Super geometry and modern physics

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Abstract

Super geometry is a generalization of differential (and algebraic) geometry where the local rings contain both commuting and anticommuting coordinates. It was discovered by the physicists in the early 1970's and it has played an important theoretical role in high energy physics since then. The anticommuting coordinates are nilpotent and so they cannot be measured. Consequently, the techniques of the Grothendieck theory of schemes have to be used in the study of super geometric objects. This talk sketches the main outlines of super geometry, its relation to spacetime at small distances, and its implications for particle classification and detection.

The evolution of the concept of space and spacetime

Euclidean space

$$\downarrow$$

Non Euclidean space

$$\downarrow$$

Riemannian space

$$\downarrow$$

Einstein spacetime

$$\downarrow$$

Super space

$$\downarrow$$

???

Riemannian geometry

• Space is built from its infinitesimal parts

• The infinitesimal distance is given by

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j$$

where the (x^i) are local coordinates and (g_{ij}) is a smooth positive definite matrix function of the local coordinates.

• The Riemann (curvature) tensor R

• $R = 0 \iff$ Euclidean space (flatness)

• Classification of homogeneous geometries (constant curvature)

• The metric of space is determined by matter.

Einstein spacetime

• Spacetime, and not space and time separately, is the object defined invariantly for all observers.

• If we exclude gravitation, spacetime is an affine manifold whose homogeneous part admits a pseudo Riemannian flat metric of signature $(1,3)$. The automorphisms of this geometry form the so called Poincaré group, given in global affine coordinates by

$$x \longmapsto x', \qquad x' = Lx + u \qquad (u \in \mathbf{R}^4, L \in \mathrm{SO}(1,3)).$$

• Gravitation forces spacetime to be a pseudo Riemannian manifold of signature $(1,3)$ whose curvature is the manifestation of gravitation. The metric of spacetime becomes a dynamic object interacting with spacetime (fulfilling Riemann's vision).

• In matter free regions the Ricci tensor vanishes (Ricci-flat). Thus non flat but Ricci-flat metrics are models of pure gravitation without matter.

Riemann's vision of space at small distances

Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena.

Göttingen inaugural address, 1854.

Current views on structure of spacetime

• At the Planck scale, no measurements are possible and so conventional models can no longer be relied upon to furnish a true description of phenomena.

• Even at energies lower than the Planck scale, a better understanding of phenomena is obtained if we assume that the geometry of spacetime is described locally by a set of coordinates consisting of the usual ones supplemented by a set of anticommuting coordinates that reflect the fermionic structure of matter.

• The transitions

$$\text{bosons} \longleftrightarrow \text{fermions}$$

require the use of a geometry that allows transformations between the commuting (bosonic) and anticommuting (fermionic) parts of spacetime.

Such transformations are called *supersymmetries* and the geometry that is the framework of supersymmetries is *super geometry.*

Bosons and Fermions

There is a fundamental dichotomy among elementary particles based on their *spin*. Unlike classical particles, quantum ones have *internal states* which move with the particle and which form an irreducible module for SU(2) of dimension $2j+1$ where $j \in (1/2)\mathbf{Z}$ is the spin. Particles with j integral (resp. half integral) are **Bosons** (resp. **Fermions**). The Hilbert space of a system of N identical particles of the same type with a Hilbert space $\mathcal{H}$ is either $\mathrm{Symm}^N(\mathcal{H})$ or $\Lambda^N(\mathcal{H})$. This is the *spin-statistics theorem.* Its origins go back to the *Pauli exclusion principle.*

Thus in QFT the one particle states form a Hilbert space $\mathcal{H}$ with a distinguished orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

where $\mathcal{H}_0$ is the space of bosonic states, $\mathcal{H}_1$ is the space of Fermionic states. Supersymmetry arose out of a desire to treat the bosonic and fermionic aspects in a unified fashion, not only at the quantized level, but even in the classical unquantized level. This meant treating manifolds with commuting and anticommuting coordinates. This approach was pioneered by **Salam** and **Strathdee**.

