Effective String Theories in Lattice Gauge Theories.

Napoli, March 2005

Based on:

M.C., M.Panero and P. Provero
"String effects in Polyakov loop correlators"
JHEP 0206 (2002) 061

M.C., M. Hasenbusch and M.Panero
"String effects in the 3d gauge Ising model"
JHEP 0301 (2003) 057 "Short distance behavior of the effective string."
JHEP 0405:032,2004 "Comparing the Nambu–Goto string with LGT results."
JHEP, submitted

M.C., M. Pepe and A. Rago
"Static quark potential and effective string corrections in the (2+1)-d $SU(2)$ Yang-Mills theory."
JHEP 0410:005,2004

Plan of the Talk

- String corrections in Wilson loop expectation values.
- String corrections in Polyakov loop correlators.
- Higher order corrections: the Nambu-Goto string.
- Comparison with MC data.
- Conclusions.

Wilson loops

Wilson loops are classically expected to obey the famous area-perimeter-constant law:

$$< W(R,L) >= e^{-(\sigma RL + p(R+L) + k)}$$

This law is indeed very well verified in the strong coupling regime (before the roughening transition).

However in the rough phase it must be modified. One must multiply it by the partition function of the $2d$ QFT describing the quantum fluctuations of the flux tube.

$$< W(R,L) >= e^{-(\sigma RL + p(R+L) + k)} Z_q(R,L)$$

For the free bosonic string $Z_q(R,L)$ is simply the partition function of $d-2$ free massless scalar fields living on the rectangle defined by the Wilson loop: $R \times T$

This partition function is given by a divergent, infinite product of eigenvalues which must be regularized. This regularization can be performed in various different ways (ζ function,momentum space cutoff, lattice cutoff....) leading to the following result:

$$Z_q(R,L) \propto \left[\frac{\eta(\tau)}{\sqrt{R}}\right]^{-\frac{d-2}{2}}$$

where $\eta(\tau)$ is the Dedekind η function and $\tau = iL/R$.

The $L \leftrightarrow R$ simmetry is ensured by this identity

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\ \eta(\tau)$$

known as the "modular" transformation of the η.

Defining the free energy as

$$F(R,L) \equiv -\log < W(R,L) >$$

and expanding the Dedekind function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad ; \quad q = e^{2\pi i \tau} \quad ,$$

(This expansion holds only if $i\tau < -1$, i.e. $L > R$) one finds

$$F(R, L) = \sigma RL + p(R + L) + k$$

$$-(d-2)\left[\frac{\pi L}{24R} + \frac{1}{4}\log R\right] + \ldots$$

The following ratio is particularly useful to single out the effective string contribution from a collection of Wilson loops (it requires a very precise knowledge of σ):

$$R(L,n) \equiv \frac{\langle W(L+n,L-n)\rangle}{\langle W(L,L)\rangle}\exp(-n^2\sigma)$$

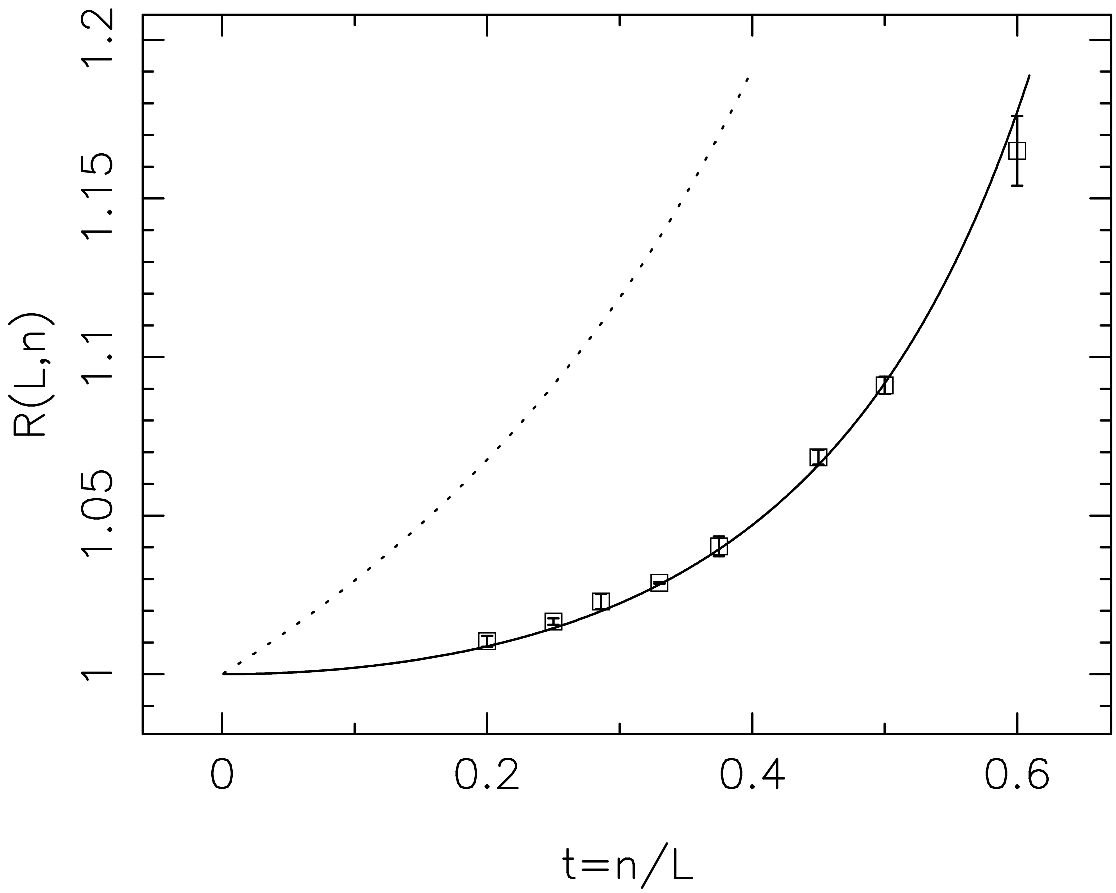
It is easy to see that $R(L,n)$ depends only on $t = n/L$:

