The fuzzy disc

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December 7, 2005

- The beat of a fuzzy drum:fuzzy Bessel functions for the disc. JHEP 2005 hep-th/0506008

- The fuzzy disc. JHEP 0308:057,2003 hep-th/0306247

- From the fuzzy disc to edge currents in Chern-Simons theory. Mod.Phys.Lett.A18:2381-2388,2003 hep-th/0309128

with F. Lizzi and A. Zampini

- A.P. Balachandran , K. S. Gupta, S. Kurkcuoglu, JHEP 0309:007,2003

Fuzzy spaces are an approximation of the abelian algebra of functions on an ordinary space with a finite rank matrix algebra, which preserves the symmetries of the original space, at the price of noncommutativity.

The idea was introduced by Madore with the fuzzy sphere: a sequence of nonabelian algebras, generated by three "noncommutative coordinates" which satisfy

$$x_i x_i = 1, \quad [x_i, x_j] = \kappa \varepsilon_{ijk} x_k$$

with κ depending on the dimension of representations of $SU(2)$.

- Brief review of the fuzzy sphere from a perspective which can be generalised to the disc case.

- The fuzzy disc out of the noncommutative plane, implementing the constraint $x^2 + y^2 \leq R^2$.

- Fuzzy Laplacian and fuzzy Bessel functions.

- Perspectives

Fuzzy sphere

We first set up a map between a space of operators and a space of functions on the sphere S^2.

We use coherent states of $SU(2)$.

Consider UIRR for $SU(2)$. On each $\mathbf{C}^N$, $N = 2L+1$, - finite dim Hilbert space - a basis is $|L, M\rangle$ -with $M = (-L, -L+1, \ldots, L-1, L)$-.

$$u \in SU(2) \overset{\widehat{R}^{(L)}}{\mapsto} \mathcal{B}\left(\mathbf{C}^N\right)$$

The second step is to fix a fiducial state. I choose the highest weight in the representation: $|\psi_0\rangle = |L, L\rangle$ with H_{ψ_0} its stability subgroup by the $\widehat{R}^{(L)}$.

If the group manifold is parametrised by Euler angles, then u represents a point whose "coordinates" range through $\alpha \in [0, 4\pi)$, $\beta \in [0, \pi)$, $\gamma \in [0, 2\pi)$. Fixed the fiducial vector, H_{ψ_0} is made by elements for which $\beta = 0$. $u \equiv u'$ if $u^\dagger u' \in H_{\psi_0}$.

It is possible to prove that:

$$SU(2) / H_{\psi_0} \approx S^2$$

identifying $\theta = \beta$ and $\varphi = \alpha \mod 2\pi$.

Chosen a representative element $\tilde{u}$ in each equivalence class of the quotient, the set of coherent states is defined as:

$$|\theta, \varphi, N\rangle = \widehat{R}^{(L)}(\tilde{u}) |L, L\rangle \ .$$

The left hand side ket now explicitly depends on N, the dimension of the space on which the representation takes place. This set of states is nonorthogonal, and overcomplete ($d\Omega = d\varphi \sin\theta \, d\theta$):

$$\begin{aligned}
\langle \theta', \varphi', N | \theta, \varphi, N \rangle &= e^{-iL(\varphi' - \varphi)} \Big[e^{i(\varphi' - \varphi)} \cos\theta/2 \cos\theta'/2 \\
&+ \sin\theta/2 \sin\theta'/2 \Big]^{2L} \ , \\
\mathbf{I} &= \frac{2L+1}{4\pi} \int_{S^2} d\Omega |\theta, \varphi, N\rangle\langle \theta, \varphi, N| \ .
\end{aligned}$$

Then

$$SU(2)/H_{\psi_0} \approx S^2$$

Chosen a representative element $\tilde{u}$ in each equivalence class of the quotient, the set of coherent states is defined as:

$$|\theta, \varphi, N\rangle = \hat{R}^{(L)}(\tilde{u})\,|L, L\rangle$$

Thus it is possible to define a map,

$$\begin{aligned} \hat{A}^{(N)} \in \mathcal{B}\left(\mathbf{C}^N\right) \approx \mathbb{M}_N \quad &\mapsto \quad A^{(N)} \in \mathcal{F}\left(S^2\right) \ , \\ A^{(N)}(\theta, \varphi) \quad &= \quad \langle \theta, \varphi, N | \hat{A}^{(N)} | \theta, \varphi, N \rangle \ . \end{aligned}$$

$A^{(N)}(\theta, \varphi)$ Berezin symbol.