Super linear algebra

- Category of super (= $\mathbf{Z}_2$-graded) vector spaces
- Supercommuting rings and modules over them
- Supercommutativity:

$$ab = (-1)^{p(a)p(b)} ba \qquad (p = \text{parity function}).$$

Examples of supercommuting algebras

- The exterior algebras

$$k[\theta^1, \theta^2, \ldots, \theta^q] \qquad \theta^i \theta^j + \theta^j \theta^i = 0.$$

- The algebra of exterior forms on a manifold.
- Matrices over a supercommuting algebra.

The Berezinian

Let R be a super commutative algebra over a field k of characteristic 0 and $R^{p|q}$ be the free module of dimension $p|q$ over R. Let $\mathrm{GL}(p|q)(R)$ be the group of invertible even morphisms of $R^{p|q}$. Then the *Berezinian* is a morphism of $\mathrm{GL}(p|q)(R)$ into $R_0^{\times}$ (the group of units of the even part R_0 of R) given by

$$x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \mathrm{Ber}(x) = \det(A - BD^{-1}C)\det(D)^{-1}.$$

$$\mathrm{Ber}(xy) = \mathrm{Ber}(x)\mathrm{Ber}(y).$$

The Berezinian or *superdeterminant* extends to super algebra the notion of the determinant. It was discovered by **F. A. Berezin**, one of the pioneers of super algebra and super analysis.

Since the entries of B and C are nilpotent, x is invertible if and only if A and D, whose entries are in the commutative ring R_0, are invertible.

The concept of a super manifold

A *super manifold* M of dimension $p|q$ is a smooth manifold $|M|$ of dimension p together with a sheaf $\mathcal{O}_M$ of super commuting algebras on $|M|$ that looks locally like

$$C^\infty(\mathbf{R}^p)[\theta^1, \theta^2, \ldots, \theta^q]$$

Comments

- Similarity with a Grothendieck scheme: the odd coordinates have vanishing square, hence nilpotent.

- Analogous definitions for complex analytic super manifolds, and super schemes.

- By taking the quotients of the local rings modulo the nilpotents we get the *reduced smooth manifold* $M^\sim$. This leads to the intuitive picture of M as $M^\sim$ surrounded by a grassmannian cloud.

Morphisms and local super geometry

• Unlike algebraic geometry, the local coordinates do not generate the full local rings *algebraically*. Nevertheless it is true that morphisms between super manifolds are determined by what happens to the local coordinates.

• The ring

$$C^\infty(t^1, \ldots, t^p)[\theta^1, \ldots, \theta^q]$$

not only has the *even* derivations $\partial/\partial t^i$ but also the *odd* derivations $\partial/\partial\theta^\alpha$. This allows the local differential geometry of a super manifold to be developed in complete analogy (no surprises) with the classical case. In particular the differential criterion for local diffeomorphisms and the local structure of immersions and submersions remain the same as in the classical case. Diffeomorphisms of super manifolds are **super symmetries**.

Example: The diffeomorphism

$$\mathbf{R}^{1|2} \simeq \mathbf{R}^{1|2} : t^1 \longmapsto t^1 + \theta^1\theta^2, \quad \theta^\alpha \longmapsto \theta^\alpha$$

is a typical supersymmetry.

Integration on super manifolds

On

$$\Lambda = \mathbf{R}[\theta^1, \dots, \theta^q]$$

the integral is a linear map

$$a \longmapsto \int a d^q\theta$$

such that

$$\int \theta^I d^q\theta = \begin{cases} 0 & \text{if } |I| < q \\ 1 & \text{if } I =: Q = \{1, 2, \dots, q\}. \end{cases}$$

Integration is also differentiation:

$$\int = \left(\frac{\partial}{\partial\theta^q}\right)\left(\frac{\partial}{\partial\theta^{q-1}}\right)\cdots\left(\frac{\partial}{\partial\theta^1}\right).$$

In local coordinates

$$\int s d^p x d^q\theta = \int s_Q d^p x \qquad (s = \sum_I s_I \theta^I).$$

The change of variables formula

For a morphism given locally as

$$\psi : (x, \theta) \longmapsto (y, \varphi)$$

we define the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial y}{\partial x} & -\frac{\partial y}{\partial \theta} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial \theta} \end{pmatrix}.$$

Then

$$\int s = \int \psi^*(s) \mathrm{Ber}(J\psi)$$

for compactly supported sections of the local ring. For arbitrary manifolds we use partitions of unity as in the classical case.

