Soliton and other travelling-wave solutions for a perturbed sine-Gordon equation encountered in superconductivity

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Introduction

The “perturbed” sine-Gordon equation ($\alpha \geq 0, \gamma \in \mathbb{R}$):

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi + \alpha\varphi_t + \gamma = 0. \qquad x \in \mathbb{R} \tag{1}$$

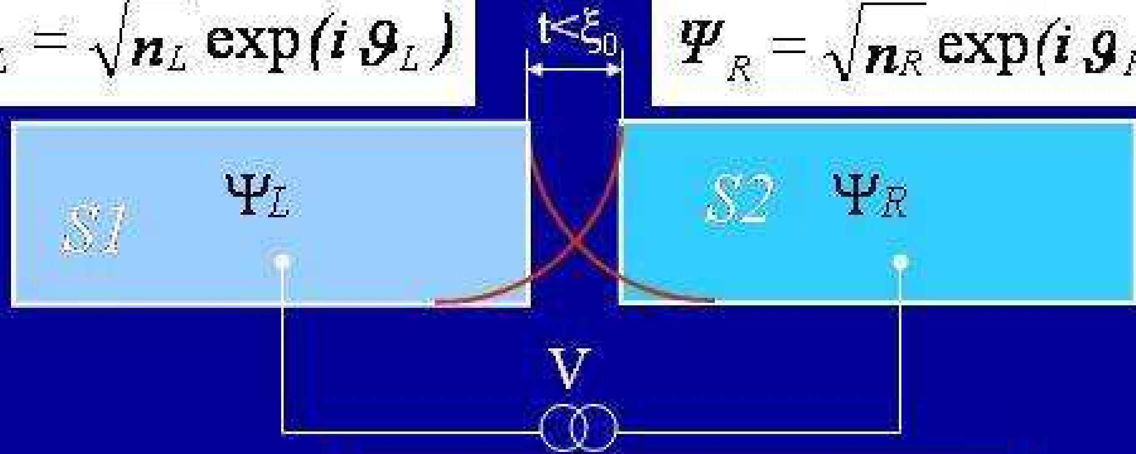
Studied analytically and numerically. Solitonic solutions of the s-G eq. are particularly important: describe stable localized entities propagating along x.

Do stable solutions survive perturbation? Do other appear? Properties?

1st question: Strong numerical, analytical and experimental indications say: **YES!** But up to our knowledge so far **no rigorous proof**.

Macroscopic wavefunctions and Josephson effect

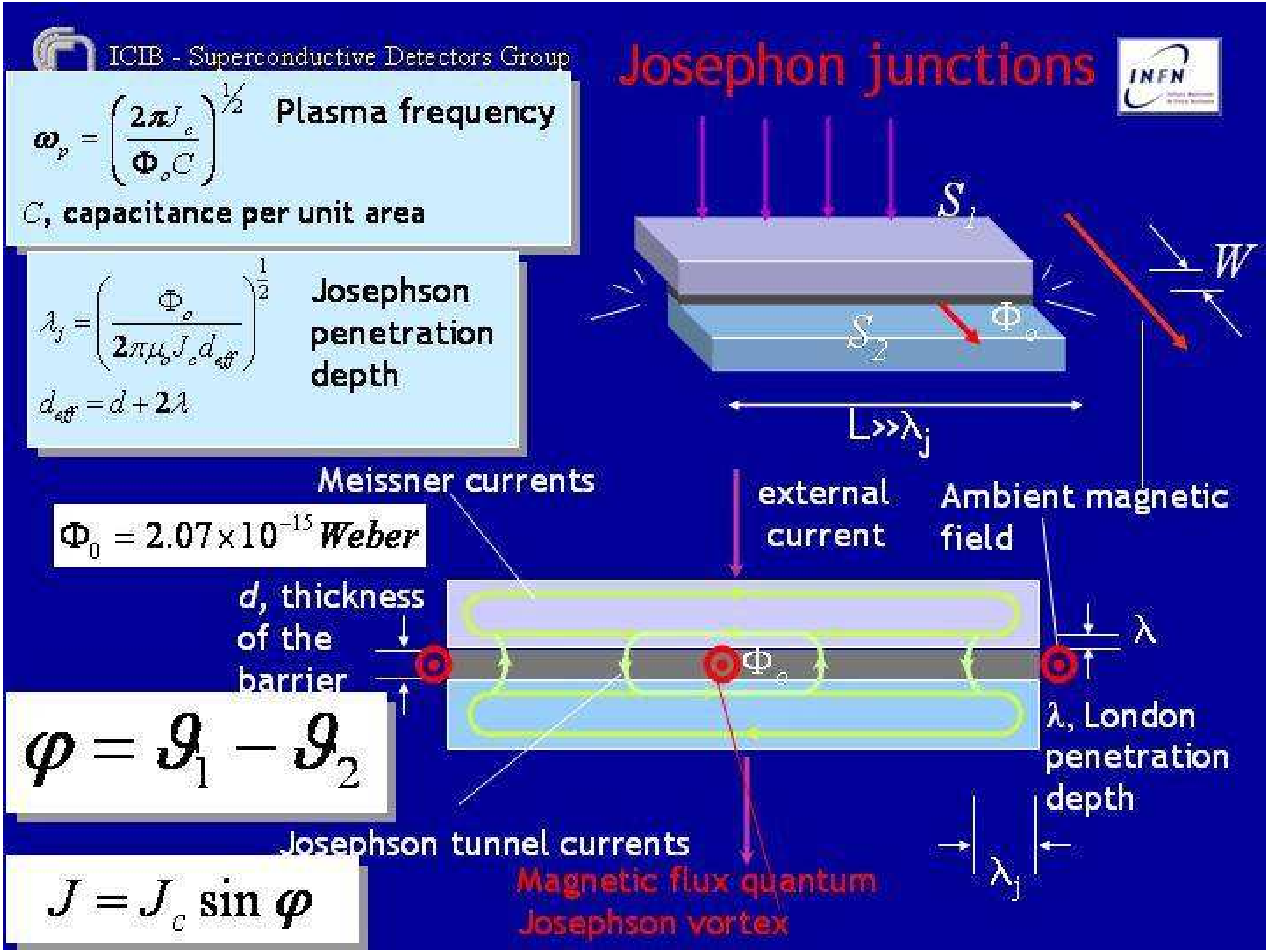
$$\Psi_L = \sqrt{n_L}\exp(i\vartheta_L) \qquad \Psi_R = \sqrt{n_R}\exp(i\vartheta_R)$$



$$\varphi = \vartheta_L - \vartheta_R$$

$$J = J_c \sin\varphi$$

$$\frac{\partial\varphi}{\partial t} = \frac{2eV}{\eta}$$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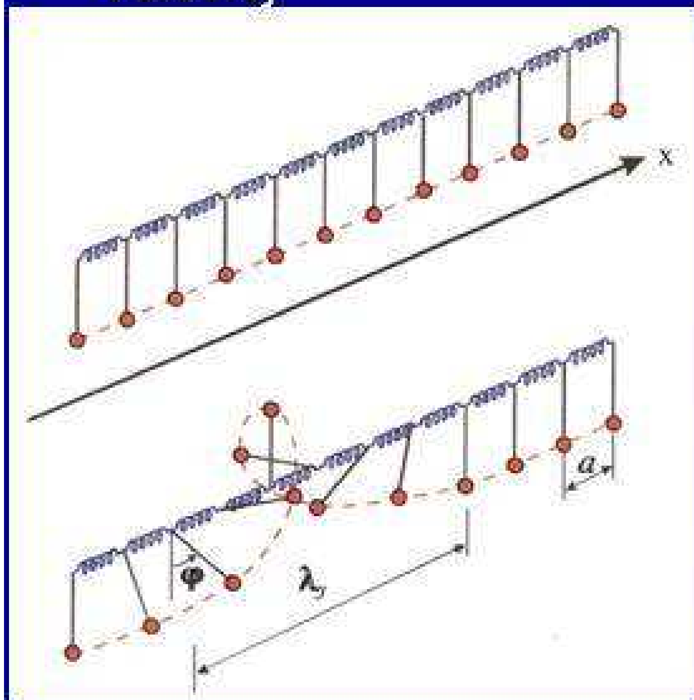
Josephon junctions
$\omega_p = \left(\frac{2\pi J_c}{\Phi_o C}\right)^{1/2}$ Plasma frequency
C, capacitance per unit area
$\lambda_j = \left(\frac{\Phi_o}{2\pi\mu_o J_c d_{eff}}\right)^{\frac{1}{2}}$ Josephson penetration depth
$d_{eff} = d + 2\lambda$
$\Phi_0 = 2.07 \times 10^{-15}$ Weber
$\varphi = \vartheta_1 - \vartheta_2$
$J = J_c \sin\varphi$
S_1
S_2
Φ_o
W
L>>λj
Meissner currents
external current
Ambient magnetic field
d, thickness of the barrier
λ
λ, London penetration depth
λj
Josephson tunnel currents
Magnetic flux quantum
Josephson vortex