$$R(L,n) = F(t) = \left[\frac{\eta(i)\sqrt{1-t}}{\eta\left(i\frac{1+t}{1-t}\right)}\right]^{1/2}$$

and does not contain any adjustable parameters.

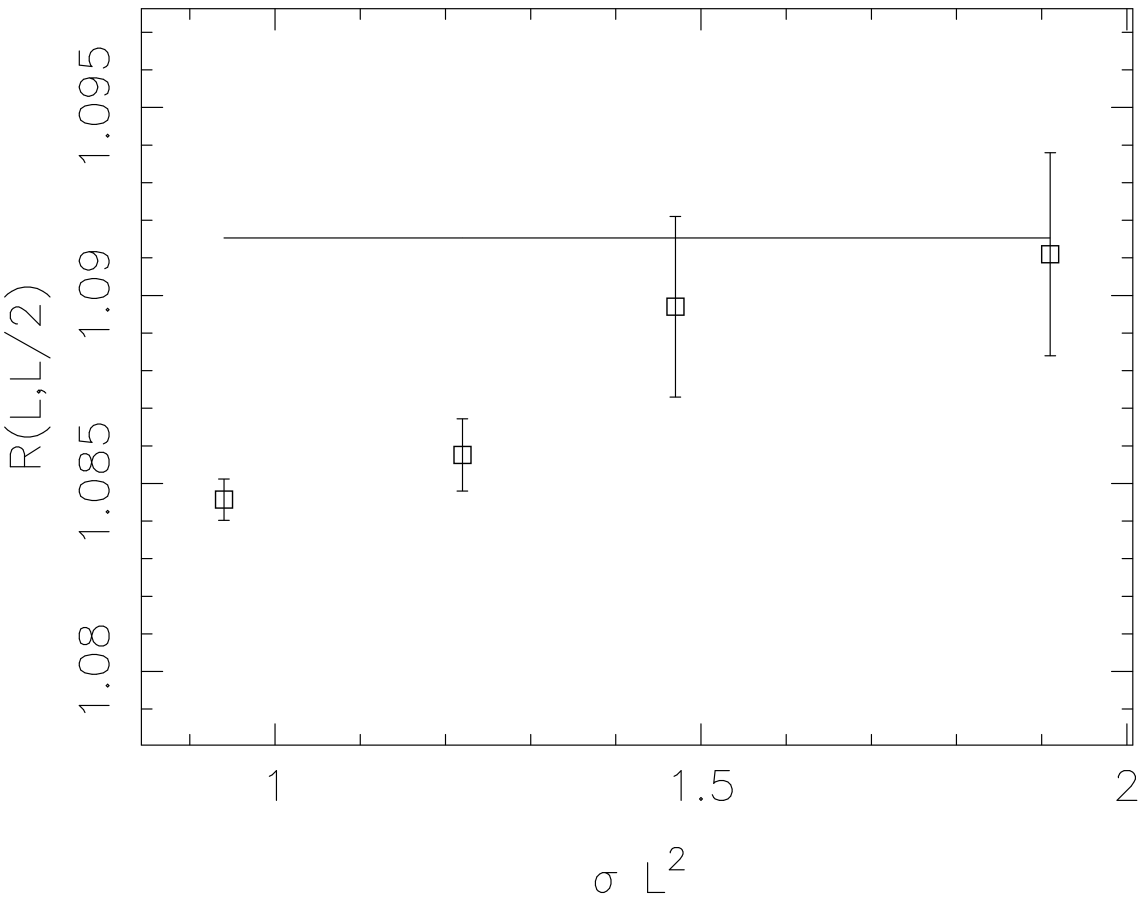
This behaviour was succesfully tested in the 3d Ising gauge case in

M.C. et al Nucl. Phys. B ***486*** *(1997) 245*



Range of validity.

By varying L we can test the range of validity of the free bosonic effective string.



Finite–size effects for small Wilson loop. The prediction of the free string model is $R(L, L/2) = 1.09153\ldots$ *(straight line).*

It turns out that L must be such that $\sigma L^2 > 1.5$. The minimum value of the interquark distance, below which the effective string picture breaks down is the flux tube thickness R_c.

Independent observations, based on the study of high T spacelike wilson loops suggest that $R_c \sim 1/T_c$. which gives $\sigma R_c^2 \sim 1$. The deviations that we observe in the range

$$1 < \sigma L^2 < 1.5$$

are most probably due to the self-interaction terms in the string action.