Among these operators, there are $\widehat{Y}^{(N)}_{JM}$ whose symbols are the spherical harmonics, up to order $2L$ ($J = 0, \ldots, 2L$ and $M = -J, \ldots, +J$):

$$\langle \theta, \varphi, N | \widehat{Y}^{(N)}_{JM} | \theta, \varphi, N \rangle = Y_{JM}(\theta, \varphi) \ ,$$

these operators are called *fuzzy harmonics*.

Why are they important?

In the spirit of NC geometry, the Laplacian carries information on the geometry of the fuzzy sphere:

$$\begin{aligned} \nabla^2 &: \quad \mathbb{M}_N \mapsto \mathbb{M}_N \ , \\ \nabla^2 \widehat{A}^{(N)} &= \left[\widehat{L}^{(N)}_s, \left[\widehat{L}^{(N)}_s, \widehat{A}^{(N)} \right] \right] \ . \end{aligned}$$

with $\hat{L}_a^{(N)}$:

$$\left[\hat{L}_a^{(N)}, \hat{L}_b^{(N)}\right] = i\epsilon_{abc}\hat{L}_c^{(N)} \ ,$$

representing the Lie algebra of the group $SU(2)$ on the space $\mathbf{C}^N$.

∇^2 is the *fuzzy Laplacian*.

Its spectrum, $\lambda = j(j+1)$, $j = 0, .., 2L$ coincides, up to order $2L$, with the one of its continuum counterpart acting on the space of functions on a sphere.
The cut-off of this spectrum is related to the dimension of the rank of the matrix algebra.

Fuzzy harmonics are the eigenmatrices of the fuzzy Laplacian

Fuzzy harmonics are a basis in each space of matrices $\mathbb{M}_N$ $\longrightarrow$

$$\widehat{F}^{(N)} = \sum_{J=0}^{2L} \sum_{M=-J}^{J} F_{JM}^{(N)} \, \widehat{Y}_{JM}^{(N)} \ ,$$

A Weyl-Wigner map can be defined mapping spherical harmonics into fuzzy harmonics:

$$\widehat{Y}_{JM}^{(N)} \ \Leftrightarrow \ Y_{JM}\,(\theta,\varphi) \ \ .$$

This map depends on N

$$\widehat{F}^{(N)} \ \ \leftrightarrow \ \ F^{(N)}\,(\theta,\varphi) = \sum_{J=0}^{2L} \sum_{M=-J}^{+J} F_{JM}^{(N)} Y_{JM}\,(\theta,\varphi)\,.$$

Given a function on the sphere,

$$f(\theta,\varphi) = \sum_{J=0}^{\infty} \sum_{M=-J}^{J} f_{JM} Y_{JM}(\theta,\varphi) .$$

Now consider the set of "truncated" functions:

$$f^{(N)}(\theta,\varphi) = \sum_{J=0}^{2L} \sum_{M=-J}^{J} f_{JM} Y_{JM}(\theta,\varphi) \quad .$$

this is made into an algebra, isomorphic to the matrix algebra $\mathbb{M}_N$, if we define a new product,

$$\left(f^{(N)} * g^{(N)}\right)(\theta,\varphi) = < \theta,\varphi,N|\widehat{f}^{(N)}\widehat{g}^{(N)}|\theta,\varphi,N >$$

The sequence of nonabelian algebras $\mathcal{A}^{(N)}\left(S^2,*\right)$ is the fuzzy sphere.

WHY

These algebras can be seen as formally generated by matrices which are the images of the norm 1 vectors in $\mathbf{R}^3$, i.e. points on a sphere.