2D+1 Perturbed sine-Gordon equation

$$\varphi_{xx} + \varphi_{yy} - \varphi_{tt} - \alpha\varphi_t + \beta\left(\varphi_{xxt} + \varphi_{yyt}\right) = \sin\varphi$$

+ B.C. determined by the value of the magnetic field at the edges of the junction

- Length in units of λ_j, time in units of ω_j, c' = $\lambda_j\omega_j$, the Swihart velocity



- A long junction has L>> λ_j, W<< λ_j (1D+1)

$$\varphi_{xx} - \varphi_{tt} - \sin\varphi = \alpha\varphi_t - \beta\varphi_{xxt} - \gamma$$

- A well known mechanical analog of the long Josephson junction is a chain of elastically coupled pendula.
- A 2π twist is identified with a fluxon or a kink. A spatially uniform bias current corresponds to a uniform torque applied to all the pendula

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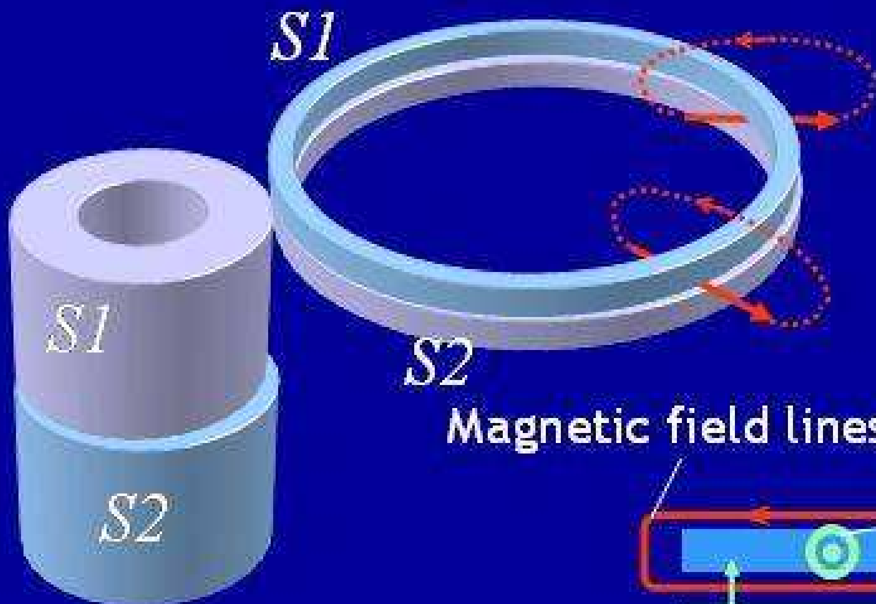
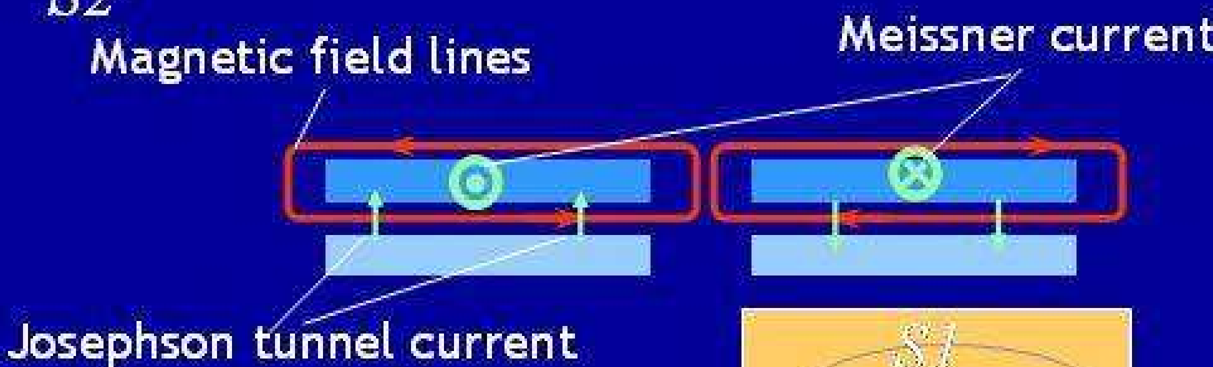
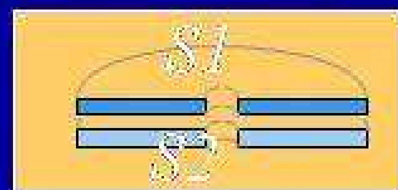
- In annular Josephson tunnel junctions Josephson effect is observed together with the flux quantization



- During the superconductive transition it is possible, on a statistical basis, to end up with one (or more) flux quantum trapped in one of the electrodes





$$\vartheta_1(x+2\pi r)-\vartheta_1(x)=2\pi$$

$$\vartheta_2=const.$$

$$\varphi=\vartheta_2-\vartheta_1$$

$$\Rightarrow \varphi(x+2\pi r)=\varphi(x)+2\pi$$

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi + \alpha\varphi_t + \gamma = 0.$$

Analytically: perturbative method inspired by ISM, based on *modulations of the unperturbed (multi)soliton solutions*. [Kaup & Newell '76, Scott & McLaughlin '77,...] Ansatz for approximate (anti)soliton:

$$\varphi(x,t) = \hat{g}_0\Big(x - t\,v(t) - x_0(t)\Big) + \gamma\varphi_1(x,t) + \dots$$

Slowly varying parameters like $x_0(t), v(t)$ and the "radiative" corrections $\gamma\varphi_1(x,t) + \dots$ have to be computed perturbatively in terms of the perturbation $\alpha\varphi_t + \gamma$. One finds in particular approximate solutions with constant velocity

$$v(t) \equiv v_\infty := \pm[1 + (4\alpha/\pi\gamma)^2]^{-\frac{1}{2}} \qquad (2)$$

characterized by power balance between the dissipative term $-\alpha\varphi_t$ and the external force $-\gamma$. They approximate *expected* exact (anti)soliton solutions

Our approach is less general and ambitious: study in detail the o.d.e. obtained from (1) by

$$\varphi(x,t) = \tilde{g}(x - vt) \tag{3}$$

(travelling-wave Ansatz). We are interested in *stable* (in the context of p.d.e.) solutions φ with bounded energy density as $x \to \pm\infty$.

We find that such kind of solutions of the pure sine-Gordon, in particular solitonic ones, are continuously deformed with γ, α, and even a new class appear. But v is no more a free parameter.