Similar results are also found in d=4 SU(3) LGT. The following figures are taken from the paper:

S. Necco and R. Sommer, Nucl.Phys. B622 (2002) 328

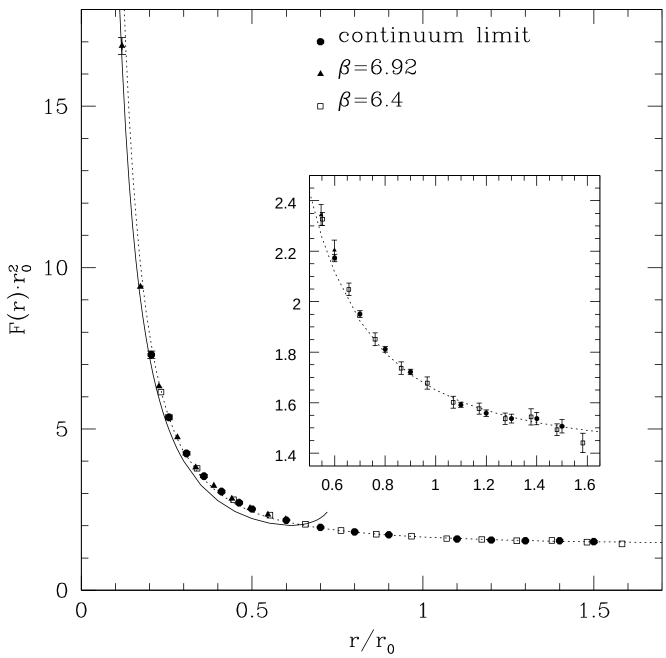
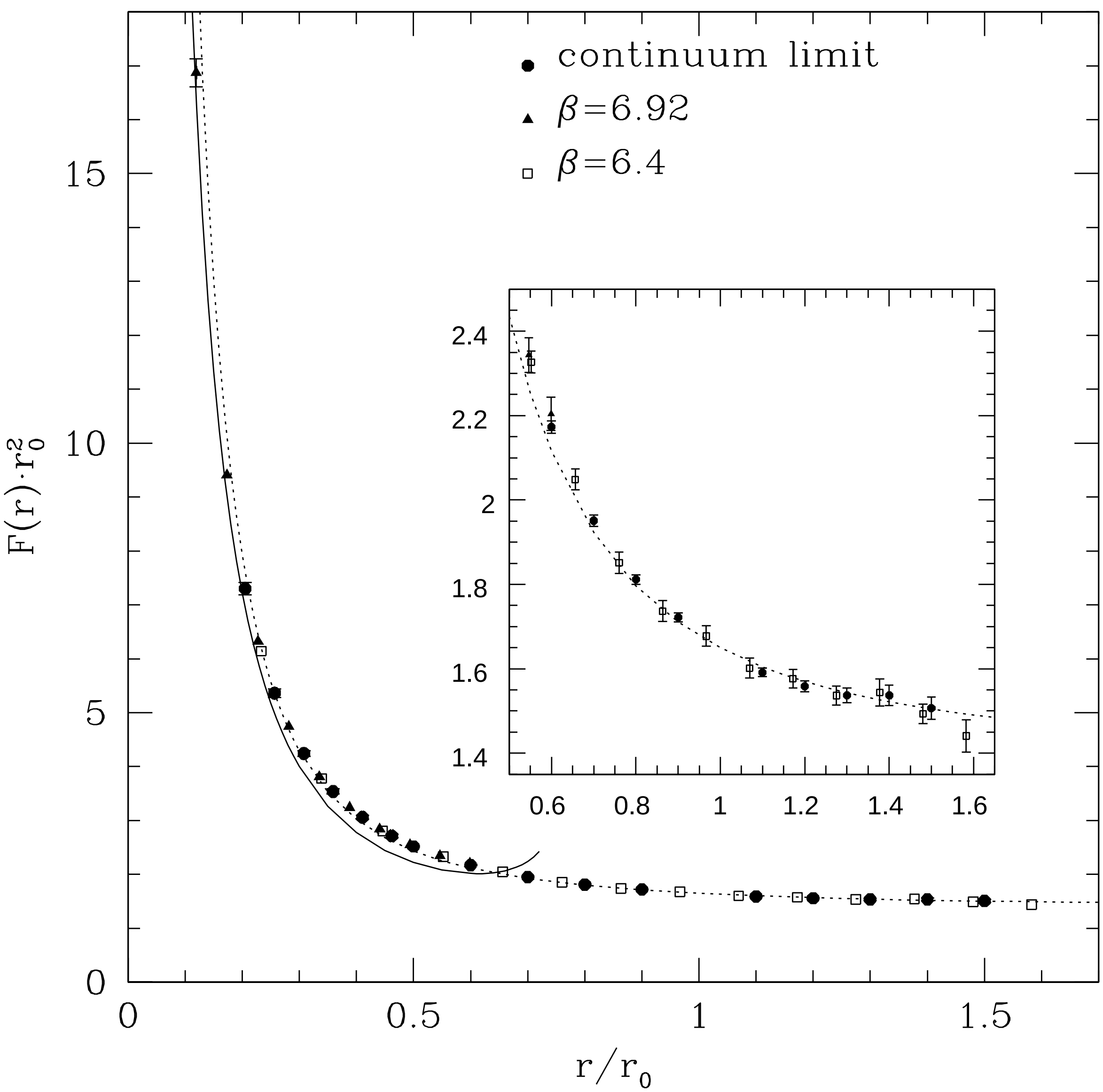