They are mapped into multiples of the generators $\widehat{L}_a^{(N)}$ of the Lie algebra:

$$\frac{x_a}{\| \vec{x} \|} \mapsto \widehat{x}_a^{(N)} \qquad \left[\widehat{x}_a^{(N)}, \widehat{x}_b^{(N)}\right] = \frac{2i\varepsilon_{abc}}{\sqrt{N^2-1}} \widehat{x}_c^{(N)} \ .$$

$$\widehat{L}_a^{(N)} = \sqrt{N^2-1}\widehat{x}_a^{(N)}$$

The commutation rules make it intuitively clear that the limit for $N \to \infty$ of this sequence is an abelian algebra.

Fuzzy disc

How do we generalise to the disc?

- To set up the Weyl-Wigner map, we use coherent states for the H-W group *plus* a truncation, which implements the constraint.

- This identifies the sequence of finite- dimensional matrix algebras which gives a good approximation to functions supported on the disc.

- The underlying geometry is introduced, as for the sphere, through a Laplacian, defined on each of the algebras of the sequence.

- The eigenvalues are seen to converge to those of the Laplacian with Dirichlet boundary conditions on the disc.

- The eigenmatrices of the fuzzy Laplacian are the fuzzy Bessel functions, their symbols converging to ordinary Bessel functions in some appropriate limit.

Bessel functions are a basis in $\mathcal{D}$.

Fuzzy Bessels are a a basis in $\mathbb{M}_N$. $\longrightarrow$ We have a WW isomorphism.

Given $\qquad f \in \mathcal{A}(D)$

- expand it in Bessel functions
- truncate the expansion
- 'quantise' by replacing Bessel functions with fuzzy Bessel functions
- consider Berezin symbols and import the nonabelian product

We obtain a sequence of nonabelian algebras converging to the algebra of functions on the ordinary disc. It is this sequence that we call fuzzy disc.

Start with the noncommutative plane

It can be defined using a Weyl-Wigner map, following again the general procedure of Berezin

- *Weyl-Wigner map*

  Given the usual coherent states of the HW group, eigenstates of the annihilation operator $\widehat{a}$, with the sole difference that

$$\left[\widehat{a}, \widehat{a}^{\dagger}\right] = \theta\, \mathbf{I},$$

  A Berezin symbol can be associated to an operator in the Fock space:

$$f\left(\bar{z}, z\right) = \langle z|\widehat{f}|z\rangle \ .$$

  with inverse

$$\widehat{f} = \int \frac{d^2\xi}{\pi\theta} \int \frac{d^2z}{\pi\theta} f\left(z, \bar{z}\right)\, e^{-\left(\bar{z}\xi - \bar{\xi}z\right)/\theta}\, e^{\xi\widehat{a}^{\dagger}/\theta}\, e^{-\bar{\xi}\widehat{a}/\theta}.$$

This quantization map can be given an interesting form. Start with functions :

$$f\left(\bar{z}, z\right) = \sum_{m,n=0}^{\infty} f_{mn}^{Tay} \bar{z}^{m} z^{n} \ .$$

which is the symbol of

$$\widehat{f} = \sum_{m,n=0}^{\infty} f_{mn}^{Tay} \widehat{a}^{\dagger m} \widehat{a}^{n} \ .$$

note the ordering

More generally we can consider operators written in a density matrix notation:

$$\widehat{f} = \sum_{m,n=0}^{\infty} f_{mn} |\psi_m\rangle\langle\psi_n| \ .$$

The Berezin symbol of this operator is the function:

$$f\left(\bar{z}, z\right) = e^{-|z|^2/\theta} \sum_{m,n=0}^{\infty} f_{mn} \frac{\bar{z}^m z^n}{\sqrt{m!n!\theta^{m+n}}} \ , \tag{1}$$

Invertibility of the Weyl map (on a suitable domain) enables to define a noncommutative product -Voros (Wick)- product

$$\left(f * g\right)\left(\bar{z}, z\right) = \langle z|\widehat{f}\widehat{g}|z\rangle \ .$$

It has an integral form similar to the Moyal product, a part from a weight factor.

