Unitary representations of super semidirect products and applications to super particle classification

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Abstract

We show that the unitary irreducible representations of super semidirect products can be classified by a generalization of the classical little group method to the super context. We apply this theory to the classification of super particles and the description of their multiplet structure.

References

1. Carmeli, C., Cassinelli, G., Toigo, A., and Varadarajan, V. S., *Unitary representations of super Lie groups and applications to the classification and multiplet structure of super particles*, preprint, hep-th 0501061, 2005.

UIR's of a classical semidirect product (SDP)

Projective UIR's of the underlying symmetry group G classify elementary particles.

- $G =$ Galilean group

  The UIR's classify Schrödinger particles of mass $m > 0$ and spin j and give rise to *mass superselection sectors*.

- $G =$ Poincaré group

  All projective UR's are ordinary and the UIR's classify Dirac particles of mass $m > 0$ and spin j, *and* Weyl particles of mass $m = 0$ and helicity j.

$G = T_0 \times' L_0$, T_0 a real (f.d) vector space amd $L_0 \subset \mathrm{SL}(T_0)$ a closed subgroup. If P is the spectral measure of T_0 in a UR of G_0, its support is the *spectrum* of the UR. $O(\lambda)$ is the orbit of λ. We assume that the orbit space T_0^*/L_0 is smooth in the Borel sense.

For $\lambda \in T_0^*$, L_0^λ = stabilizer (little group) of λ in L_0^λ; $G_0^\lambda = T_0 L_0^\lambda$. A UR of G_0^λ is λ-*admissible* if T_0 acts as the character $e^{i\lambda}$.

$$\text{UIR's of } G_0 \text{ with spectrum} = O(\lambda)$$
$$\Longleftrightarrow \text{ admissible UIR's of } G_0^\lambda.$$

Super semidirect products (SSDP)

and super Poincaré groups

$(G_0, \mathfrak{g})$ is a *super semidirect product* if:

- $G_0 = T_0 \times' L_0$, T_0 a real (f.d) vector space, $L_0 \subset \mathrm{SL}(T_0)$ a closed subgroup
- T_0 acts trivially on $\mathfrak{g}_1$ and $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{t}_0 := \mathrm{Lie}(T_0)$

$(G_0, \mathfrak{g})$ is a *super Poincaré group* if:

- T_0 is a Minkowski space of signature $(1, D-1)$ and $L_0 = \mathrm{Spin}(1, D-1)$ (2-fold cover of $\mathrm{SO}(1, D-1)^0$)
- L_0 acts on $\mathfrak{g}_1$ spinorially, i.e., its complexification splits as a direct sum of spin modules.

Theorem *Given any spinorial module V there is a super Poincaré group $(G_0, \mathfrak{g})$ with $\mathfrak{g}_0 = \mathrm{Lie}(G_0), \mathfrak{g}_1 = V$. The bracket on $\mathfrak{g}_1$ is projectively unique if V is irreducible.*

UIR's of a SSDP

For $\lambda \in T_0^*$, $S^\lambda = (G_0^\lambda, \mathfrak{g}^\lambda)$ is the *little* (*super*)*group* at λ: $G_0^\lambda = T_0 L_0^\lambda$ and $\mathfrak{g}^\lambda = \mathfrak{t}_0 \oplus \mathfrak{l}_0^\lambda \oplus \mathfrak{g}_1$. It is a special sub super Lie group. For a UR (π, ρ^π) of $G = (G_0, \mathfrak{g})$, P is the spectral measure of $\pi\big|_{T_0}$. *Since T_0 acts trivially on $\mathfrak{g}_1$, the $\pi(t)$ commute with the $\rho^\pi(X)$.* If the UR is irreducible (UIR), then P is concentrated on an orbit.

If $\lambda \in T_0^*$, a UR of G is λ-*admissible* if $\pi(t) = e^{i\lambda(t)} I (t \in T_0)$. λ itself is *admissible* if there is a λ-admissible UIR.

Theorem. *For any $\lambda \in T_0^*$, the super imprimitivity theorem gives an equivalence of categories from the category of λ-admissible* UR*'s of S^λ with the* UR*'s of $(G_0, \mathfrak{g})$ whose spectra are contained in the orbit of λ. In particular, a* UIR *has spectrum in the orbit of λ if and only if λ is admissible, and then we have a bijection between the sets of equivalence classes of* UIR*'s of G and S^λ.*

Remark. This theorem is significantly different from the classical one, because there is a *selection rule* for the orbits: admissibility.

Admissibility as the positive energy condition

Let λ be admissible and (σ, ρ^σ) be a λ-admissible UIR for S^λ. Then

$$-id\sigma(Z) = \lambda(Z)I \qquad (Z \in \mathfrak{t}_0).$$

- $Q_\lambda(X) = (1/2)\lambda([X, X])$ is a L_0^λ-invariant quadratic form on $\mathfrak{g}_1$
- $\rho^\sigma(X)^2 = Q_\lambda(X)I$ on $C^\infty(\sigma)$

It follows, as $\rho^\sigma(X)$ is essentially self adjoint on $C^\infty(\sigma)$, that

- Q_λ is *nonnegative* and the ρ^σ are *bounded.*

Theorem. *Let $\lambda \in T_0^*$. Then the following are equivalent.*

(i) λ is admissible

(ii) $Q_\lambda(X) \geq 0$ for all $X \in \mathfrak{g}_1$.

