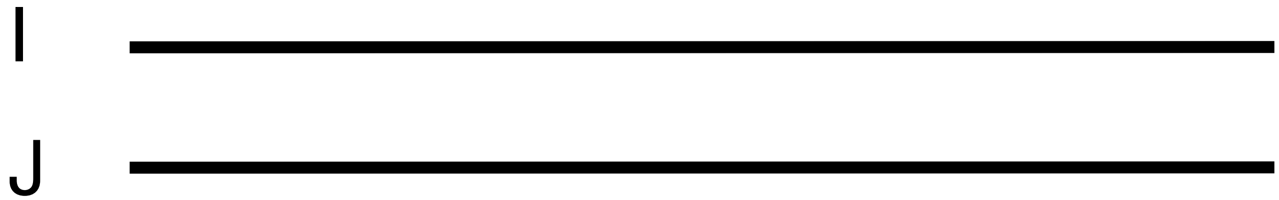
$$\Delta_{AB} = -\frac{\delta^{ab}}{2p^2}\varepsilon_{\mu\nu\alpha\lambda}\, p^\alpha,$$

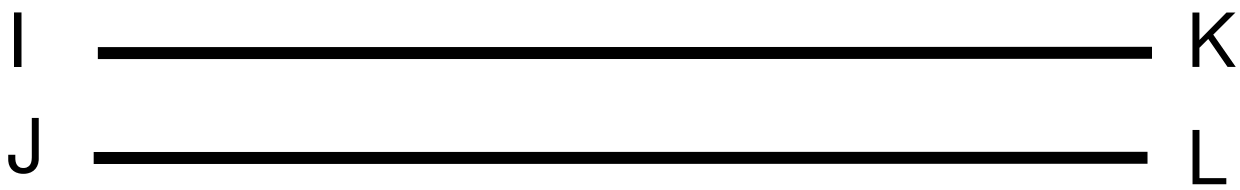
....

Gravitational ‘t Hooft notation

The ‘t Hooft notation can be introduced as before, being the physical fields in the adjoint of the gauge group. However, there is an interesting difference: being the fundamental representation of SO(**N**-1,1) (which, in the gravitational case, is the gauge group) real and the adjoint the tensor product of the fundamental by itself, there is no need to use the arrows. Physically, the “gravitational charge” is always positive.