Figure 1: The force in the continuum limit and for finite resolution, where the discretization errors are estimated to be smaller than the statistical errors. The full line is the perturbative prediction with $\Lambda_{\overline{\mathrm{MS}}}\, r_0 = 0.602$. The dashed curve corresponds to the bosonic string model normalized by $r_0^2 F(r_0) = 1.65$.



continuum limit
β=6.92
β=6.4
$F(r)\cdot r_0^2$
r/r_0

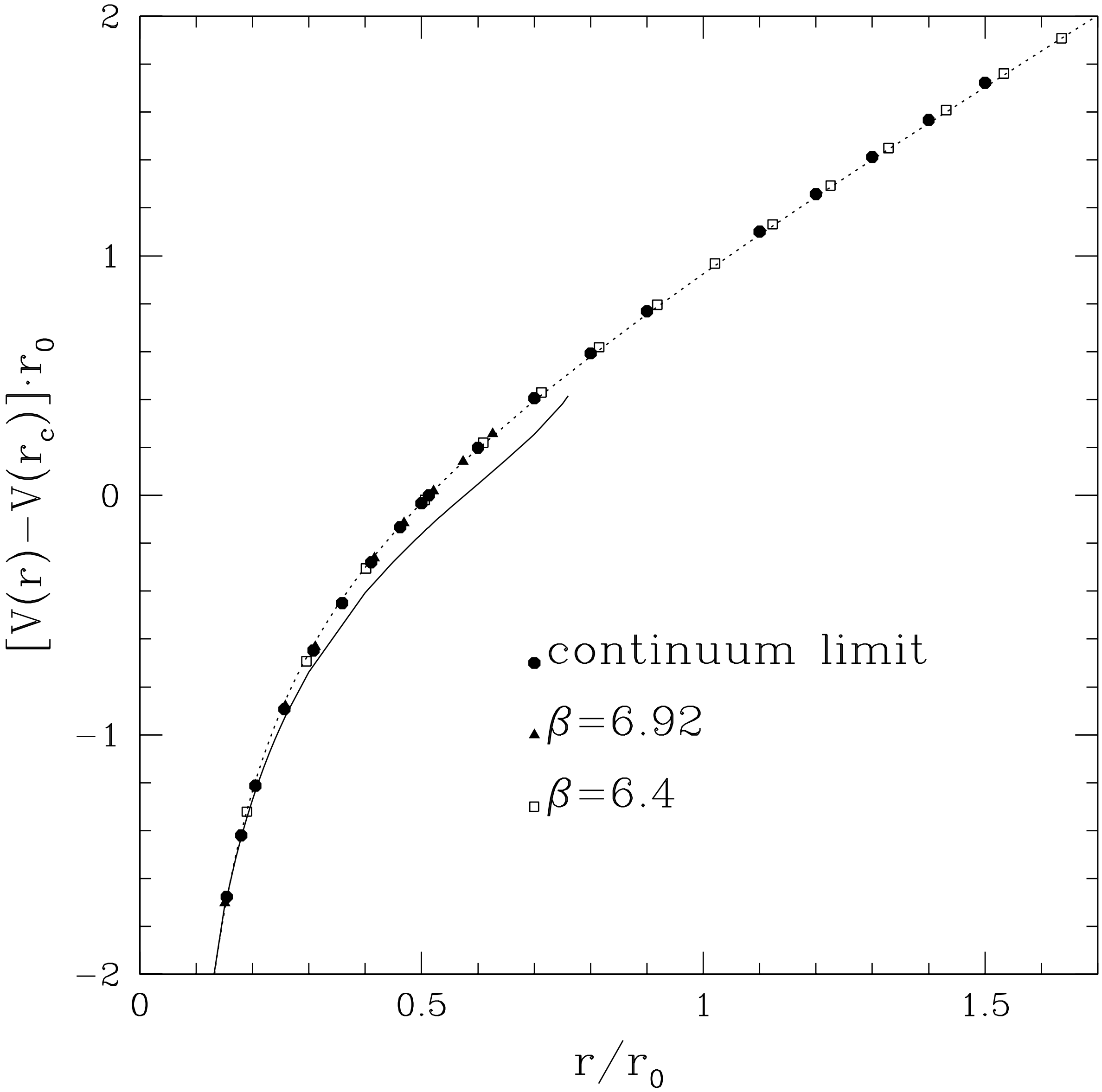


Figure 2: The static potential. The dashed line represents the bosonic string model and the solid line the prediction of perturbation theory as detailed in the text.

Polyakov loop correlators

The peculiar geometry of the Polyakov loop correlators implies that they are perfect tools to explore the range of scales where deviations with respect to the free bosonic effective string appear.

Important observation:

In the Wilson loop geometry $(T = 0)$ RL is simply the area of the loop. One can always choose large enough Wilson loops so as to reach the free string limit.

In the finite temperature geometry $L = 1/T$. The free string limit is reached only for very low temperatures. In particular at intermediate temperatures (say, $T \geq T_c/3$) higher order effects (which encode the self-interaction of the bosonic fields) become important and cannot be neglected.