- This beautiful formula goes back to Berezin. The justification for the peculiar definition of integration in the anticommuting variables is the change of variables formula.

Super Lie groups and Lie algebras

• Super Lie groups are group objects in the category of super manifolds. For a super Lie group G,

$$T \longmapsto G(T) = \mathrm{Morph}(T, G), \ (T \text{ a super manifold }).$$

is a contravariant functor from the category of super manifolds into the *category of groups.* Thus super Lie groups are to be thought of as group valued functors rather than actual groups of points.

• As in the theory of group schemes one can define the super Lie algebra $\mathrm{Lie}(G)$ of a super Lie group G.

• A *super Harish-Chandra pair* is a pair $(G_0, \mathfrak{g})$ where G_0 is a Lie group in the usual sense, $\mathfrak{g}$ is a super Lie algebra with $\mathrm{Lie}(G_0) = \mathfrak{g}_0$, and G_0 acts on $\mathfrak{g}$ with differential equal to the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}$. If G is a super Lie group and G_0 is its reduced Lie group, then $(G_0, \mathrm{Lie}(G))$ is a super Harish-Chandra pair. The functor

$$G \longmapsto (G_0, \mathrm{Lie}(G))$$

is an equivalence of categories.

Super space times and super Poincaré groups

• Let V_0 be a flat Minkowski space of signature $(1, n)$, and V_1 a module for $\mathrm{Spin}(1, n)$, composed (over $\mathbf{C}$) of the spin representations. Then $\mathfrak{t} = V_0 \oplus V_1$ is a super Lie algebra if $[V_0, V_0] = [V_0, \mathfrak{t}] = 0$ and the bracket on V_1 is defined by a nonzero symmetric $\mathrm{Spin}(1, n)$-map (which always exists)

$$[\cdot, \cdot] : V_1 \otimes V_1 \longrightarrow V_0.$$

The super Lie group T of $\mathfrak{t}$ is a *super Minkowski space time*, and the semi direct product

$$G = T \times' \mathrm{Spin}(1, n)$$

is a *super Poincaré group.*

• All of the machinery are now in place for introducing the Lagrangians and doing susy field theory on flat super Minkowski spacetime: **Wess-Zumino** (susy electrodynamics) and **Ferrara-Zumino** (susy Yang-Mills).

• The susy extension of Einstein spacetime is more complicated and was first done by **Ferrara, Freedman**, and **van Nieuwenhuizen**, and also by **Deser** and **Zumino**, in 1976. It is called *supergravity.*

Classification of super particles

In quantum theory the unitary irreducible representations (UIR) of the Poincare group classify free elementary particles. This is done by the **Frobenius-Wigner-Mackey** method of *little groups.*

A UR of a super Harish-Chandra pair $(G_0, \mathfrak{g})$, is a UR of G_0 on a super Hilbert space on which there is a compatible action of $\mathfrak{g}$. (WARNING: $\mathfrak{g}$ will act by unbounded operators and so care is needed in the above definitions).

- The UIR's of a super Poincaré group classify elementary super particles. Each super particle, when viewed as a UR of the reduced Lie group, which is a Poincaré group, is a collection of ordinary particles, called a *multiplet.* The members of a multiplet are called *susy partners. Unlike the classical case, the positivity of energy is a consequence of supersymmetry.*

- It is the hope of many that the new super collider being readied at CERN will create the super partners of the usual elementary particles. Of course super symmetricians have drawn up contingency plans in case the super partners are not found.

References

1. Varadarajan, V. S., *Supersymmetry for Mathematicians: An Introduction*, AMS-Courant Institute Lecture Notes, Providence, R.I., 2004.

2. Deligne, P., and Morgan, J., Notes on supersymmetry (following Joseph. Bernstein), *Quantum fields and strings: a course for mathematicians*, vol. I, 41-97, American Mathematical Society, Providence, R.I.,1999.

3. Ferrara, S., ed., *Supersymmetry*, vols. 1,2. World Scientific, Singapore, 1987.

4. C. Carmeli, G. Cassinelli, A. Toigo, and V. S. Varadarajan, *Unitary representations of super Lie groups and applications to the classification and multiplet structure of super particles*, preprint, hep-th 2005.

5. Ferrara, S., *Present status of Supersymmetry*, preprint, 2005.