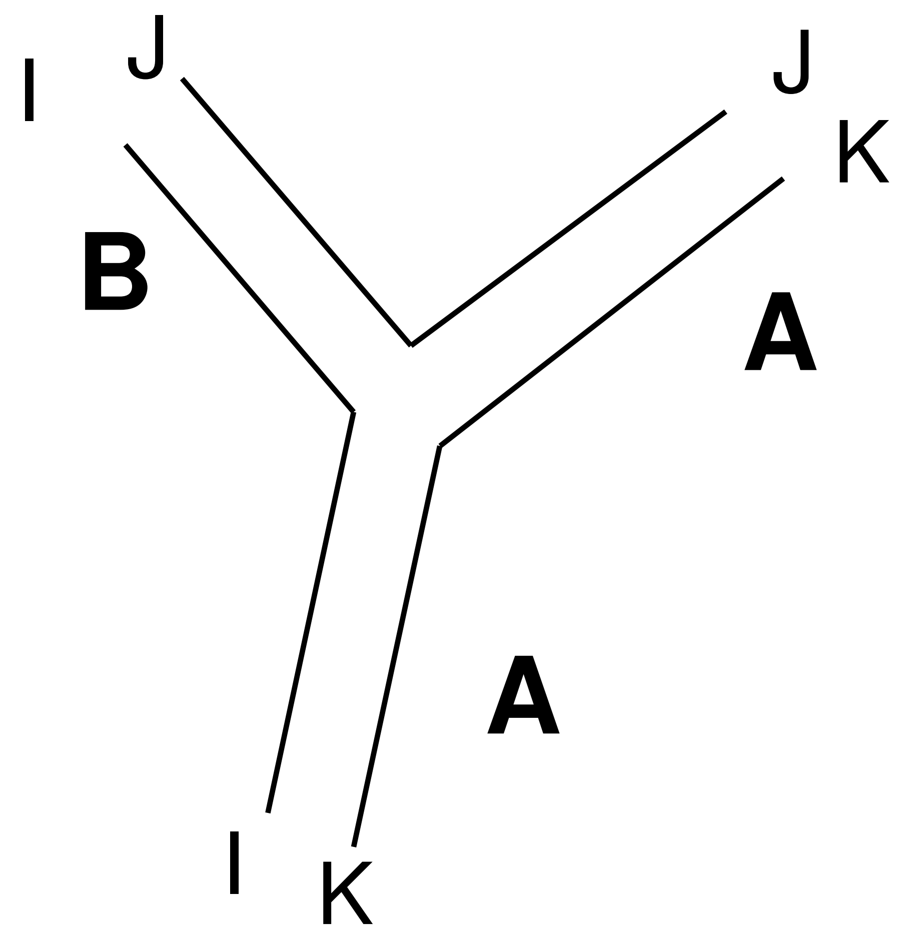
A,B→A,B

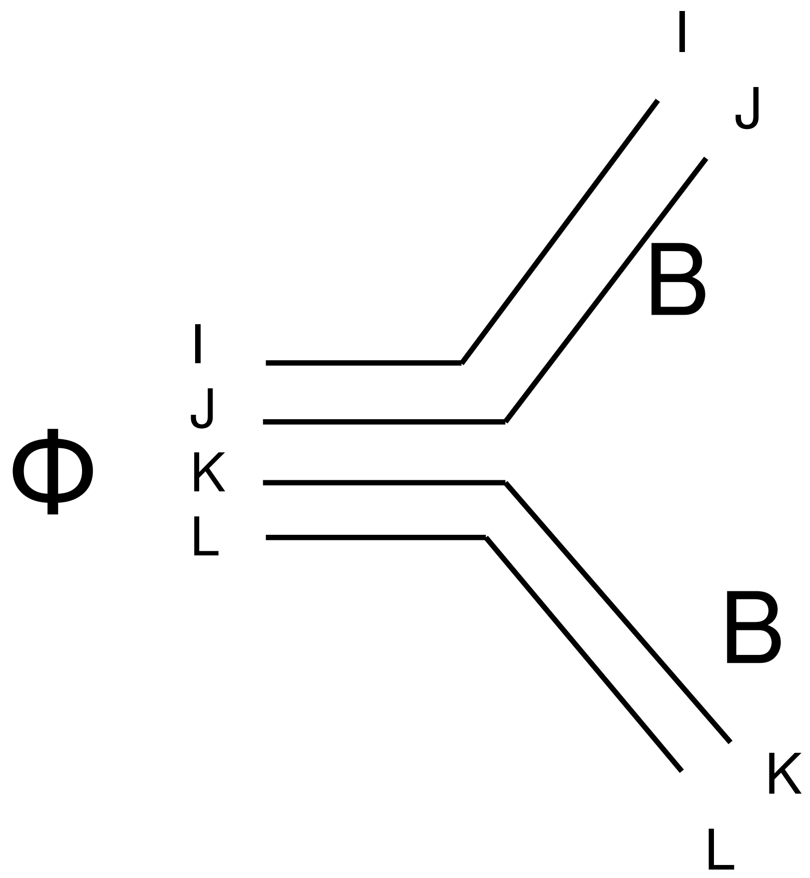




$$\delta_{IK}\delta_{JL}$$

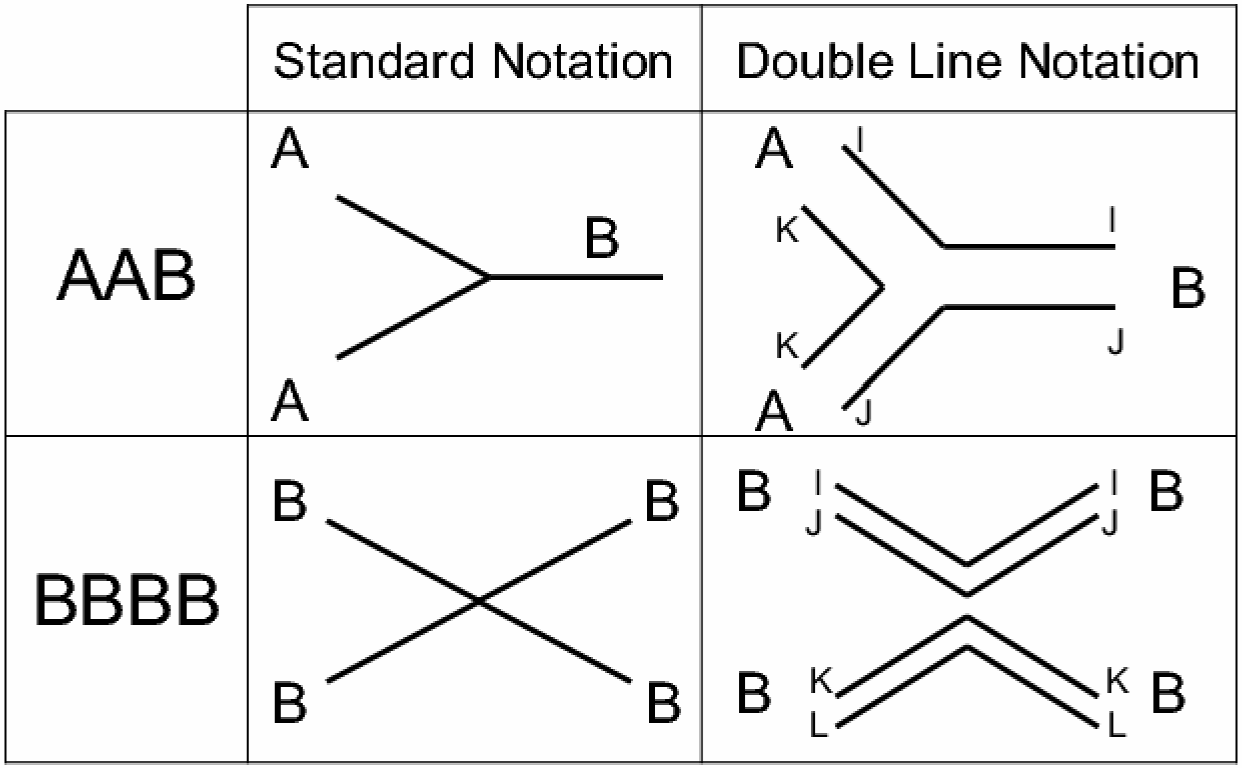
Vertices in the ‘t Hooft notation

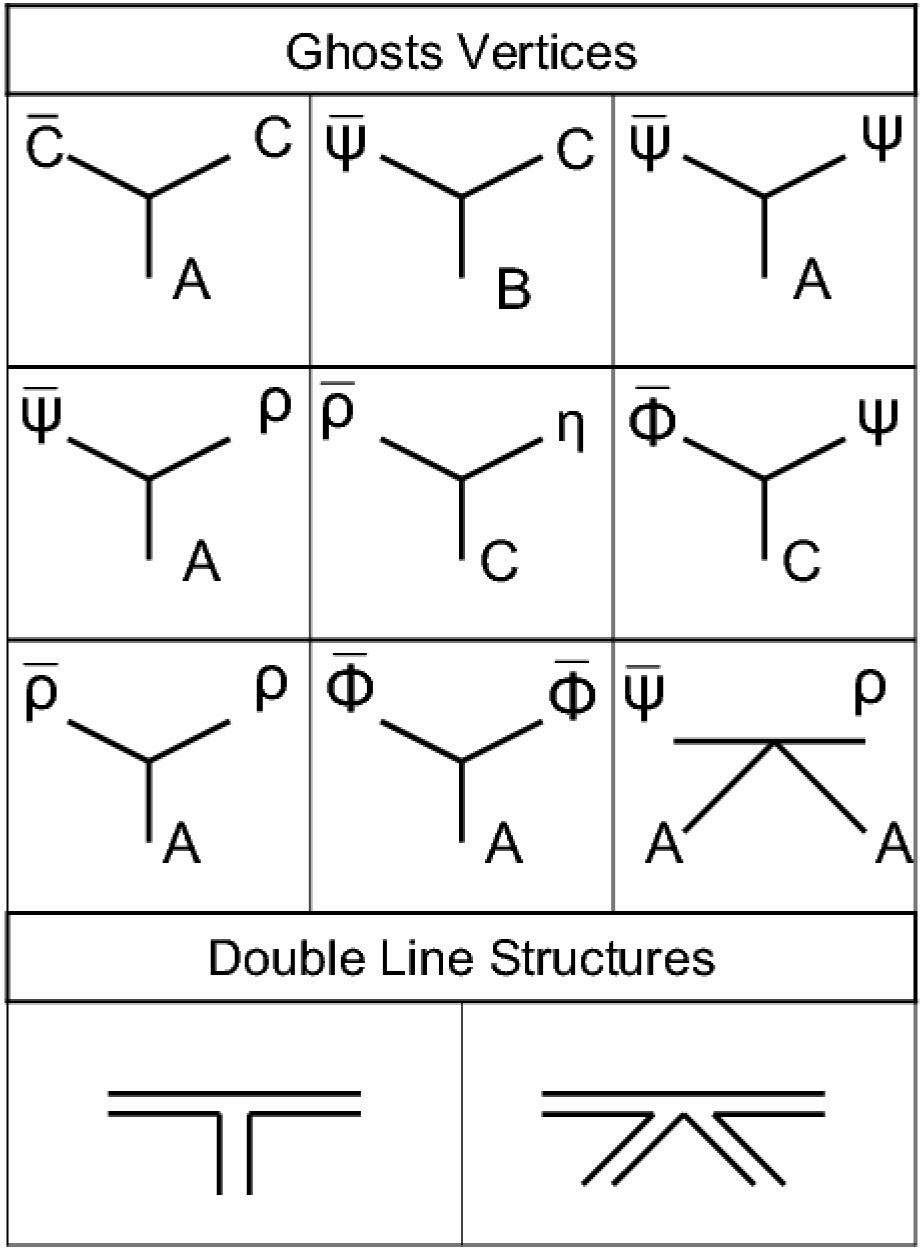




This vertex gives rise to a "disconnected" 4-uple vertex for B.