If symbols are expressed in the form (1), then the product acquires a matrix form:

$$(f * g)_{mn} = \sum_{k=0}^{\infty} f_{mk} g_{kn} \ .$$

This is an important feature of Voros product.

$\mathcal{A}_\theta = \mathcal{F}(\mathbb{R}^2, *)$ is a nonabelian algebra, a noncommutative plane.

It is isomorphic to an algebra of operators -infinite dimensional matrices-

- *Sequence of non abelian algebras*

A fuzzy space has been presented as a sequence of finite rank matrix algebras converging to an algebra of functions.

Here -differently form the sphere- there is no natural definition of finite dimensional matrix algebras, the HW group being noncompact.

$$\Rightarrow$$

– Consider $\mathcal{A}_\theta$ as a matrix algebra made up by formally infinite dimensional matrices.

– Define a set of finite dimensional matrix algebras truncating $\mathcal{A}_\theta$.

Truncation is formalised via the introduction of a set of projectors:

$$\widehat{P}_\theta^{(N)} = \sum_{n=0}^{N} |\psi_n\rangle\langle\psi_n|$$

in the space of operators. Their symbols are projectors in the algebra $\mathcal{A}_\theta$ of the non-commutative plane. With $z = re^{i\varphi}$:

$$\begin{aligned} P_\theta^{(N)}(r,\varphi) &= e^{-r^2/\theta} \sum_{n=0}^{N} \frac{r^{2n}}{n!\theta^n} \\ P_\theta^{(N)} * P_\theta^{(N)} &= P_\theta^{(N)} \ . \end{aligned}$$

This finite sum can be performed yielding a rotationally symmetric function:

$$P_\theta^{(N)}(r,\varphi) = \frac{\Gamma\left(N+1, r^2/\theta\right)}{\Gamma(N+1)} \ .$$

The function P_θ^N for $N = 10^2$

– In the limit $N \to \infty$, θ fixed, and nonzero, $P_\theta^{(N)}(r,\varphi) \to 1$

in this limit one recovers the whole non-commutative plane.

– In the limit $N \to \infty$, $N\theta = R^2$

$$P_\theta^{(N)} \to \begin{bmatrix} 1 & r < R \\ 1/2 & r = R \\ 0 & r > R \end{bmatrix} = Id\,(r) \ .$$

The sequence of projectors converges to a step function in the radial coordinate r, the characteristic function of a disc . Thus a sequence of subalgebras $\mathcal{A}_\theta^{(N)}$ can be defined by:

$$\mathcal{A}_\theta^{(N)} = P_\theta^{(N)} * \mathcal{A}_\theta * P_\theta^{(N)} \ .$$

Given a generic $f \in \mathcal{F}(\mathbf{R}^2)$:

$$f_\theta^{(N)} = P_\theta^{(N)} * f * P_\theta^{(N)} = e^{-|z|^2/\theta} \sum_{m,n=0}^{N} f_{mn} \frac{\bar{z}^m z^n}{\sqrt{m!n!\theta^{m+n}}} \ .$$

Profile of the spherically symmetric function $\Pi_\theta^N\left(\frac{1}{\pi\alpha}e^{-\frac{r^2}{\alpha}}\right)$ for the choice $R^2 = N\theta = 1,\ N = 10^3$

- On every subalgebra $\mathcal{A}_\theta^{(N)} \approx \mathbb{M}_{N+1}$, $P_\theta^{(N)}(r,\varphi)$ is the identity.

- Note that the rotation group on the plane, $SO(2)$, acts in a natural way on these subalgebras.

  Its generator is the truncated number operator $\widehat{N}^{(N)} = \sum_{n=0}^{N} n\theta|\psi_n\rangle\langle\psi_n|$.

- Cutting at a finite N the expansion provides an infrared cutoff.

  This cutoff is "fuzzy" in the sense that functions in the subalgebra are still defined outside the disc of radius R, but are exponentially damped.