Other preliminaries

- Space and time translations (2 parameters) map solutions into solutions
- The total Hamiltonian and its density

$$H := \int_{-\infty}^{+\infty} h(x,t)dx, \qquad h := \frac{\varphi_t^2}{2} + \frac{\varphi_x^2}{2} - \cos\varphi + \gamma\varphi + K \tag{4}$$

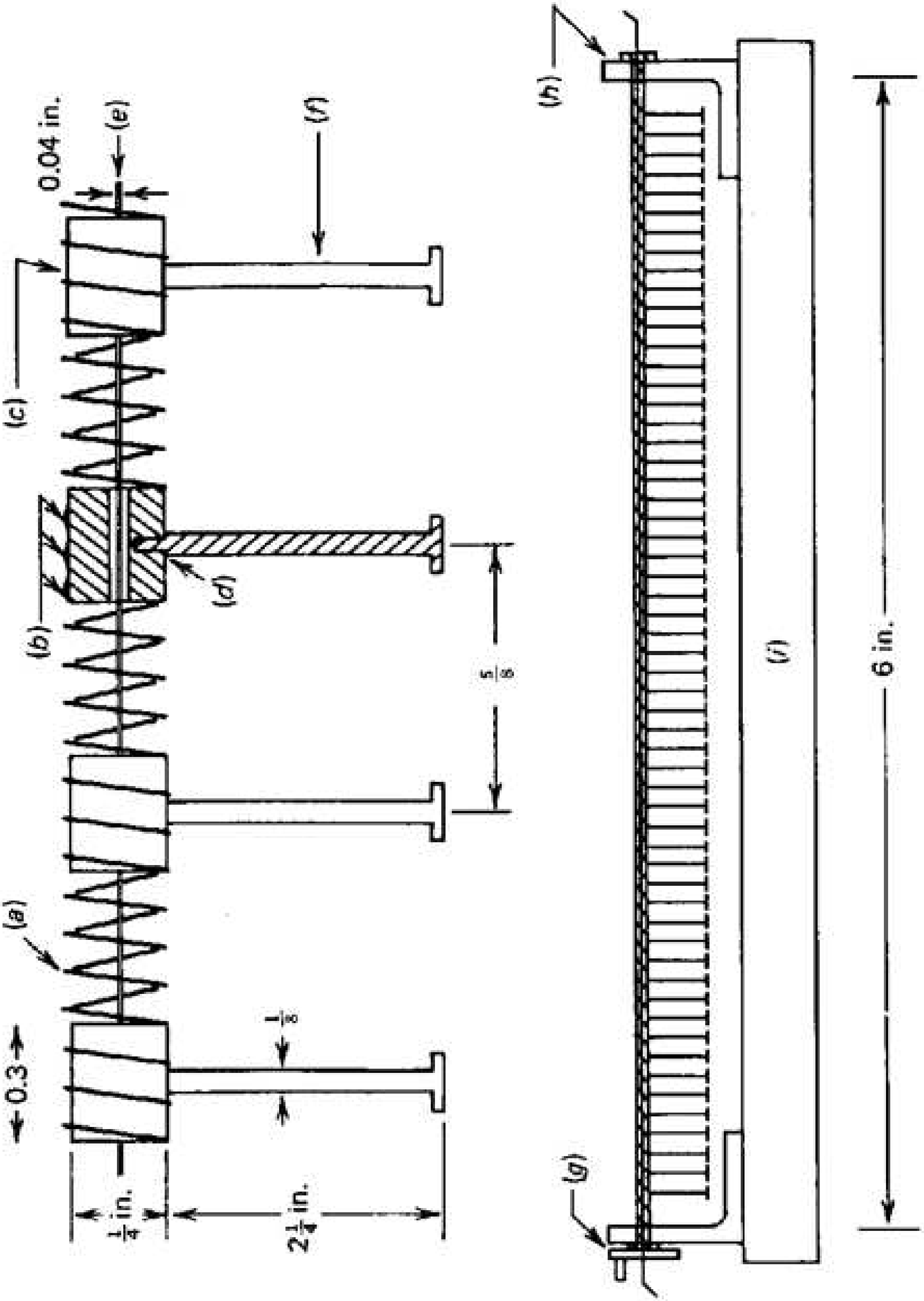
fulfill (with $j := \varphi_x\varphi_t \equiv$energy current density)

$$\dot{H} = -\int_{-\infty}^{\infty} \alpha\varphi_t^2 \leq 0, \qquad \partial_t h - \partial_x j = -\alpha\varphi_t^2. \tag{5}$$

- Our definition of a soliton solution: φ is a *stable travelling-wave solution with* h *differing from some minima only in some localized regions*. Then mod. 2π it must be

$$\lim_{x \to -\infty} \varphi(x,t) = -\sin^{-1}\gamma, \qquad \lim_{x \to +\infty} \varphi(x,t) = -\sin^{-1}\gamma + 2n\pi \tag{6}$$

  As we shall see, again only $n = 1, -1, 0$ possible (soliton, antisoliton and constant solution $\varphi(x,t) \equiv \sin^{-1}\gamma$).
- Without loss of generality $\gamma \geq 0$ (if necessary replace $\varphi \to -\varphi$). If $\gamma > 1$ no φ as above can exist; if $\gamma = 1$ such a φ exists but is unstable.
- Stability: many unstables solutions recognizable “at sight” from the pendula chain model.


0.04 in.
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
0.3
1/4 in.
2 1/4 in.
1/8
5/8
6 in.

Transforming the pde into a ode

If $v^2 = 1$: unstable φ. So $v^2 \neq 1$. We refine the Ansatz:

$$\xi := \frac{x-vt}{-\text{sign}(v)\sqrt{v^2-1}} \qquad \varphi(x,t) = -g(\xi) \qquad \text{if } v^2 > 1,$$

$$\xi := \frac{x-vt}{\text{sign}(v)\sqrt{1-v^2}} \qquad \varphi(x,t) = -g(\xi) + \pi \qquad \text{if } v^2 < 1,$$

$$\xi := x \qquad \varphi(x,t) = -g(\xi) + \pi \qquad \text{if } v = 0.$$

Replacing in (**??**) we find in all 3 cases the 2nd order o.d.e.

$$g'' + \mu g' + U_g(g) = 0, \tag{7}$$

$$U(g) = -(\cos g + \gamma g), \qquad \mu := \alpha / \sqrt{|v^{-2} - 1|}. \tag{8}$$

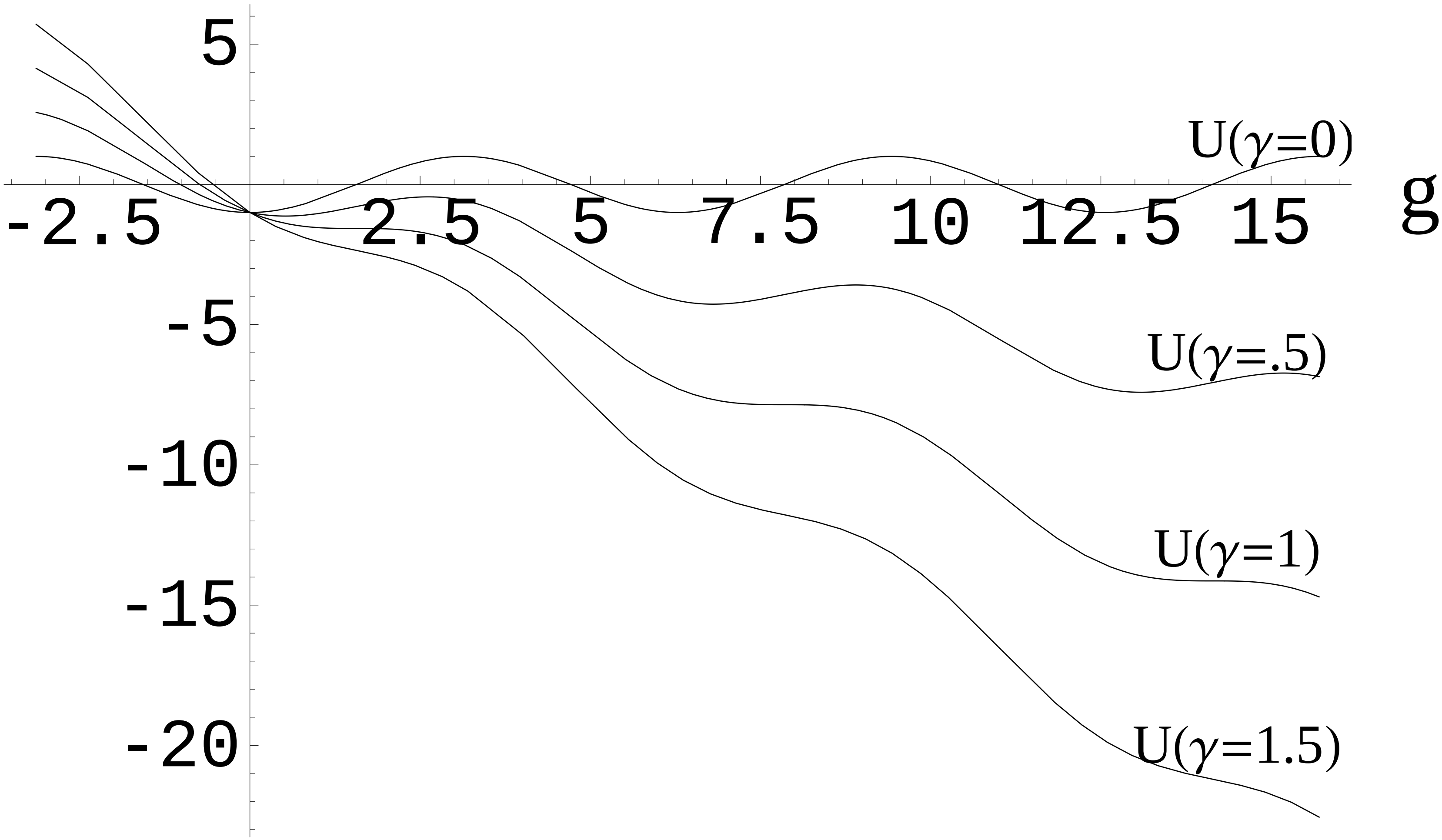
Note: α, v appear only through μ. Studied by [Tricomi, Amerio,Urabe,...]. E.o.m. w.r.t. ‘time’ ξ of a particle... with unit mass, position g, subject to a ‘washboard’