It appears a 4-uple disconnected vertex:




Ghosts Vertices
$\bar{C}$ C A
$\bar{\Psi}$ C B
$\bar{\Psi}$ Ψ A
$\bar{\Psi}$ ρ A
$\bar{\rho}$ η C
$\bar{\Phi}$ Ψ C
$\bar{\rho}$ ρ A
$\bar{\Phi}$ $\bar{\Phi}$ A
$\bar{\Psi}$ ρ A A
Double Line Structures

In fact…

- At a first glance, it seems that the vertex with the Lagrangian multiplyer Φ gives rise to a standard 4-uple vertex for **B**.
- In fact, as it can be seen graphycally, this is not completely true. This fact has an interesting physical interpretation.

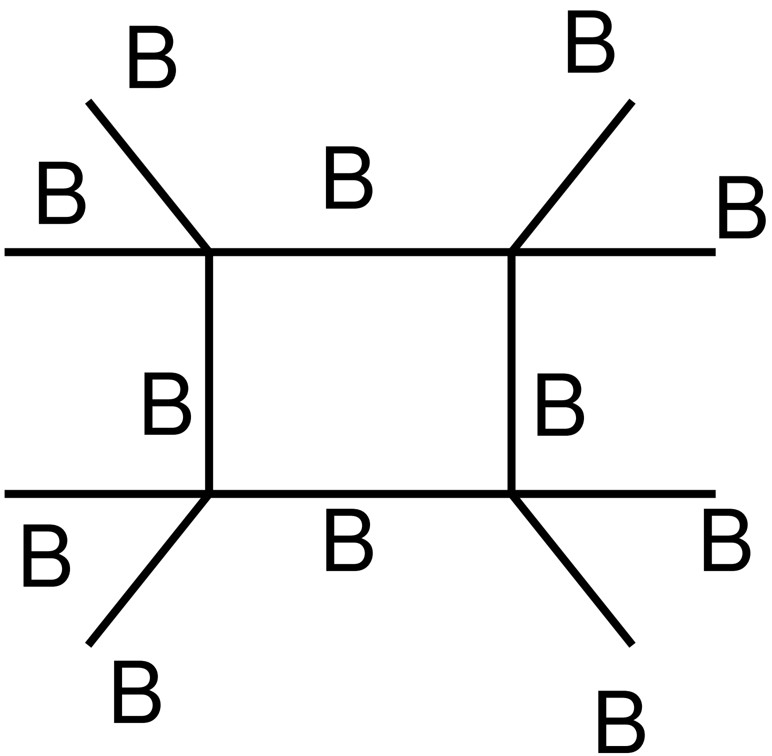
Perturbative non renormalizability

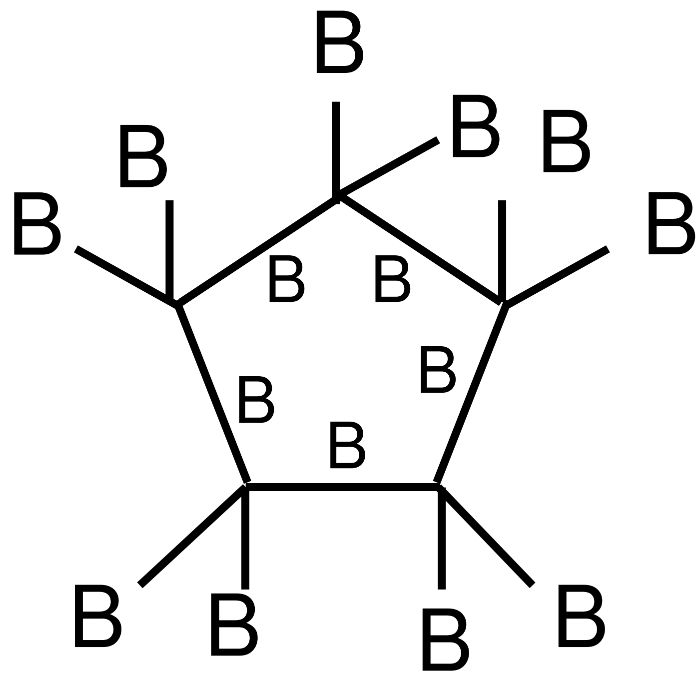
- The perturbative non renormalizability of (super)gravity was an important result: there are many examples (such as gravity in three dimensions) of theories which are trivially renormalizable (being BRS-exact) and, in fact, would not appear in this way by power counting. The results about the one-loop finiteness of gravity lead to the expectations that the symmetries of gravity could give rise to some "miracle”: in the perturbative formulation, such a miracle does not occur.

- In the BF formulation the perturbative non renormalizability comes from the 4-uple **B** vertex: the **B** propagator has a bad UV behavior as in the YM case. In fact, in the YM case, loops with only **B** propagators do not occur because one only has the **AAB** vertex.

- In BF gravity the 4-uple **B** vertex gives rise to loops with an arbitrary number of **B** propagators inside so that new infinities at each perturbative order appear.

Some divergent diagrams in the usual perturbation theory





Matter coupling

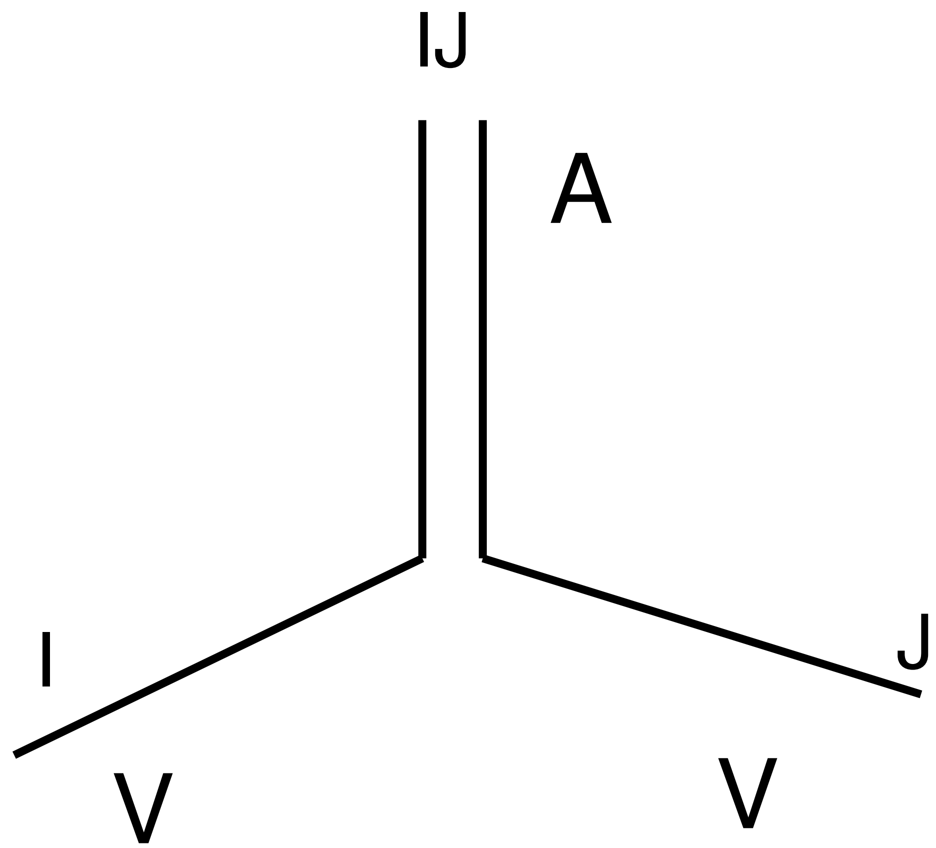
- Levi-Civita covariant derivative (in which it enters the gravitational connection) couples to tensorial indices.
- Thus, in this scheme, vector and spinor fields should be seen as scalar field with an internal index.
- However, it is not clear how to introduce matter fields because of the lacking of the metric which, in this scheme, is not a fundamental field.