Let us define the free energy as

$$G(R) = \langle P(0) P^\dagger(R) \rangle = \exp\left[-F(R,L)\right]$$

$F(R,L)$ depends on the inverse temperature $L \equiv 1/T$ ($i.e.$ the lattice size in the compactified time direction) and the distance R, and is given by a classical and a quantum contribution:

$$F(R,L) = F_{\rm cl}(R,L) + F_{\rm q}(R,L)$$

The classical term corresponds to the area law:

$$F_{\rm cl}(R,L) = \sigma_0 L R + k(L)$$

while the quantum term turns out to be:

$$F_{\mathrm{q}}(R,L) = (d-2)\log\eta(\tau) \qquad \tau \equiv \frac{iL}{2R}$$

where η is again the Dedekind function

M. Minami, Prog. Theor. Phys. **59** *(1978) 1709.*

P. de Forcrand, G. Schierholz, H. Schneider and M. Teper, Phys. Lett. B **160** *(1985) 137.*

M. Flensburg and C. Peterson, Nucl. Phys. B **283** *(1987) 141.*

Important observation: Due to the modular transformation

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\ \eta(\tau)$$

the asymptotic expansion is different in the two regimes:

$2R < L$

$$F_q(R,L) = (d-2)\left[-\frac{\pi L}{24R} + \sum_{n=1}^{\infty} \log(1 - e^{-\pi n L/R})\right]$$

$2R > L$

$$F_q(R,L) = (d-2)\left[-\frac{\pi R}{6L} + \frac{1}{2}\log\frac{2R}{L}\right]$$

$$+(d-2)\left[\sum_{n=1}^{\infty} \log(1 - e^{-4\pi n R/L})\right]$$

Comments:

- For $R > L/2$ the string correction is linear in R and acts as a finite temperature renormalization of the string tension:

$$\sigma(T) = \sigma_0 - (d-2)\frac{\pi T^2}{6}$$

- For $R \sim L$ the log term cancels the linear one and the string correction vanishes. This explains why it is so difficult to observe string corrections in the Polyakov geometry

String self-interaction terms.

Simplest option: Nambu-Goto string.

$$A\left[\phi\right]=\int_0^{L_1}dx\int_0^{L_2}dy\,\sqrt{1+\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2}\quad.$$

where ϕ denotes the transverse displacement of the surface with respect to the plane joining the two Polyakov loops. There are at least two arguments in favour of this choice:

- Within the framework of the Nambu Goto action one obtains (*R. D. Pisarski and O. Alvarez, Phys. Rev. D* **26** *(1982) 3735. P. Olesen, Phys. Lett. B* **160** *(1985) 408.*)

  $$T_c^2=\frac{3\sigma_0}{(d-2)\pi}$$

  which turns out to be in good agreement with MC results for various LGT both in d=3 and d=4.

- In the dual problem of the interface behaviour in the 3d Ising spin model it correctly describes higher order (short range) corrections in the interface free energy (*M.C et al. Nucl. Phys. B* **432** *(1994) 590*)

Next to leading order.

The next to leading order in $1/\sigma$ of the free energy can be evaluated in the framework of the zeta function regularization (*K. Dietz and T. Filk, Phys. Rev. D* **27** *(1983) 2944.*) the result in $d = 3$ is:

$$F_q^{(NLO)}(R, L) = -\frac{\pi^2 L}{1152\ \sigma R^3}\left[2E_4(\tau) - E_2^2(\tau)\right]$$

where E_2 and E_4 are the second and fourth order Eisenstein functions.

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n \quad (1)$$

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \quad (2)$$

$$q \equiv e^{2\pi i\tau}\ , \quad (3)$$

where $\sigma(n)$ and $\sigma_3(n)$ are, respectively, the sum of all divisors of n (including 1 and n), and the sum of their cubes.

Nambu–Goto action to all orders.

More than 20 years ago Arvis proposed the following expression for the energy spectrum of the Nambu-Goto string (with Dirichelet boundary conditions):

$$E_n = \sigma R \sqrt{1 - \frac{\pi}{12 \sigma R^2} + \frac{2\pi n}{\sigma R^2}} \tag{4}$$

(where we have fixed $d = 2 + 1$).

From the Arvis spectrum it is easy to construct (at least formally) the partition function of the Nambu-Goto string to all orders.

$$Z = \sum_{n}^{\infty} w_n e^{-E_n L} \tag{5}$$

where the weights w_n are the number of partitions of the integer n.

Comparison with MC simulations

- **Ising model**

  We studied the 3d gauge Ising model at

  $\beta = 0.75180$ which corresponds to $T_c = 8$

  For this value of β the string tension is known with very high precision $\sigma = 0.0105241(15)$. We performed simulations at $T = \frac{T_c}{3}, \frac{T_c}{2}, \frac{2T_c}{3}, \frac{4T_c}{5}$ corresponding to $L = 24, 16, 12, 10$ respectively with a new algorithm based on dual transformations. For a wide range of values of R we studied two types of ratios:

$$Q_q(R) \equiv \ln \frac{G(R)}{G(R+1)} - \sigma_0(\beta) L$$

  which is chosen so as to cancel the classical part of the free energy.

$$Q_q(R) = F_q(R+1, L) - F_q(R, L)$$

We also studied the combination

$$\Gamma(R) - \Gamma_{LO}(R)$$

where

$$\Gamma(R) \equiv \frac{G(R)}{G(R+1)}$$

and $\Gamma_{LO}(R)$ is its prediction in the free bosonic string approximation.
This observable is the best tool to magnificate the next to leading corrections and test if they agree with the Nambu-Goto expectation.