Eq. (**??**) is equivalent to the autonomous first order system

$$g' = u,$$
$$u' = -\mu u - \sin g + \gamma. \quad (9)$$

The rhs's are functions of g, u with bounded continuous derivatives. By the Peano-Picard Thm:

- All solutions are defined on all $-\infty < \xi < \infty$ (existence).
- The paths [≡trajectories in phase space (g, u)] do not intersect (uniqueness). So, each is uniquely identified by any of its points (g_0, u_0).
- The solutions are continuous functions of μ, γ, (g_0, u_0) (≠singular point), uniformly in every compact subset.



8-1

- $U(g)$ has local min. (resp. max.) only if $0 \leq \gamma < 1$, in

$$g_k^m := \sin^{-1}\gamma + 2k\pi \qquad \text{(resp. } g_k^M := \pi - \sin^{-1}\gamma + 2k\pi\text{)};$$

  the corresponding values of U coincide if $\gamma = 0$,
  linearly decrease with k if $\gamma > 0$.

- For $\gamma = 1$ $g_k^m = g_k^M = (2k+12)\pi$ are inflections points.
- For $\gamma > 1$ $U_g < 0$ everywhere.

Singular points in phase space exist only for $\gamma \leq 1$ and are

$$\begin{aligned} &A_k = (g_k^M, 0), \qquad B_k = (g_k^m, 0), \qquad &\gamma < 1 \\ &C_k = \big((2k+1/2)\pi, 0\big) \qquad &\gamma = 1 \end{aligned} \tag{10}$$

The pure sine-Gordon case

First recall the case $\gamma = 0 = \alpha$ (See e.g. [Barone et al. '71, Scott et al. '73]).

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi = 0, \qquad \text{sine-Gordon eq.,}$$

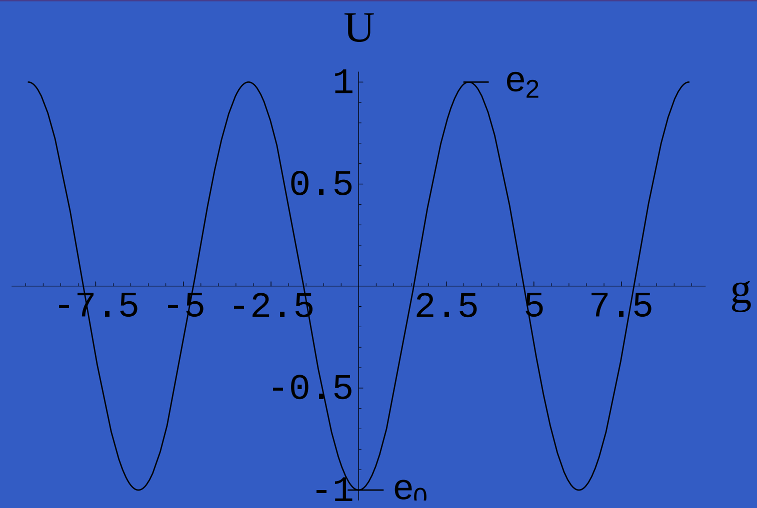
$$g'' + \sin g \equiv g'' + U_g = 0 \qquad \text{pendulum eq.}$$

Conservation of the "pendulum total energy" e:

$$\frac{d}{d\xi}\left[\frac{g'^2}{2} + U\right] = 0 \qquad \Rightarrow \qquad \mathsf{e} := \frac{g'^2}{2} + U = \text{const} \geq -1.$$

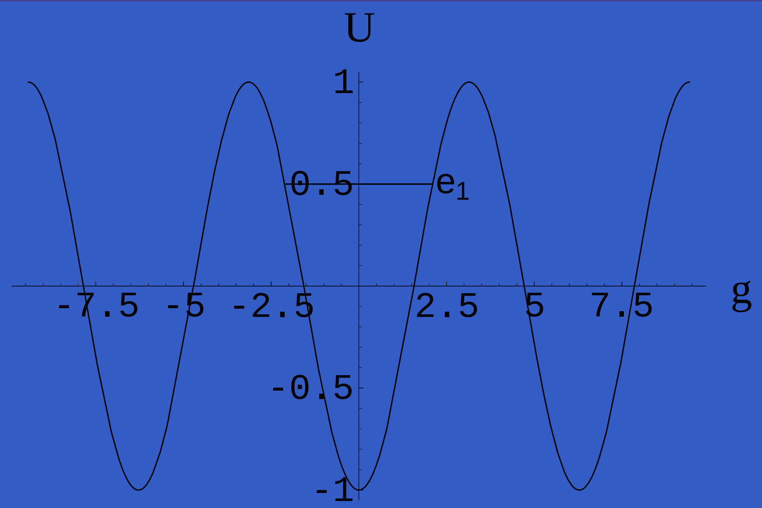
Thus we can express the kinetic energy as a function $z(g)$ of the position g and of the parameter e. Plot U for an im-

Constant solutions

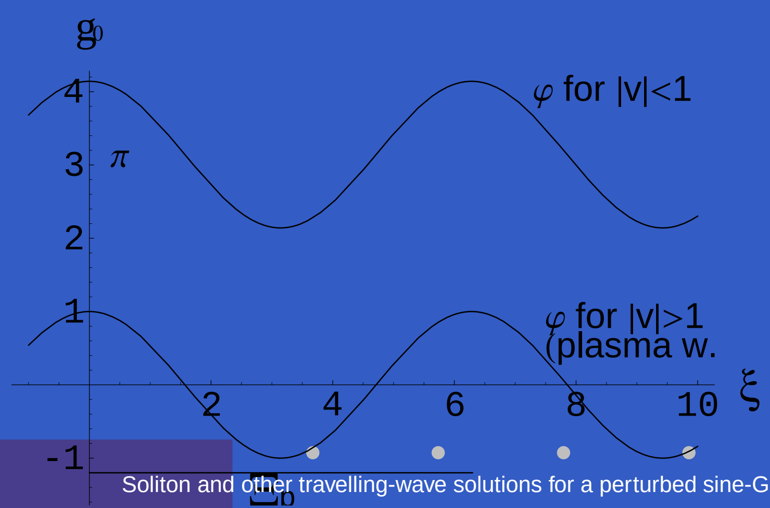


- if e $=$ e$_0 := -1$ then $g_0(\xi) \equiv 0$ (mod. 2π)
- if e $=$ e$_2 = 1$ then $g_0(\xi) \equiv \pi$ (mod. 2π).