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\rho} V^\rho \Leftrightarrow$$

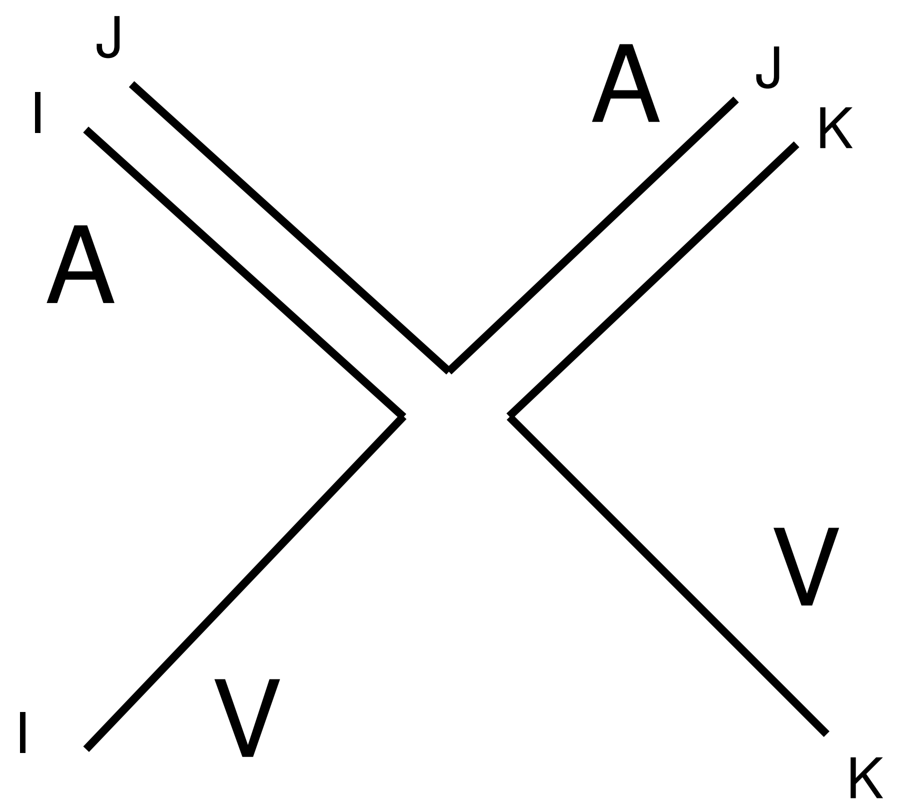
$$(\nabla_A V)^I = \partial_\mu V^I + (A_\mu)^I{}_K V^K$$

$$V^\mu \rightarrow (V)^I$$

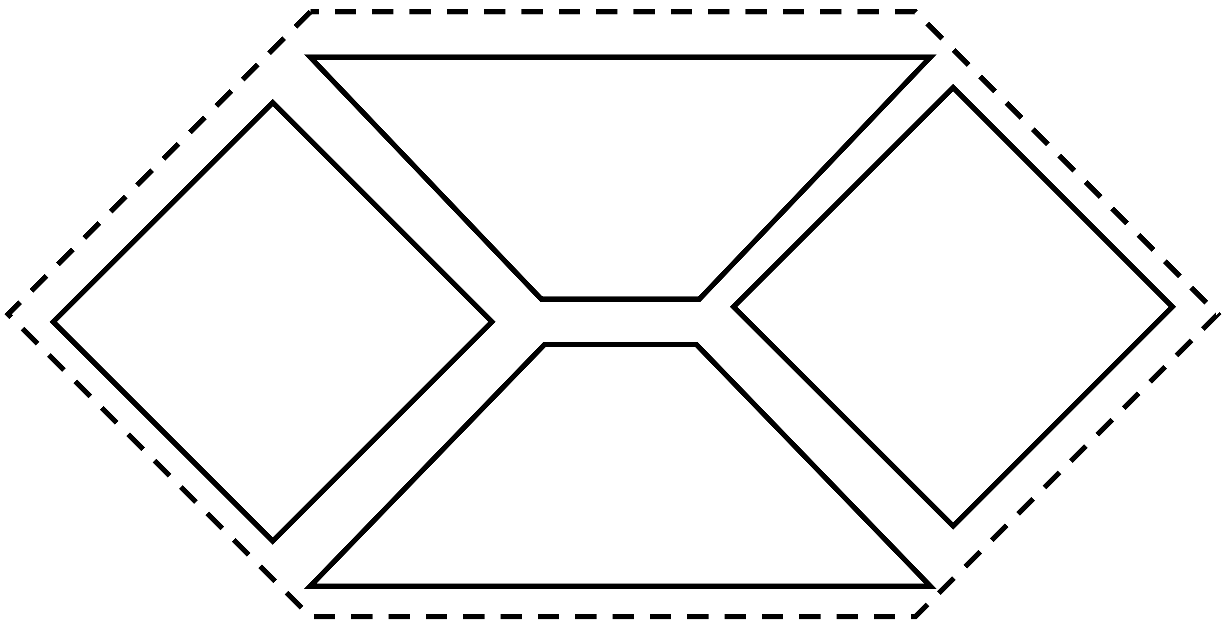
Matter vertices



Thus, as suggestd by the Yang-Mills analogy, matter fields could be dealt as scalar fields carrying an internal index: the counting will be also similar. This matter vertex and the previous one would come from a covariant kinetic term for a scalar field with an internal "gravitational" index.