- **SU(2) model**

  We studied the 3d SU(2) model at

  $\beta = 9.0$ which corresponds to $T_c \sim 6$

  For this value of β the string tension is known with very high precision $\sigma = 0.025900(12)$. We performed simulations at $T = \frac{T_c}{10}$ and $\frac{3T_c}{4}$ corresponding to $L = 60, 8$ respectively.

- **SU(3) model**

  For the SU(3) model we used the data reported in

  M. Lüscher and P. Weisz,
  "Quark confinement and the bosonic string"
  JHEP 0207 (2002) 049.

In this cases we studied the ratio:

$$Q(R) = -\frac{1}{L} \log \left(\frac{G(R+1)}{G(R)} \right)$$

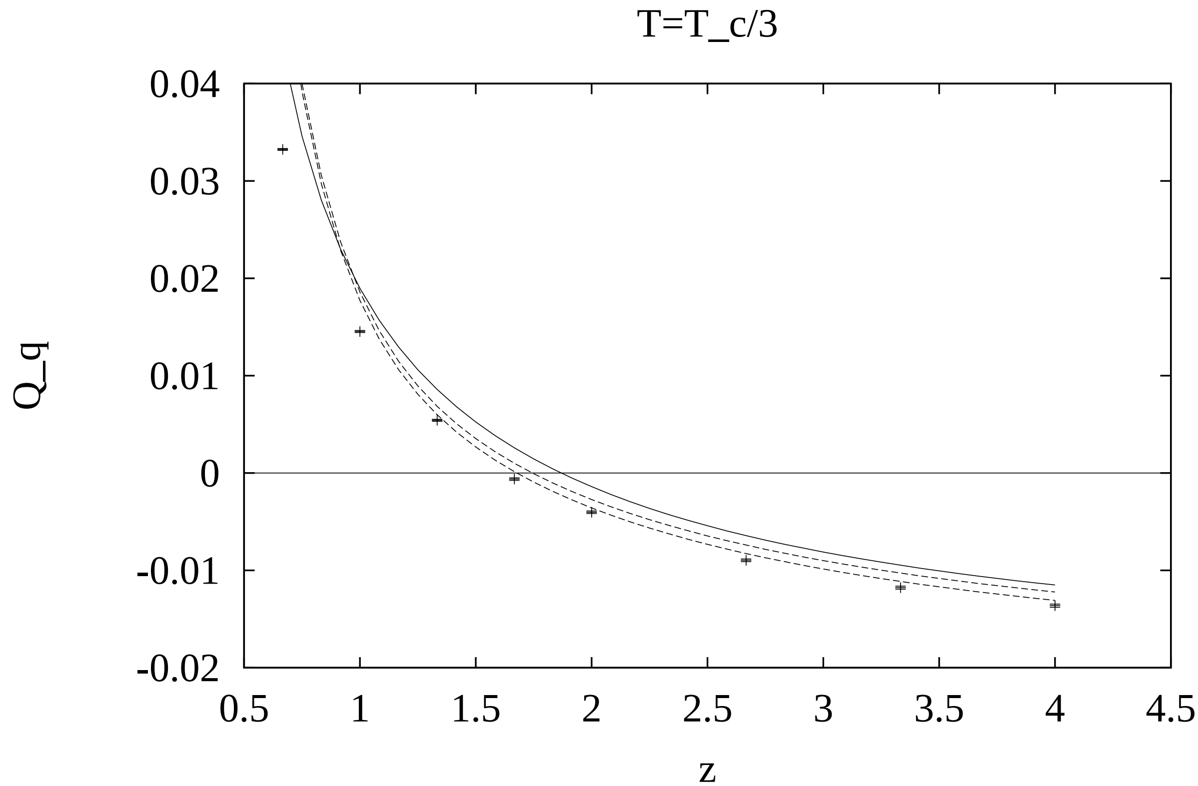


Figure 3: Q_q for $N_t = 24$ (i.e. $T = T_c/3$) at $\beta = 0.75180$. The continuous line corresponds to the free bosonic string prediction, while the two dashed lines correspond to the first Nambu-Goto corrections. The difference between the two dashed lines keeps into account the uncertainty in the estimate of σ