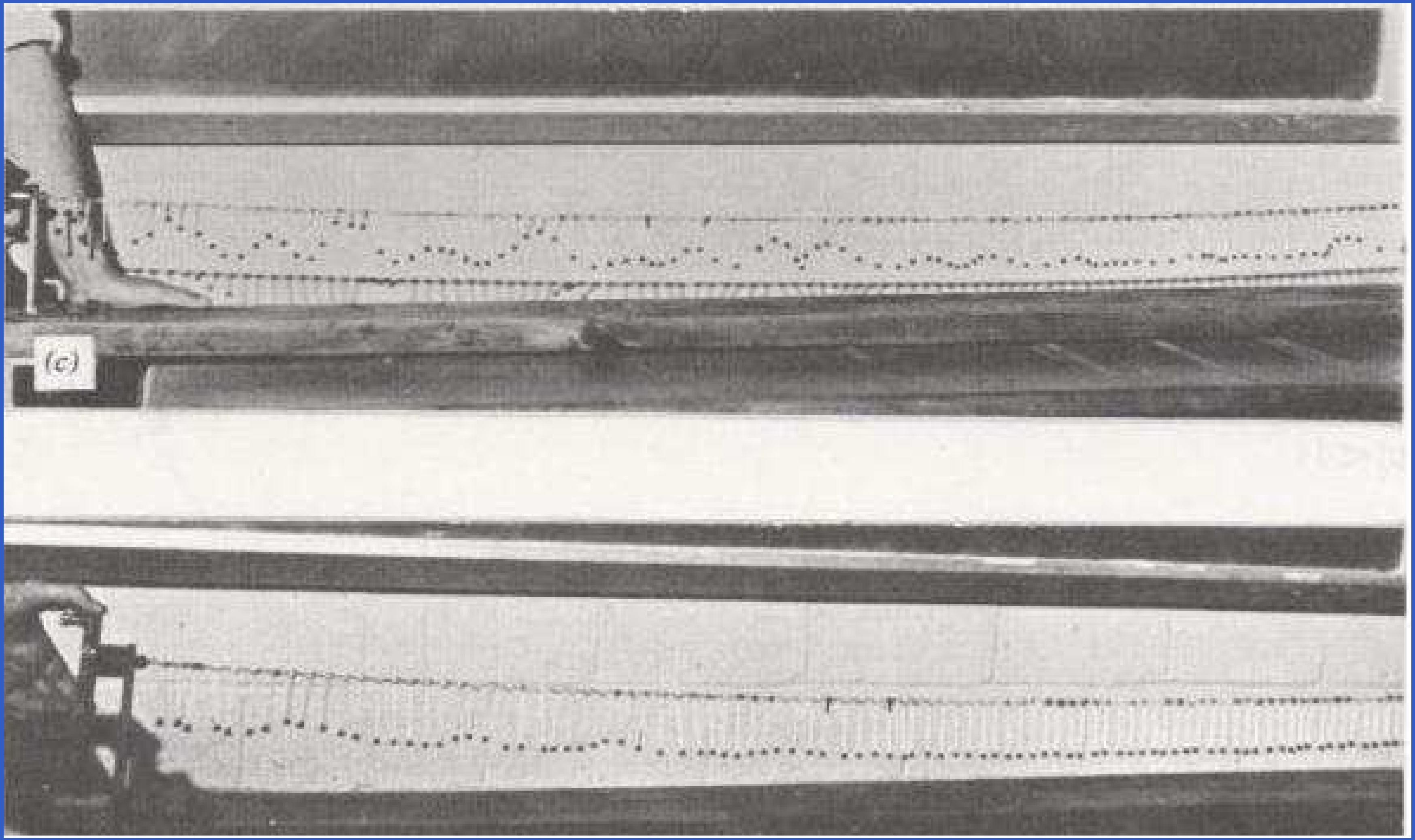
$\varphi^s(x,t), \equiv 0$	stable: all pendula hang down
$\varphi^u(x,t) \equiv \pi$	unstable: all pendula stand up.

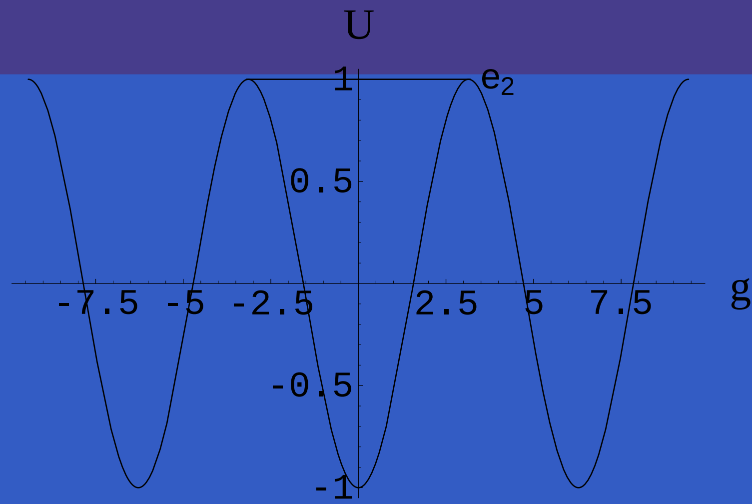


- $g_0(\xi)$ oscillates around $g = 0$ with some period $\Xi_p(\mathrm{e})$.

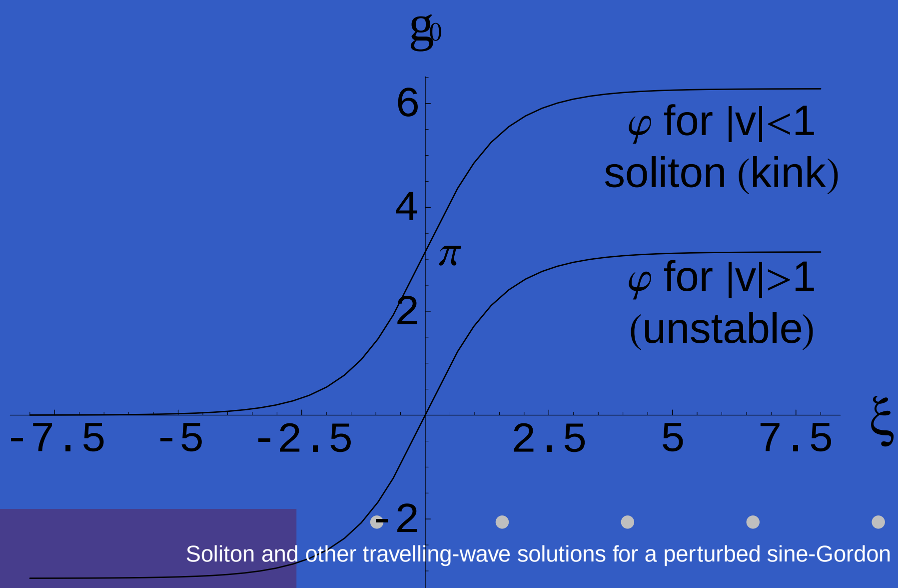


Large and small amplitude “Plasma waves”.