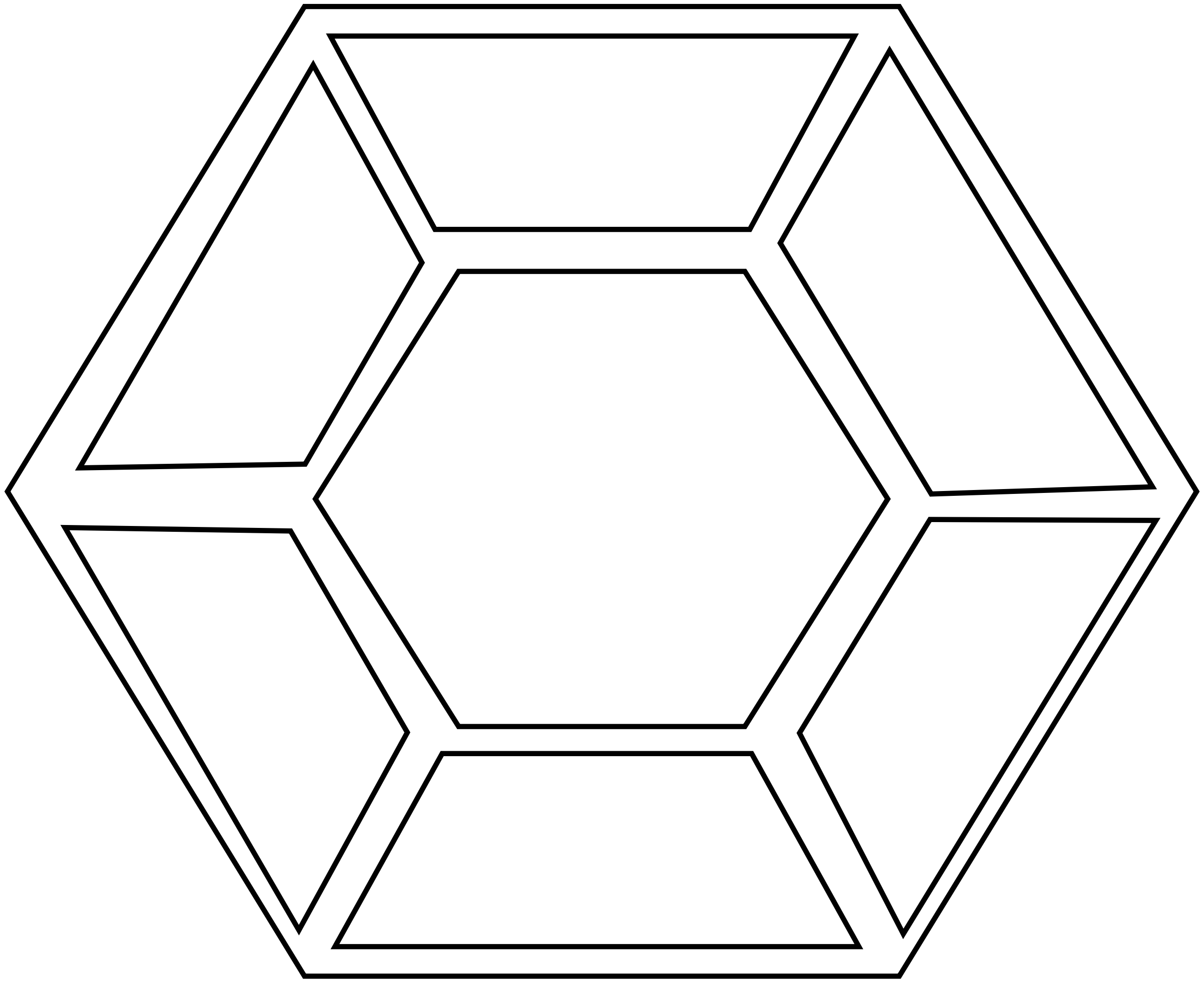


An example of a graph contributing to the free energy with one matter loop, four color loops, four matter vertices and two gravitational vertices.