We shall see that if G is a super Poincaré group, condition (ii) is essentially the condition that energy is positive. Hence we refer to (ii) as the *positive energy condition.* We shall sketch an outline of the proof assuming L_0^λ is connected. This is satisfied for super Poincaré groups.

Clifford algebras associated to positive energy orbits

Let $\mathcal{C}_\lambda$ be the algebra generated by $\mathfrak{g}_1$ with the relations

$$X^2 = Q_\lambda(X)1(X \in \mathfrak{g}_1).$$

Even though Q_λ may have a nonzero radical we call $\mathcal{C}_\lambda$ the *Clifford algebra* of $(\mathfrak{g}_1, Q_\lambda)$. If

$$\mathfrak{g}_{1\lambda} := \mathfrak{g}_1/\text{rad } Q_\lambda$$

then Q_λ is strictly positive on $\mathfrak{g}_{1\lambda}$ and there is a natural map

$$\mathcal{C}_\lambda \longrightarrow \mathcal{C}_\lambda^\sim = \text{ Clifford algebra of } \mathfrak{g}_{1\lambda}$$

with kernel as the ideal generated by the radical of Q_λ.

We wish to build a UIR (σ, ρ) of the little group S^λ with

- ρ a representation of $\mathcal{C}_\lambda$ by *bounded operators*, $\rho(X)$ *self adjoint and odd* for all $X \in \mathfrak{g}_1$; ρ is called a *self adjoint representation.*
- σ is an even UR of L_0^λ such that

$$\sigma(t)\rho(X)\sigma(t)^{-1} = \rho(tX) \qquad (t \in L_0^\lambda, X \in \mathfrak{g}_1)$$

Simply connected little super groups

We shall assume that L_0^λ is simply connected. This is satisfied if G is a super Poincaré group and $D \geq 4$. Since Q_λ is L_0^λ-invariant we have a map

$$L_0^\lambda \longrightarrow \mathrm{SO}(\mathfrak{g}_{1\lambda})$$

which lifts to a map

$$L_0^\lambda \longrightarrow \mathrm{Spin}(\mathfrak{g}_{1\lambda}).$$

There is an *irreducible* self adjoint representation τ_λ of $\mathcal{C}_\lambda$, finite dimensional, unique if $\dim(\mathfrak{g}_{1\lambda})$ is odd, unique up to parity reversal otherwise. The spin representation of $\mathrm{Spin}(\mathfrak{g}_{1\lambda})$ lifts to an even UR κ_λ of L_0^λ, with

$$\kappa_\lambda(t)\tau_\lambda(X)\kappa_\lambda(t)^{-1} = \tau_\lambda(tX) \qquad (t \in L_0^\lambda, X \in \mathfrak{g}_1).$$

The assignment

$$r \longmapsto \theta_{r\lambda} = (\sigma, \rho), \qquad \sigma = e^{i\lambda} r \otimes \kappa_\lambda, \qquad \rho = 1 \otimes \tau_\lambda$$

is an equivalence of categories from the category of purely even UR's r of L_0^λ to the category of λ-admissible UR's of the little super group S^λ. It gives a bijection (up to equivalence) between UIR's of L_0^λ and UIR's of S^λ.

When the little group is only connected

If L_0^λ is connected but not simply connected, we assume that it is of the form

$$L_0^\lambda = A \times' T \qquad (A \text{ simply connected }, T \text{ a torus}).$$

Then there is a 2-fold cover

$$T^\sim \longrightarrow T$$

such that

$$L_0^\lambda \longrightarrow \mathrm{SO}(\mathfrak{g}_{1\lambda})$$

lifts to

$$p : L_0^\sim \longrightarrow \mathrm{Spin}(\mathfrak{g}_{1\lambda}), \quad L_0^\sim = A \times' T^\sim, \quad p(1, \xi) = -1$$

where ξ is the non trivial element in the kernel of $T^\sim \longrightarrow T$. We can lift the spin representation of $\mathrm{Spin}(\mathfrak{g}_{1\lambda})$ to a UR κ'_λ of $L_0^\sim$. If we take a character χ of $T^\sim$ with $\chi(\xi) = -1$, and view it as a character of $L^\sim$, then

$$\kappa_\lambda = \chi\kappa'_\lambda$$

takes $(1, \xi)$ to 1, hence may be viewed as a UR of L_0^λ. From now on the development is the same as before.

The fundamental multiplet

The theory now gives a bijection

$$r \longleftrightarrow \theta_{r\lambda} \longleftrightarrow \Theta_{r\lambda}$$

between UIR's r of L_0^λ and UIR's $\Theta_{r\lambda}$ of G with spectrum in the orbit of λ. The $\Theta_{r\lambda}$ represent the *super particles.* The corresponding UR's of G_0 are *not* irreducible and their irreducible constituents define the so-called *super multiplets.* The members of the multiplet are the ordinary particles that correspond to the orbit of λ and the irreducible constituents of $r \otimes \kappa_\lambda$. When r is the trivial representation we obtain the *fundamental multiplet.* They are the ordinary particles defined by the orbit of λ and the irreducible constituents of κ_λ. In the case of super Poincaré groups κ_λ can be explicitly determined and its decomposition into irredeucibles described (in principle). When $D = 4$ this was done using the R-group in the paper of Ferrara, Savoy, and Zumino.