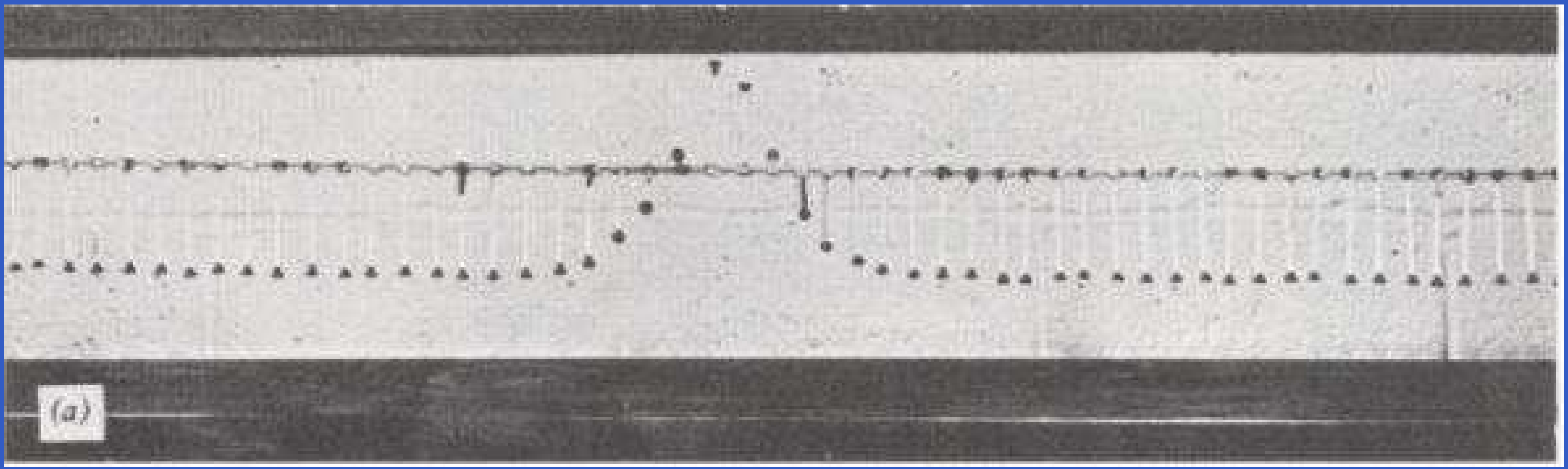


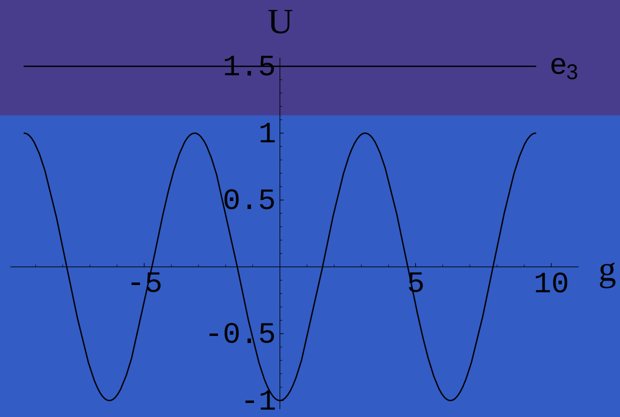


- In addition, $g_0(\pm\xi) = 4\tan^{-1}\left[\exp(\pm\xi)\right] - \pi$. Whence,



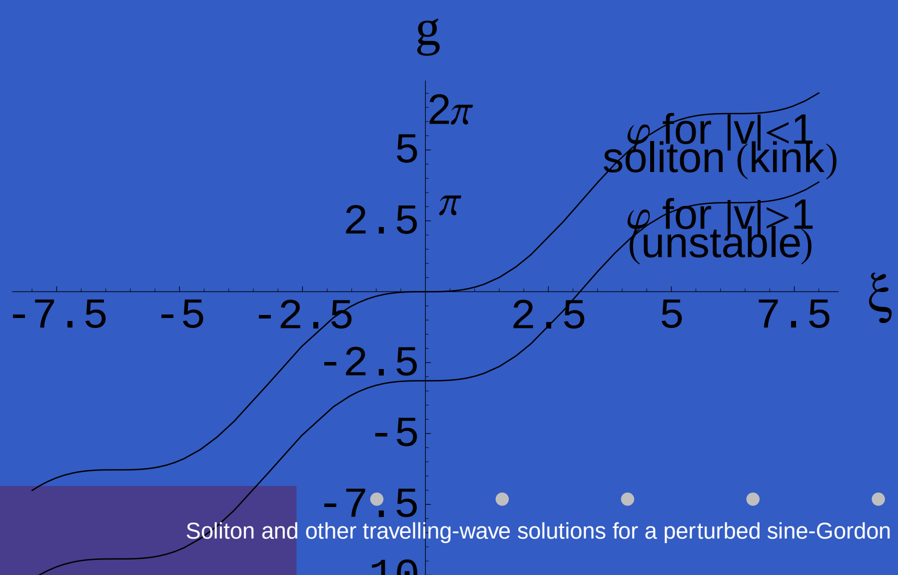
The soliton (kink) solution.

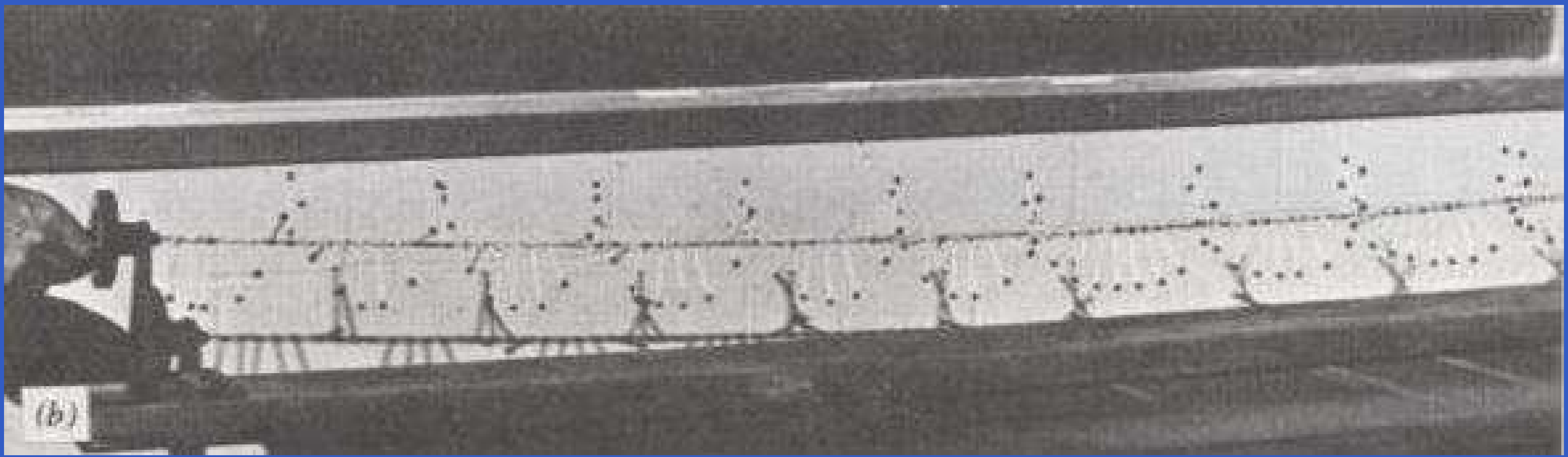




- $$\check{g}_0(\xi+\Xi_0)=\check{g}_0(\xi)+2\pi \qquad \check{g}_0{}'(\xi+\Xi_0)=\check{g}_0{}'(\xi). \qquad (11)$$

(The period $\Xi_0 = \Xi_0(\mathsf{e}) \to \infty$ as $\mathsf{e} \downarrow 1$). Correspondingly,





Choose any $m \in \mathbb{N}$ and set $L := m\Xi_0\sqrt{1-v^2}$. Eq. (**??**) implies

$$\check{\varphi}_0(x+L,t) = \check{\varphi}_0(x,t) + 2\pi m \qquad (12)$$

This makes sense also as a stable solution of the s-G equation **on a circle of length** L! m parametrizes topological sectors: pendula chain twisted around the circle m times!

Transforming the 2nd order problem into a 1st order one

$g(\xi)$ can be inverted in any interval X where $u = g'$ keeps its sign ϵ. There the 'velocity' u and the 'kinetic energy' $z = u^2/2$ of the 'particle' expressed as functions of its 'position' g. The second order problem (**??**)+ (**??**) splits into two 1st order ones:

- $$uu_g(g) + \mu u + \sin g - \gamma = 0, \qquad u(g_0) = u_0, \tag{13}$$

which has to be solved first (eq. invariant under $g \to g + 2\pi$!);

- $$g'(\xi) = u\big(g(\xi)\big), \qquad g(\xi_0) = g_0. \tag{14}$$

The second is integrated by quadrature: implicitly yields

Transforming the 1st order problem (??) into the integral equatio

$$z(g) = z_0 + U(g_0) - U(g) - \epsilon \int_{g_0}^{g} ds\, \mu \sqrt{2z(s)} \quad (16)$$

When $\mu = 0$ (no dissipation) the integral disappears and the rhs gives the solutions explicitly. This amounts to the statement of conservation of the total energy of the particle. It applies in particular to the pure sine-Gordon case ($\gamma = \alpha = 0$).

Monotonicity properties

- Fixed g_0, g, the velocity $u(g; g_0, u_0; \mu, \gamma)$ and the kinetic energy $z(g; g_0, u_0; \mu, \gamma)$ are stricly monotonic functions of u_0, μ, γ.

Let $G =]g_-, g_+[\equiv$maximal domain of $u(g)$
$g_k := g_0 + 2\pi k$, $K := \{k \in \mathbb{Z} \mid g_k \in \bar{G}\}$,

- The sequence $\{u(g_k)\}$ is either constant (with $K = \mathbb{Z}$) or strictly monotonic. As $k \to \infty$ it will: diverge if $u < 0$; either converge or stop if $u > 0$.