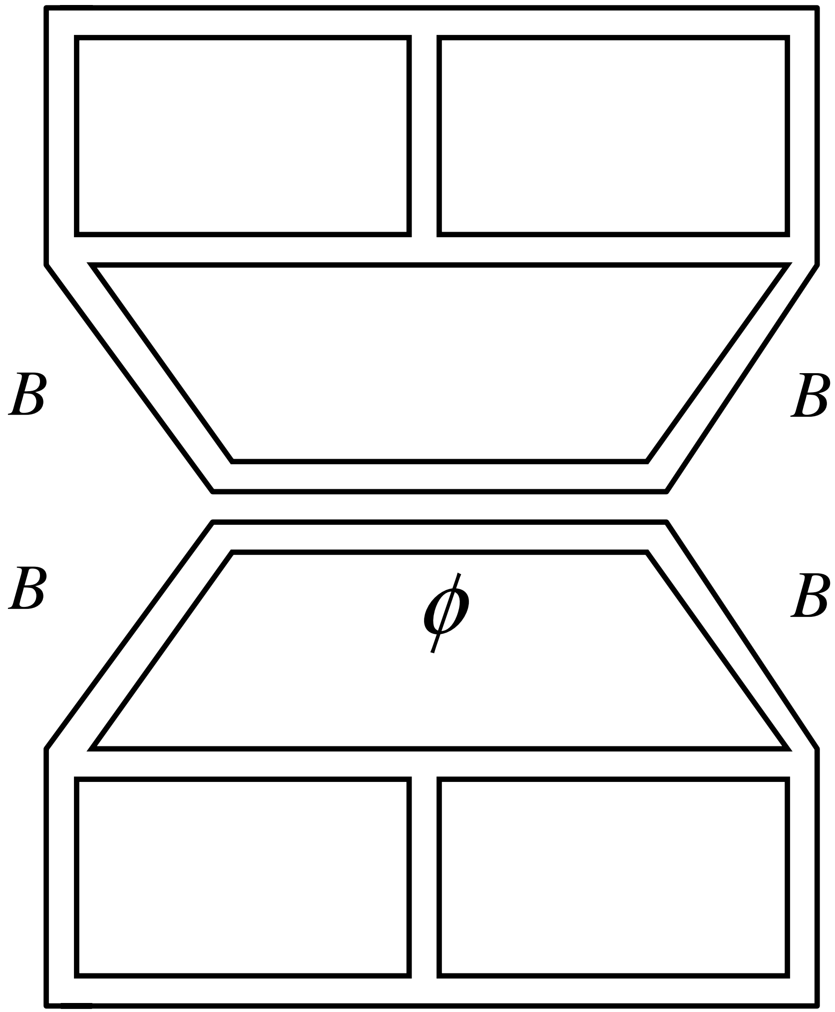


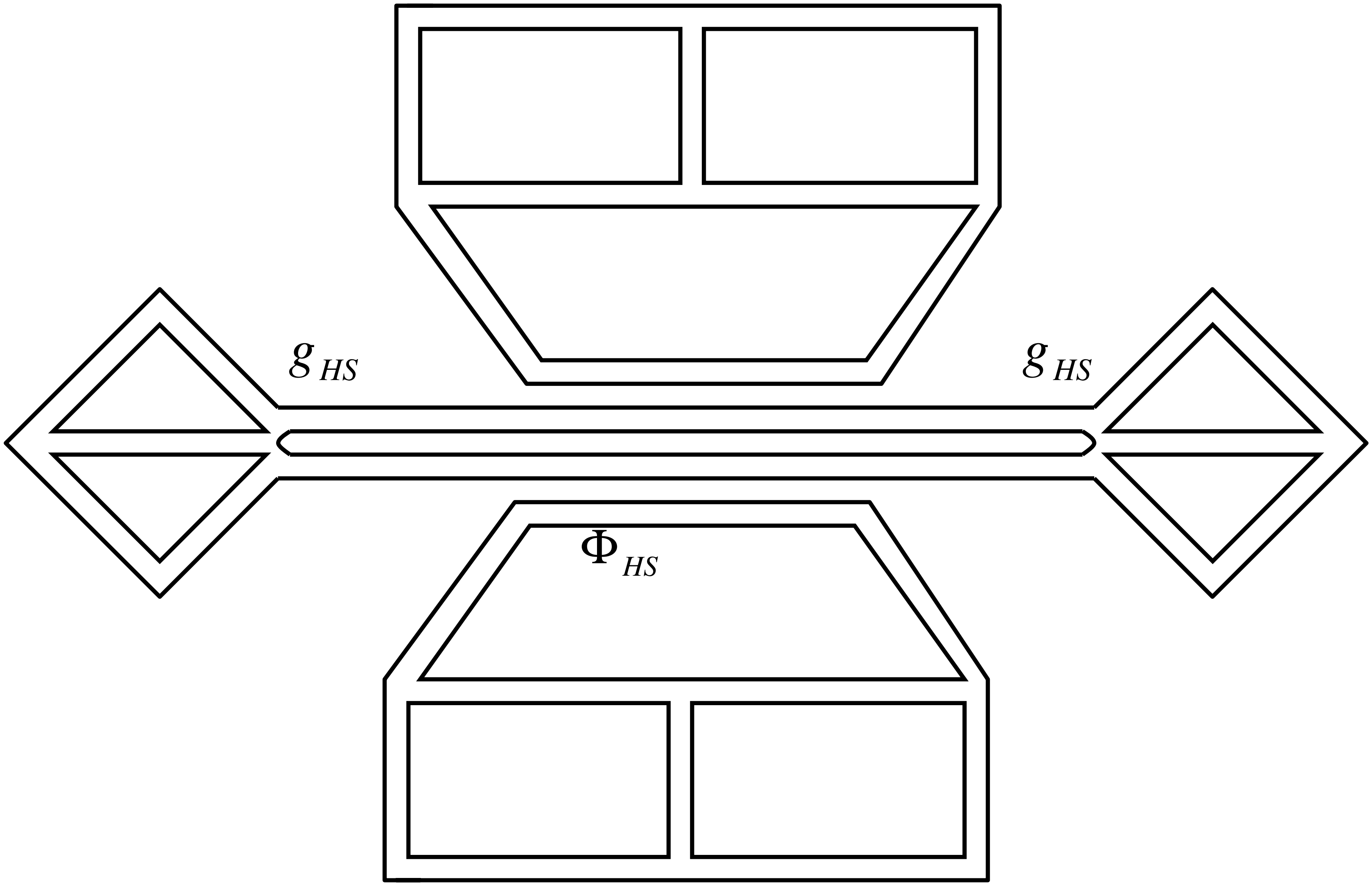
$$\approx N^4 N_f$$

An example of a graph contributing to the free energy with eight color loops.



$$\approx N^8$$


B
B
B
B
ϕ


g_{HS}
g_{HS}
Φ_{HS}

Here (one of) the difference(s)…

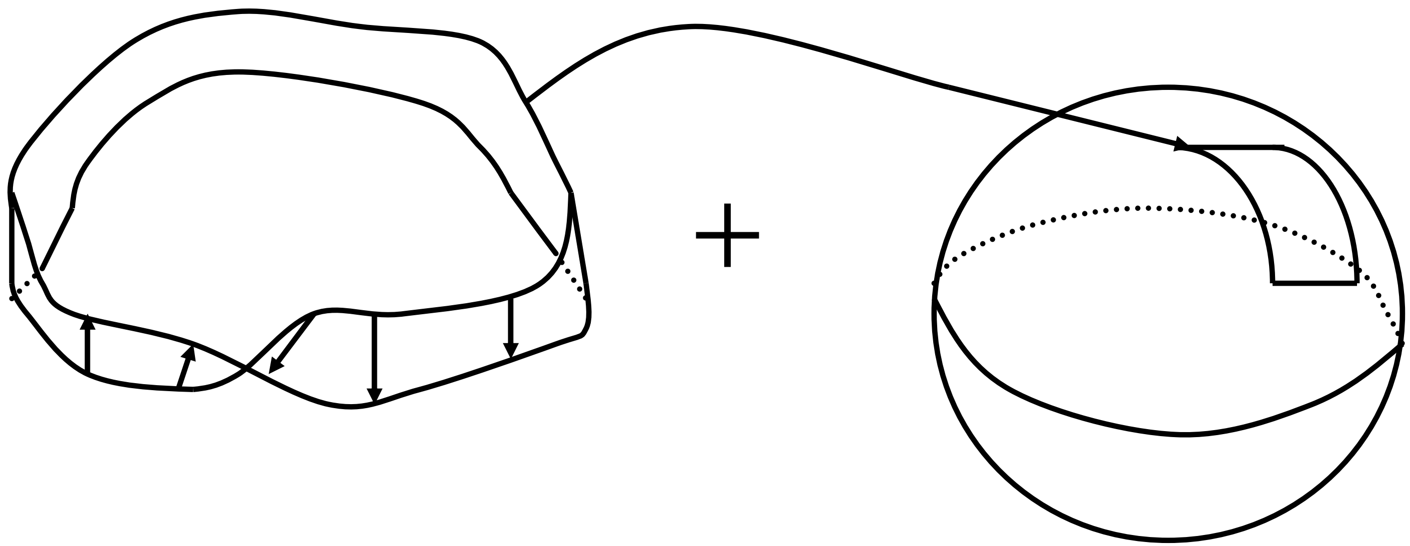
- The previous kind of graphs, which only pertain to gravity, would have no role in a gauge theory with fields carrying only one and two internal indices and connected vertices.
- In fact, in gravity, it naturally appears a “disconnected” vertex.
- Such a vertex is likely to have the role to decrease the entropy with repsect to the gauge theory case. This is the first quantistic argument supporting the holographic principle.

Another difference…

- There is another difference related to the group-theoretical structure of gravity: internal lines carry no arrow. This implies that, in the topological expansion, non orientable surfaces cannot be omitted.
- Thus, purely gravitational loops, unlike gauge theory, are suppressed as 1/**N** instead of 1/$\mathbf{N}^2$.

The point is that the Euler formula $2g-2=E-V-F$ also holds in this case provided the genus *g assumes half-integer values too.*

Gluing the boundaries



- The interpretation of this result is that, unlike gauge fields, the gravitational field is also able to imitate matter fields. In fact, this should not be too surprising: besides the Kaluza-Klein mechanism, exact solutions of vacuum Einstein equations carrying spin ½ and spin 1 have been found. This property could also be kept in the would be quantum theory.

- Eventually, it is interesting to note that, in some sense, scalar fields behave as baryons in QCD: this could explain (at least at a qualitative level) why they are so heavy and weakly interacting.

- At a first glance, there is a contraddiction: the "disconnected" vertex is likely to be responsible of the decreasing of the degrees of freedom in the strongly coupled phase of gravity.

- On the other hand, it is precisely the disconnected vertex the "guilty" of the perturbative non renormalizability of gravity.

- How is it possible that the same vertex is responsible both for the bad UV-behavior and for the decreasing of the degrees of freedom in the UV?

- The large N epansion provides with a very natural answer.
- One first has to formulate correctly the large N expansion. Then, one finds that in gravity the large N power counting is different from the perturbative power counting.
- In gauge theories the power counting is the same in both expansions but there are well known examples in which this is not true.