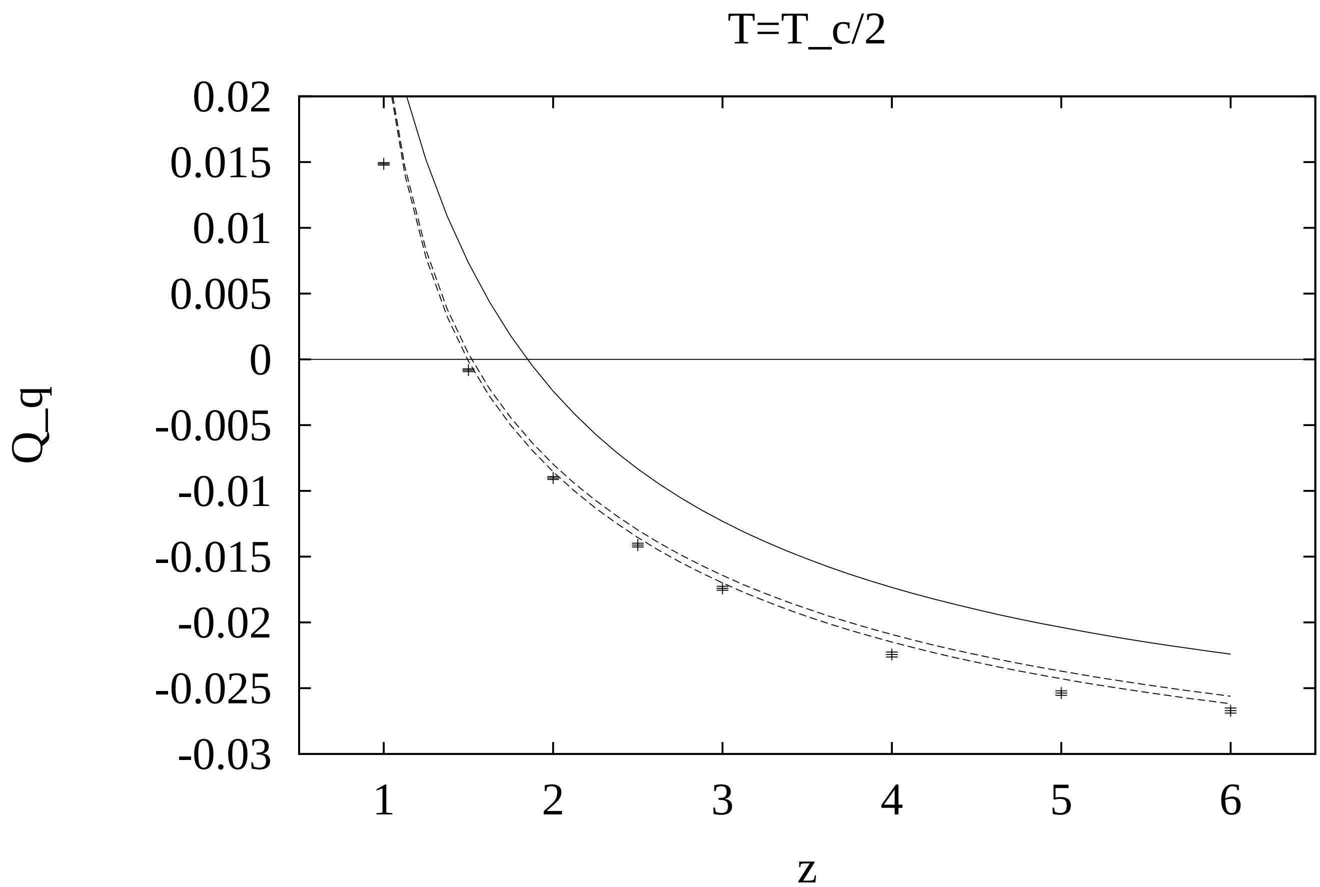


Figure 4: Same as above, but for $N_t = 16$(i.e. $T = T_c/2$)