Proposition 1 *As functions of* z_0*:* $z = u^2/2$ *is stricly increasing;* g_+ *is increasing and* g_- *decreasing (strictly if* $g_\pm \neq \infty$*).*

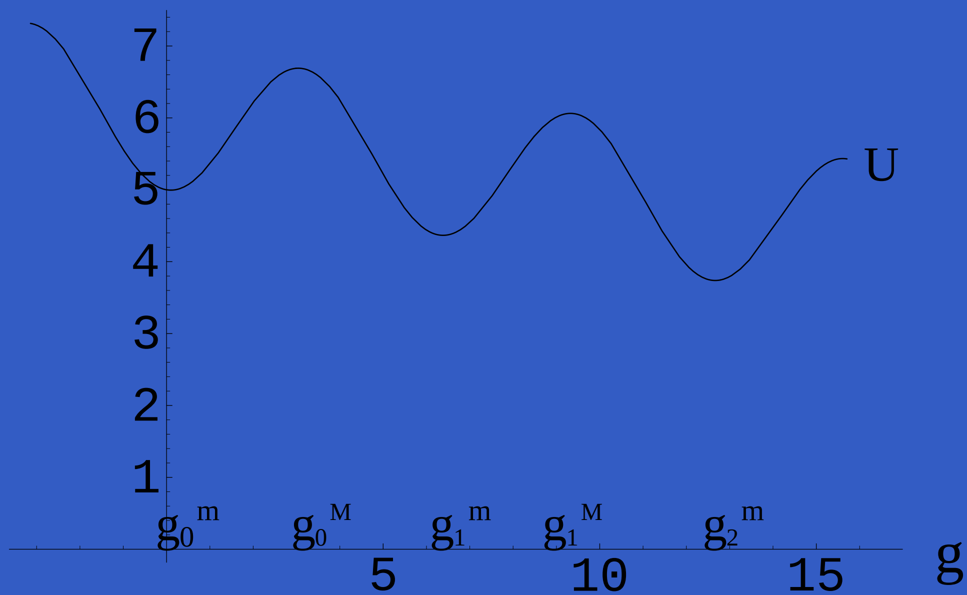
Proposition 2 *As a function of* μ $u(g;\mu)$ *is continuous and strictly decreasing (resp. increasing) for* $g \in]g_0, g_+[$ *(resp.* $g \in]g_-, g_0[$*). If* $\epsilon > 0$ *(resp.* < 0*), then* $g(\xi,\mu)$ *is a decreasing (resp. increasing) function of* μ*, and so is* $g_\pm(\mu)$ *(strictly if* $g_\pm \neq \pm\infty$*).*

Let $I_k := \int_{g_k}^{g_{k+1}} ds\sqrt{2z(s)}$ if $k, k+1 \in K$.

Proposition 3 $\forall g_0$ *the sequences* $\{z(g_k)\}, \{I_k\}$ *are either constant (with* $K = \mathbb{Z}$*) or strictly monotonic.*

Solutions zoology

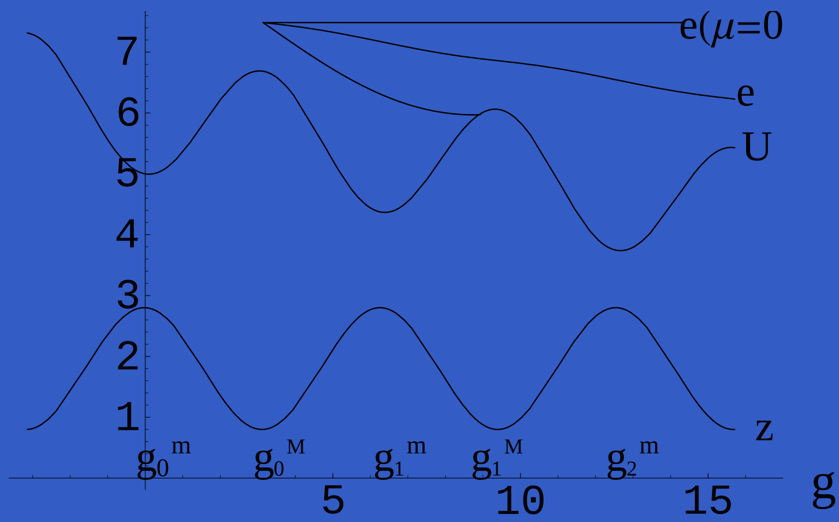
When $\gamma, \mu > 0$, many kind of solutions $g(\xi)$! Many are to be discarded.



The relevant solutions

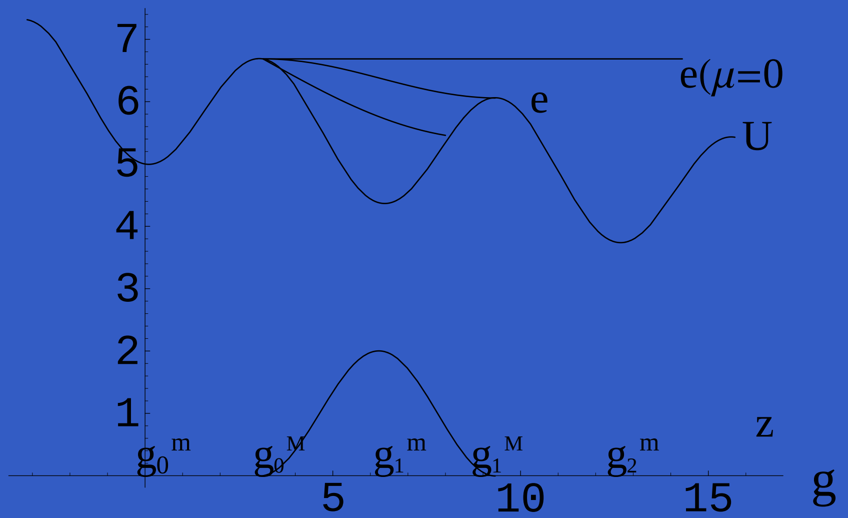
So we stick to solutions where $\epsilon = g'(\xi) > 0$ in all $\mathbb{R}$.

Take $g_0 = g_0^M$. Two parameters μ, $z_M \equiv z_0 \geq 0$ left.



$\exists!\ \check{\mu}(\gamma, z_M)$, $\check{z}(g)$ s. t.: $\check{z}(g_k^M) = z_M = \check{z}(g_{k-1}^M)$.

Take $z_M = 0$.



$\exists!$ $\hat{\mu}(\gamma, z_M)$, $\hat{z}(g)$ s. t.: $\check{z}(g_0^M) = 0 = \check{z}(g_1^M)$.

The range of $\check{\mu}$ as z_M spans $[0, \infty[$ is $]0, \hat{\mu}]$.

Theorem *Mod. 2π, travelling-wave solutions of* **(??)** *(where $\gamma > 0$ and $\alpha \geq 0$) having bounded energy density at infinity and not manifestly unstable are only of the following types (with $\xi := (x - vt)/\sqrt{|1 - v^2|}$):*

- *If $\gamma < 1$, the static solution $\varphi^s(x,t) \equiv -\sin^{-1}\gamma$.*
- *If $\gamma < 1$, the soliton/antisoliton solution $\hat{\varphi}^{\pm}(x,t) = \hat{g}(\pm\xi;\gamma)$*

$$\lim_{x \to -\infty} \hat{\varphi}^{\pm}(x,t) = -\sin^{-1}\gamma, \qquad \lim_{x \to \infty} \hat{\varphi}^{\pm}(x,t) = -\sin^{-1}\gamma \pm 2\pi$$

travelling rightwards/leftwards resp. with velocity

$$v = \pm\hat{v}, \qquad \hat{v} := \frac{\hat{\mu}(\gamma)}{\sqrt{\alpha^2 + \hat{\mu}^2(\gamma)}} < 1; \tag{18}$$

$\hat{\mu}(\gamma)$ can be determined with arbitrary accuracy [see Thm. **??***], and at lowest order in γ is given by $\hat{\mu}(\gamma) = \pi\gamma/4 + \ldots$*