EX: 5D interacting scalar field

$$L = \frac{1}{2}\left(\partial_\mu \phi_I \partial^\mu \phi_I + m^2 \phi_I \phi^I\right) + \frac{\lambda}{8}\left(\phi_I \phi^I\right)^2 \Leftrightarrow$$

$$L = \frac{1}{2}\left(\partial_\mu \phi_I \partial^\mu \phi_I + \left(m^2 + i\sigma\sqrt{\lambda}\right)\phi_I \phi^I\right) + \frac{1}{2}\sigma^2$$

$$\Delta_\sigma(p) = 1; I = 1,.., N$$

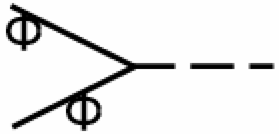
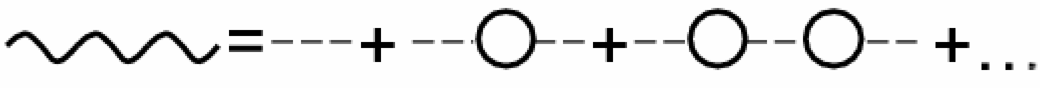
$$\Delta^{(N)}{}_\sigma(p) = \frac{1}{1 + g\Pi(p)},$$

$$\Pi(p) \xrightarrow[p \to \infty]{} p; g = \lambda N$$

In this case, a perturbatively non renormalizable 5D scalar theory becomes renormalizable at large N due to the resummation of the bubble diagrams: *Green functions are non analytic anymore in the coupling constant(s).*

Summing bubble diagrams

O(N) Φ^4 model example

Feynman Rules	
Φ-propagator	
σ-propagator	
ΦΦσ - vertex	
Improved σ-propagator	
$\Delta_\sigma =$ 	

- The seemingly "magic" properties of the large **N** resummations are related in a very simple way to physical properties of the models. In the above mentioned cases, the large **N** expansion is able to explore the strongly coupled phase of the theory (vertices and propagators are not anymore analytic in the “old” coupling constant) in which non perturbative phenomena (such as the appearance of "color-less" bound states in the spectrum, spontaneous breaking of symmetries and so on) occur.

- Thus, a theory which is renormalizable at large **N** and is not renormalizable in the standard expansion is not “wrong”. It is simply formulated in terms of variables which are not able to describe small excitations around the strongly coupled vacuum. In many cases, the large **N** expansion is able to capture such non perturbative features.

Large **N** power counting

- In the standard perturbative formulation, the basic objects for the power counting are bare vertices and propagators: that is, "loop-less" vertices and propagators. With these objects one can construct loop-integrals and study the UV behavior.
- Similarly, at large N the basic objects are "loop-less" vertices and propagators. Now, "loop-less" means "without closed color loops".
- In the YM case (connected vertices), it is not possible to construct diagrams which *do have* standard "Feynman" loops and, in fact, *do not have* closed color loops. The UV-counting is the same.

Setting of the expansions

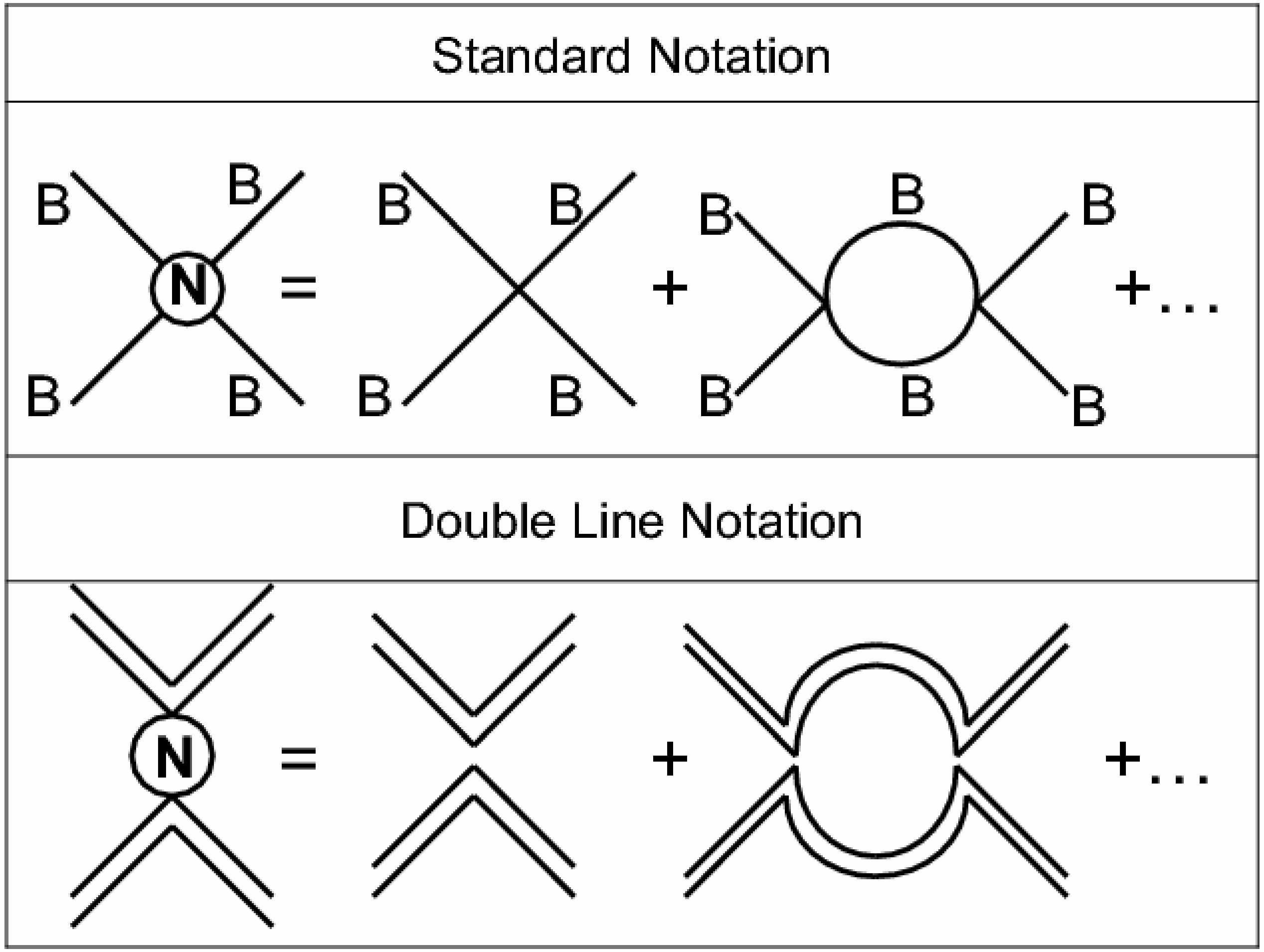
$$O_F = \sum_{L=0}^{\infty} \eta^L O^{(L)}{}_F ,$$

$$O_N = \sum_{g} N^{2-2g} \sum_{L_c} (\lambda_{eff})^{L_c} O_{(g, L_c)}$$

$$L = 0 \Leftrightarrow L_c = 0$$

The role which in the standard perturbative expansion is played by the tree diagrams, at large **N** is played by the diagrams without closed color loops. Such diagrams are the building blocks of large **N** power counting.