- Functions are close to $f_\theta^{(N)}$ if they are mostly supported on a disc of radius $R = \sqrt{N\theta}$, otherwise they are exponentially cut.

- If they present oscillations of too small wavelength (compared to θ) the projected function becomes very large on the boundary of the disc.

Profile of the spherically symmetric function $\Pi_\theta^N(\left(\frac{1}{\pi\alpha}e^{-\frac{r^2}{\alpha}}\right))$ for the choice $R^2 = N\theta = 1$, $N = 10^2$

- *Fuzzy derivatives and fuzzy Laplacian*

  So far the Weyl-Wigner formalism and the projection procedure have provided a way to associate to functions a sequence of finite dimension $(N+1) \times (N+1)$ matrices.

  The next step is the analysis of the geometry these algebras can formalize.

  We need derivations and a Laplacian.

  In the full algebra $\mathcal{A}_\theta$

$$
\begin{aligned}
\partial_z f &= \frac{1}{\theta}\langle z| \left[\widehat{f}, \widehat{a}^\dagger\right] |z\rangle \ , \\
\partial_{\overline{z}} f &= \frac{1}{\theta}\langle z| \left[\widehat{a}, \widehat{f}\right] |z\rangle \ .
\end{aligned}
$$

We define a fuzzified version as

$$\partial_z f_\theta^{(N)} \equiv \frac{1}{\theta}\langle z|\widehat{P}_\theta^{(N)}\left[\widehat{P}_\theta^{(N)}\widehat{f}\widehat{P}_\theta^{(N)},\widehat{a}^\dagger\right]\widehat{P}_\theta^{(N)}|z\rangle$$
$$\partial_{\bar{z}} f_\theta^{(N)} \equiv -\frac{1}{\theta}\langle z|\widehat{P}_\theta^{(N)}\left[\widehat{P}_\theta^{(N)}\widehat{f}\widehat{P}_\theta^{(N)},\widehat{a}\right]\widehat{P}_\theta^{(N)}|z\rangle\ .$$

This is really a derivation on each $\mathcal{A}_\theta^{(N)}$

Let us come to the Laplacian operator.

This additional structure carries information about the geometry of the space underlying $\mathcal{A}_\theta$.

We can say we have succeeded only if we have been able to define a fuzzy Laplacian whose spectrum approaches that of the ordinary Laplacian on the disc when $N \to \infty$.

Starting from the exact expressions:

$$\nabla^2 f(\bar{z}, z) = 4\partial_{\bar{z}}\partial_z f = \frac{4}{\theta^2}\langle z| \left[\hat{a}, \left[\hat{f}, \hat{a}^\dagger\right]\right] |z\rangle$$

we define, in each $\mathcal{A}_\theta^{(N)}$:

$$\nabla^2_{(N)} \hat{f}_\theta^{(N)} \equiv \frac{4}{\theta^2}\hat{P}_\theta^{(N)} \left[\hat{a}, \left[\hat{P}_\theta^{(N)} \hat{f} \hat{P}_\theta^{(N)}, \hat{a}^\dagger\right]\right] \hat{P}_\theta^{(N)} \ .$$

The eigenvalues of this Laplacian can be numerically calculated.

They are seen to converge to the spectrum of the continuum one for functions on a disc, with boundary conditions on the edge of the disc of Dirichlet homogeneous kind.

Exact spectrum (Dirichlet boundary conditions):

All eigenvalues are negative, and their modules λ solve the implicit equation:

$$J_n\left(\sqrt{\lambda}\right) = 0 \ ,$$

where n is the order of the Bessel functions.

Those related to J_0 are simply degenerate, the others are doubly degenerate: so there is a sequence of eigenvalues $\lambda_{n,k}$ where k indicates that it is the k^{th} zero of the function.

The eigenfunctions are:

$$\Phi_{n,k} = e^{in\varphi} J_{|n|} \left(\sqrt{\lambda_{|n|,k}} r \right) \ .$$

Fuzzy spectrum

Computed numerically

It is in good agreement with the spectrum of the continuum case, even for low values N of the dimension of truncation, as can be seen in figure.