- *For any* $\mu > 0$ *if* $\gamma > 1$*, for any* $\mu \in]0, \hat{\mu}(\gamma)[$ *if* $\gamma \leq 1$*, the "array of solitons/antisolitons" solution* $\check{\varphi}^{\pm}(x,t) = \check{g}(\pm\xi;\mu,\gamma)$*, where* $\check{g}$ *fulfills*

$$\check{g}(\xi+\Xi) = \check{g}(\xi) + 2\pi, \qquad \Xi(\mu,\gamma) = \int_{g}^{g+2\pi} \frac{ds}{\sqrt{2\check{z}(s)}} \in]0,\infty[,$$

travelling rightwards/leftwards resp. with velocity

$$v = \pm\check{v} \qquad \check{v} = \frac{\mu}{\sqrt{\alpha^2+\mu^2}} \in [0,\hat{v}[. \tag{19}$$

- *If $\gamma < 1$, for any $\mu \in]0, \hat{\mu}(\gamma)[$ the "half-array of solitons/antisolitons" solution $\bar{\varphi}^{\pm}(x,t) = \bar{g}(\pm\xi; \mu, \gamma)$, with $\bar{g}$ fulfilling*

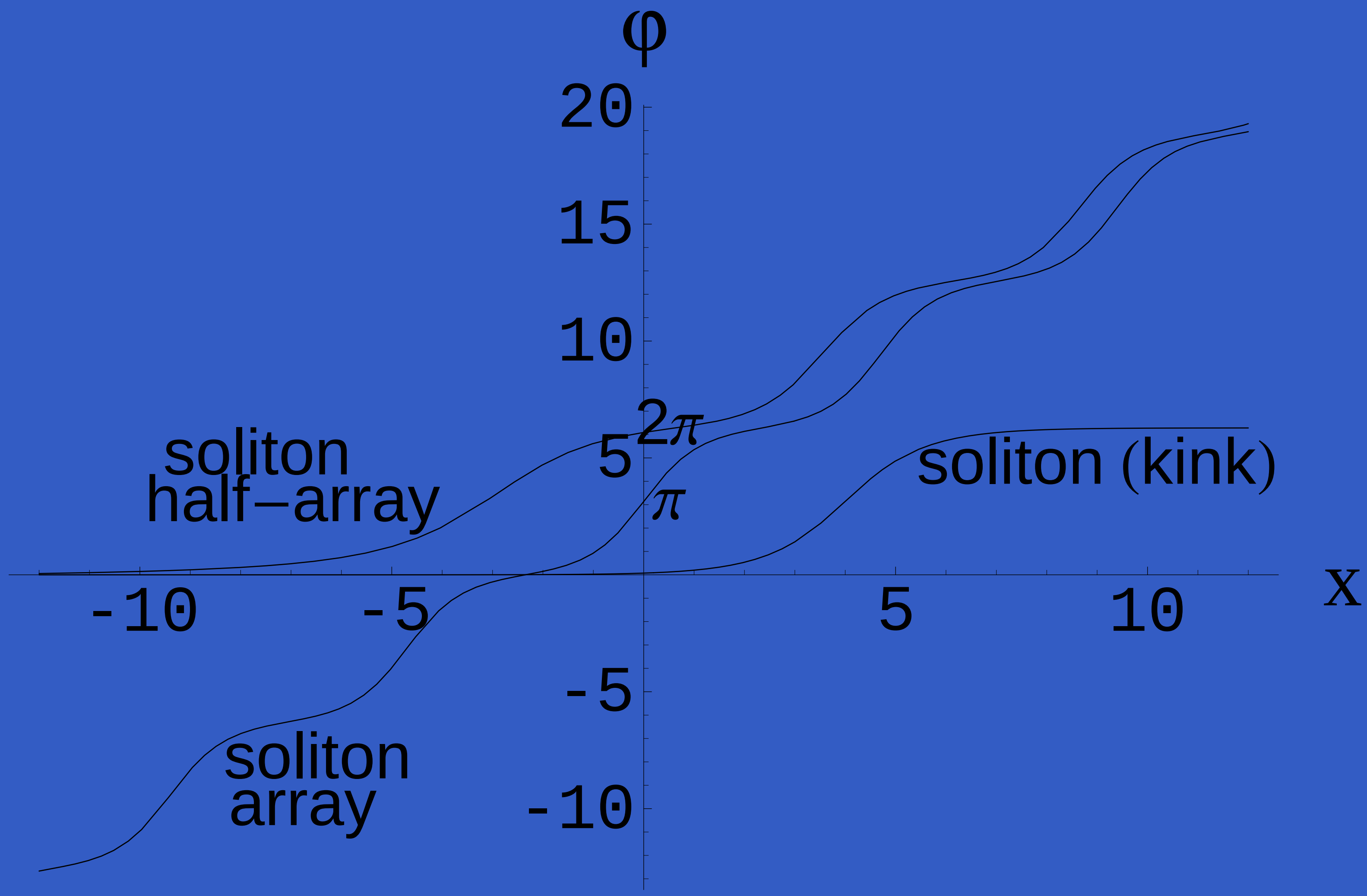
$$\lim_{\xi \to -\infty} \bar{g}(\xi) = -\sin^{-1}\gamma \qquad \lim_{\xi \to \infty} [\check{g}'(\xi) - \bar{g}'(\xi)] = 0^{-}, \tag{20}$$

$$\lim_{\xi \to \infty} [\bar{g}(\xi + \Xi) - \bar{g}(\xi)] = 2\pi^{-} \tag{21}$$

travelling rightwards/leftwards resp. with velocity

$$v = \pm\check{v} \qquad \check{v} = \frac{\mu}{\sqrt{\alpha^2 + \mu^2}} \in [0, \hat{v}[.$$

Also $|v|$ or Ξ can parametrize the solutions 3,4 instead of μ.


φ
20
15
10
5
-5
-10
-10
-5
5
10
x
2π
π
soliton
half−array
soliton (kink)
soliton
array

Convergence of the method successive approximations

Let $y := (g - g_{k-1}^{M})/2\pi$. The integral equation for the soliton solution $\hat{z}$ can be written as the fixed point equation

$$A\hat{z} = \hat{z},$$

where

$$Az(y) := U(0;\gamma) - U(y;\gamma) - \tilde{\mu}(z) \int_0^y ds \sqrt{2z(s)}$$

$$\tilde{\mu}(z) := \frac{2\pi\gamma}{\int_0^{2\pi} ds \sqrt{2z(s)}} \qquad (22)$$

As a nontrivial application of the fixed point Thm we find

Theorem 1 *Let* $\hat{z}_n := A^n \hat{z}_0$*,* $\mu_n := \tilde{\mu}(\hat{z}_n)$*. The sequences* $\{\hat{z}_n\}$*,* $\{\mu_n\}$ *converge respectively to the soliton solution* $\hat{z}$ *in the norm*

$$\|z\| = \sup_{y\in[0,2\pi]} \left| \frac{2z(y)}{\hat{z}_0(y)} \right| \tag{23}$$

and to the corresponding $\hat{\mu}(\gamma)$ *at least for* $\gamma \in [0, .18[$*. The errors are bound by*

$$\|\hat{z}_n - \hat{z}\| \leq \frac{\lambda^n}{1-\lambda} \|\hat{z}_1 - \hat{z}_0\| \qquad |\mu_n - \hat{\mu}| \leq \frac{\pi\gamma}{4a^3} \frac{\lambda^n}{1-\lambda} \|\hat{z}_1 - \hat{z}_0\| \tag{24}$$

where

$$\lambda(\gamma) = \frac{\pi\gamma}{4} \left[1 + 2\pi\gamma + O(\gamma^2)\right].$$

First improved approximation

$$\hat{z}_1(y) = \sqrt{1-\gamma^2} 2\sin^2 \pi y + \gamma \left[\pi(2y - 1 + \cos \pi y) - \sin 2\pi y\right] \quad (25)$$

$$\mu_1 = \frac{1}{4}\pi\gamma \quad (26)$$

$$\hat{\mathrm{e}}_1(y) = \gamma + \text{const}\,\pi(\cos \pi y - 1) \quad (27)$$

$$\hat{v}(\gamma, \alpha) = \frac{1}{\sqrt{1 + (4\alpha/\pi\gamma)^2}} = \frac{\pi\gamma}{4\alpha} + O(\gamma^2). \quad (28)$$

In a similar way one can determine iteratively solutions of type 3 ($\mu, \check{z}$); we shall deal with this elsewhere. In Fig. **??** we just plot the potential, kinetic and total energies corresponding to families 2,3 at first improved approximation.

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