Summing Tree-diagrams at large **N**...



$$g_4 \to \frac{M^2{}_P}{M^2{}_P - p^2}$$

$$M_P{}^2 = \frac{1}{(G\Lambda)^2}$$

$$GN \xrightarrow[N \to \infty]{} const$$

$$\frac{\Lambda}{N} \xrightarrow[N, \Lambda \to \infty]{} const$$

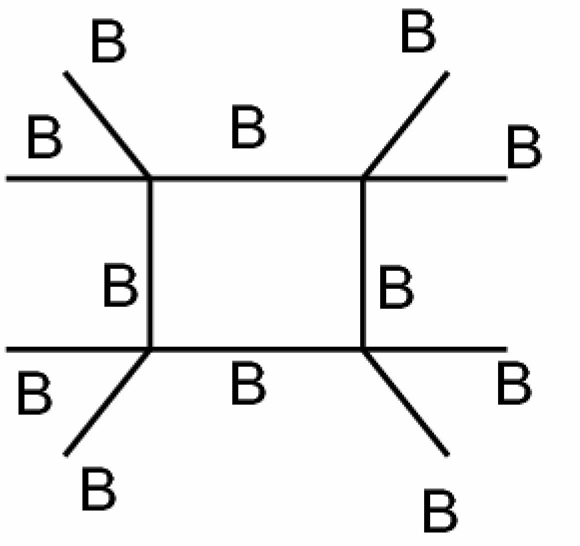
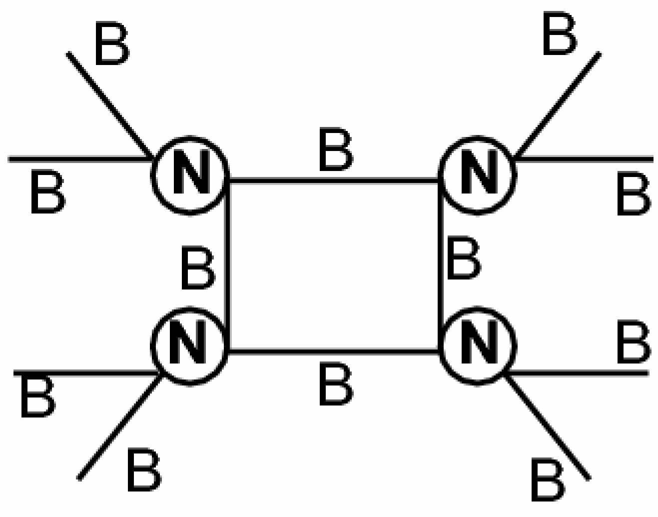
$$g_4^{(0)} \rightarrow g_4(p) = g^{(0)}{}_4 \frac{1}{1-\Pi(p)}$$

$$\Pi(p) = G^2 \left(\varepsilon^4\right)_{\alpha\beta}{}^{\xi\eta} \int_\Lambda d^4q \frac{q^\alpha q^\beta (q-p)_\xi (q-p)_\eta}{q^2 (q-p)^2},$$

$$\left(\varepsilon^4\right)_{\alpha\beta}{}^{\xi\eta} = \varepsilon_{\mu\nu\alpha\lambda} \varepsilon_{\gamma\rho\beta\lambda} \varepsilon^{\mu\nu\xi\chi} \varepsilon^{\gamma\rho\eta\chi}$$

Actually, also a tadpole term has to be removed "by hand" from the denominator of the effective coupling constant. In simpler models the tadpole is already embodied in the so called *gap equation*.

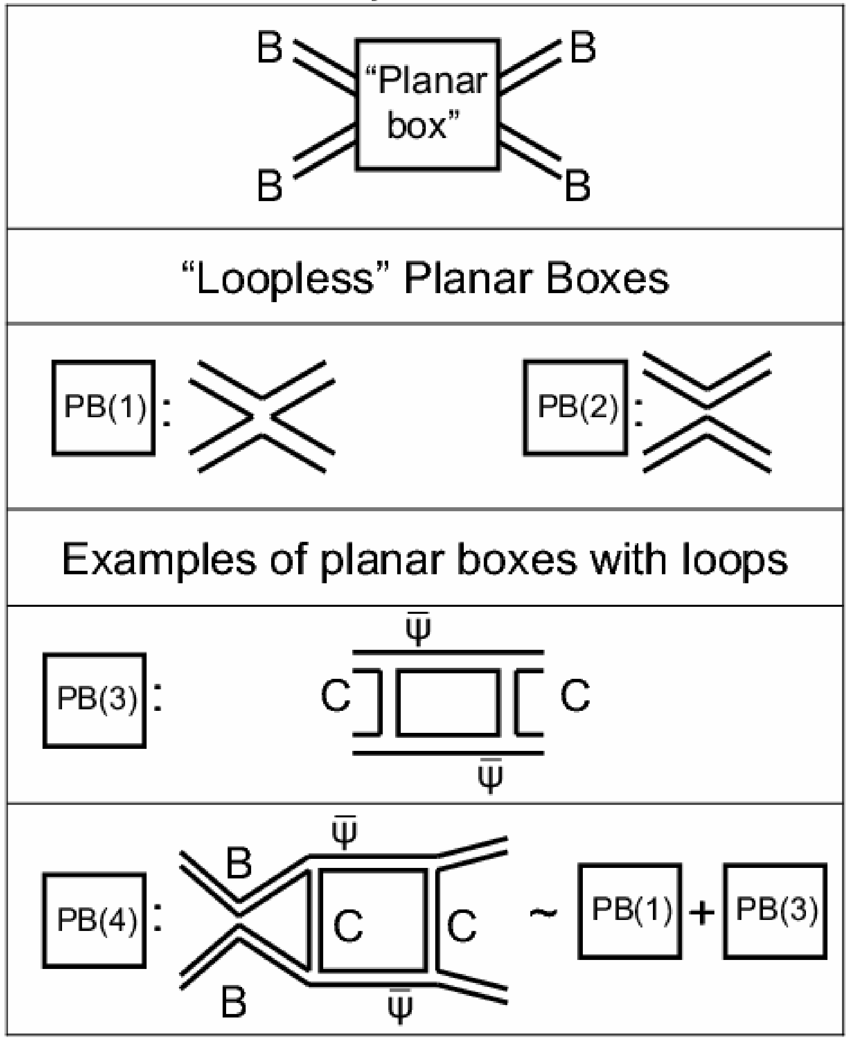
UV-softening…

Divergent in the usual perturbation theory	Convergent at large N
	

Ghosts loops effects:

Thus, it is not possible to construct a ghost contribution to the 4-uple **B** vertex without closed color loops.

Leading order large N contribution to the 4-uple B-vertex



Physical interpretation

A natural interpretation is the following: “bad” amplitude in the UV are connected to higher spin particles.
Thus, if beyond a certain energy, only scalar particles are left in the spectrum, the UV infinities would disappear. This is quite consistent with the gauge/gravity duality and could have interesting cosmological consequences (inflation, reheating,…).

$$A_J(s,t) \approx \frac{s^J}{t - M_J^{\ 2}} \Rightarrow$$

$$s,t \geq E^2{}_0 \rightarrow J = 0 \Rightarrow$$

$$"UV - OK"$$

Conclusion

- The large N expansion in gravity seems to be a good tool to explore the strongly coupled regime of gravity. It is qualitative consistent with known results, it also provides the holographic principle with a field theoretical basis and leads to a surprising UV-softening.
- Renormalization in 1/N, cosmology, fermions…

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