It reproduces correctly the degeneracy pattern

Eigenvalues of the fuzzy Laplacian (circles) / exact Laplacian (crosses). $N = 10$

$$N = 20$$

$$N = 30$$

- *Fuzzy Bessel functions* We now introduce a basis on the fuzzy disc.

  We cannot use representation theory unlike for the sphere.

  But we have a well defined Laplacian for each $\mathcal{A}_\theta^{(N)}$.

  We answer the problem looking at the eigen-functions of the fuzzy Laplacian.

– Consider exact eigenfunctions $\Phi_{n,k}$

– map to operators $\widehat{\Phi}_{n,k}$.

$$\widehat{\Phi}_{n,k} = \left(\frac{\lambda_{n,k}}{4}\right)^{n/2} \sum_{j=n}^{\infty} \sum_{s=0}^{j-n}$$

$$\left(-\frac{\theta\,\lambda_{n,k}}{4}\right)^{s} \frac{\theta^{n/2}}{s!\,(s+n)!} \frac{\sqrt{j!\,(j-n)!}}{(j-s-n)!} |\psi_{j-n}\rangle\langle\psi_j| \ .$$

– truncate the series (fuzzify)

In the paper we prove that $\widehat{\Phi}^{(N)}_{n,k}$ are eigenfunctions of the fuzzy Laplacian , namely

$$\nabla^2_{(N)}\, \widehat{\Phi}^{(N)}_{n,k} = -\lambda^{(N)}_{n,k}\, \widehat{\Phi}^{(N)}_{n,k}$$

if and only if the eigenvalues are solutions of:

$$\sum_{k=0}^{N-n+1} (-1)^{k+N-n+1}$$

$$\frac{(N+1)!\,(N-n+1)!}{k!\,(n+k)!\,(N-n-k+1)!} \left(\frac{\lambda^{(N)}_{n,k}}{4N}\right)^k = 0 \ .$$

Finally, we prove that solutions of this equation are exactly the eigenvalues.

This completes the proof: Eigenfunctions of the fuzzy Laplacian coincide, up to the order of the truncation, with ordinary Bessel functions

Since the fuzzy Bessels play a role similar to fuzzy harmonics for the fuzzy sphere algebra, we can now make describe the process of approximating the algebra of functions on a disc with matrices more precise.

If f is square integrable with respect to the standard measure on the disc $d\Omega = r dr d\varphi$, it can be expanded in terms of Bessel functions:

$$f(r,\varphi) = \sum_{n=-\infty}^{+\infty} \sum_{k=1}^{\infty} f_{nk} e^{in\varphi} J_{|n|}\left(\sqrt{\lambda_{|n|,k}}\, r\right)$$

and it is possible to truncate:

$$f^{(N)}(r,\varphi) = \sum_{n=-N}^{+N} \sum_{k=1}^{N+1-|n|} f_{nk} e^{in\varphi} J_{|n|}\left(\sqrt{\lambda_{|n|,k}}\, r\right)$$

$$= \sum_{n=-N}^{+N} \sum_{k=1}^{N+1-|n|} f_{nk}\, \Phi_{n,k}(r,\varphi) \ .$$

This set of functions is a vector space, but it is no more an algebra, with the standard definition of sum and pointwise product, as the product of two truncated functions will get out of the algebra. The mapping from truncated functions into finite rank matrices:

$$f^{(N)} \to \widehat{f}_{\theta}^{(N)} = \sum_{n=-N}^{+N} \sum_{k=1}^{N+1-|n|} f_{nk} \widehat{\Phi}_{n,k}^{(N)}$$

endows the set of functions with a noncommutative product, inherited from the matrix product, which makes it into a non abelian algebra. The formal limit $N \to \infty$ with the constraint $N\theta = 1$ is the abelian algebra of functions on the disc. The sequence of nonabelian algebras $\mathcal{A}_{\theta}^{(N)}$ is what we call the *fuzzy disc*.