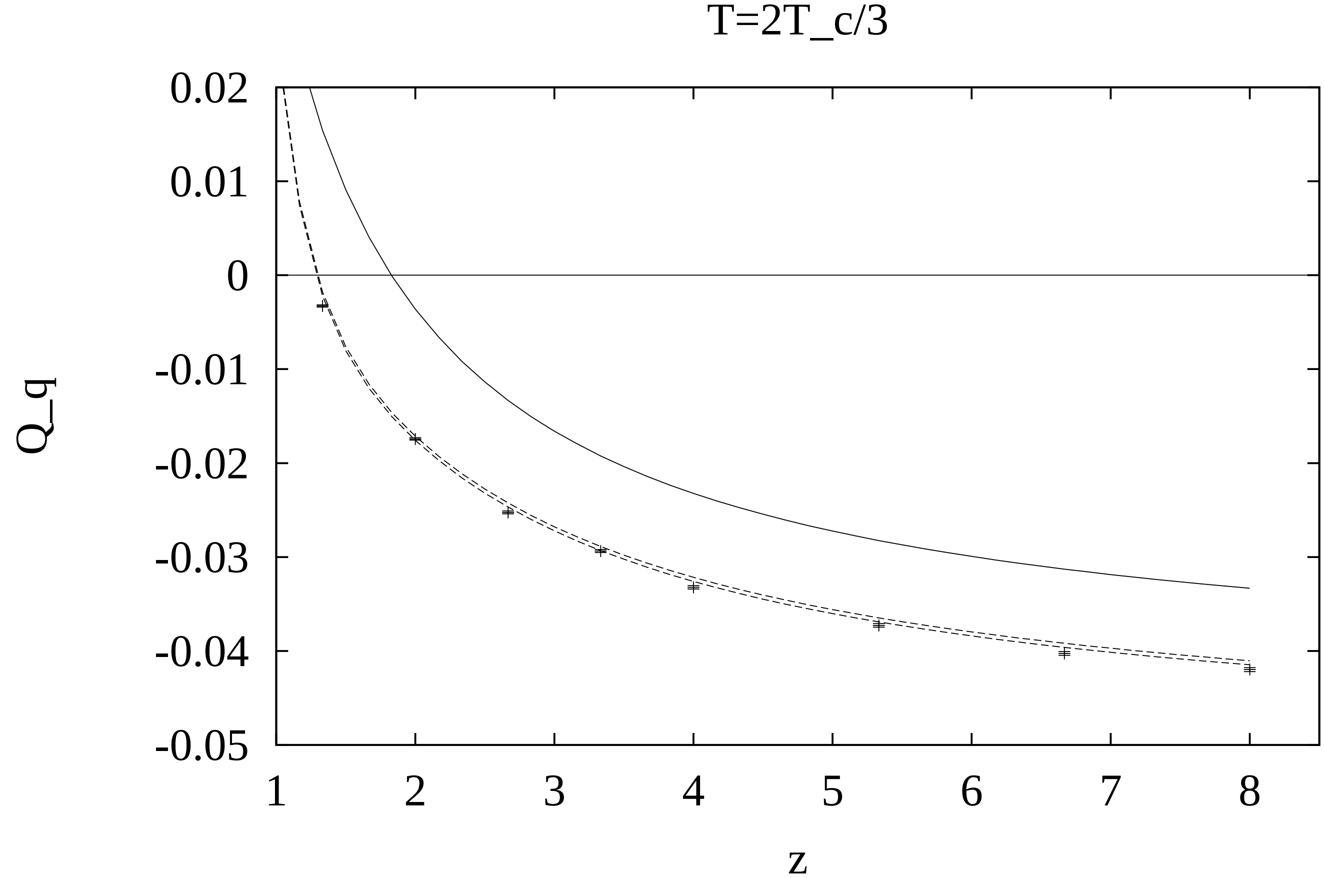


Figure 5: Same as above, but for $N_t = 12$(i.e. $T = 2T_c/3$)

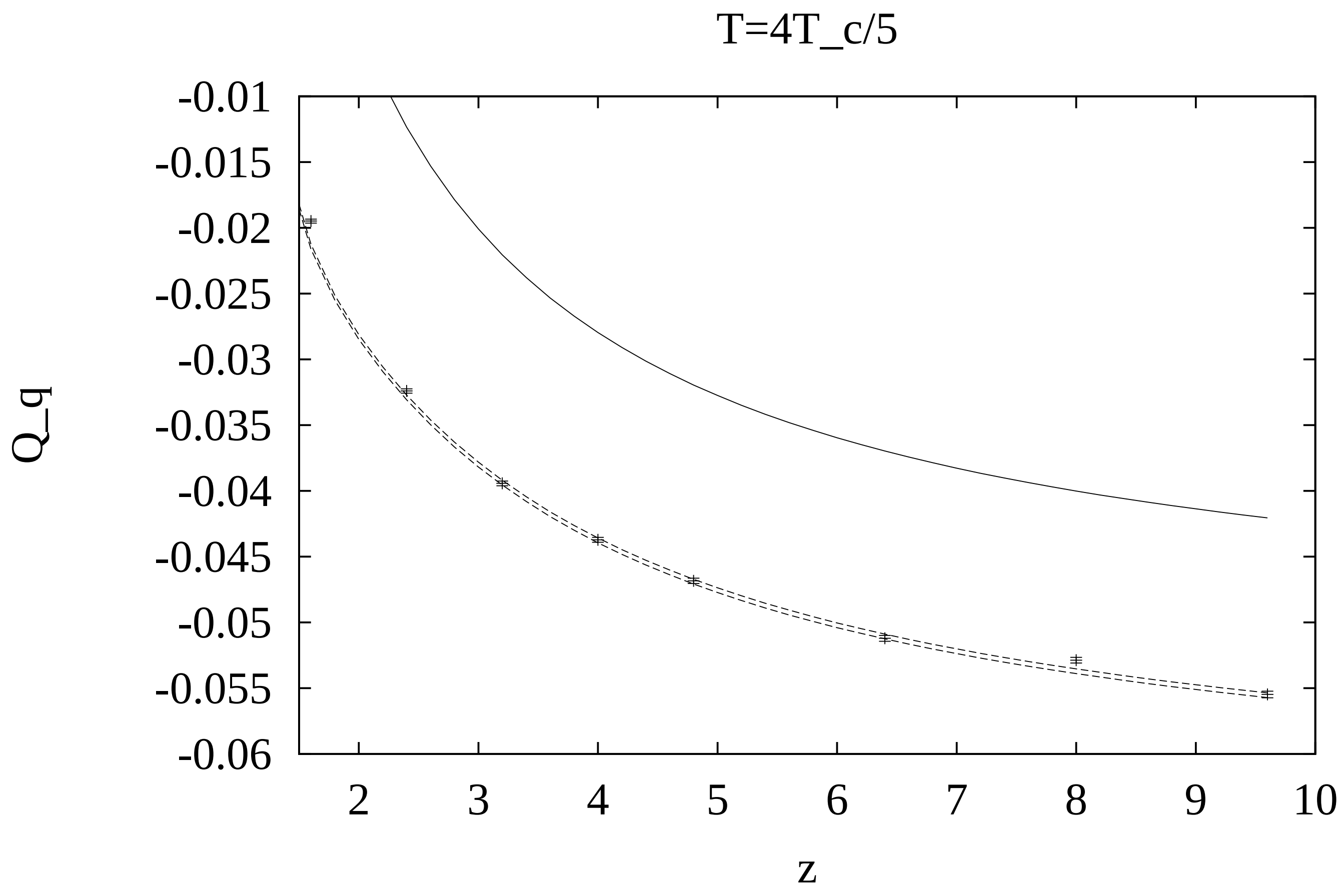


Figure 6: Same as above, but for $N_t = 10$(i.e. $T = 4T_c/5$)

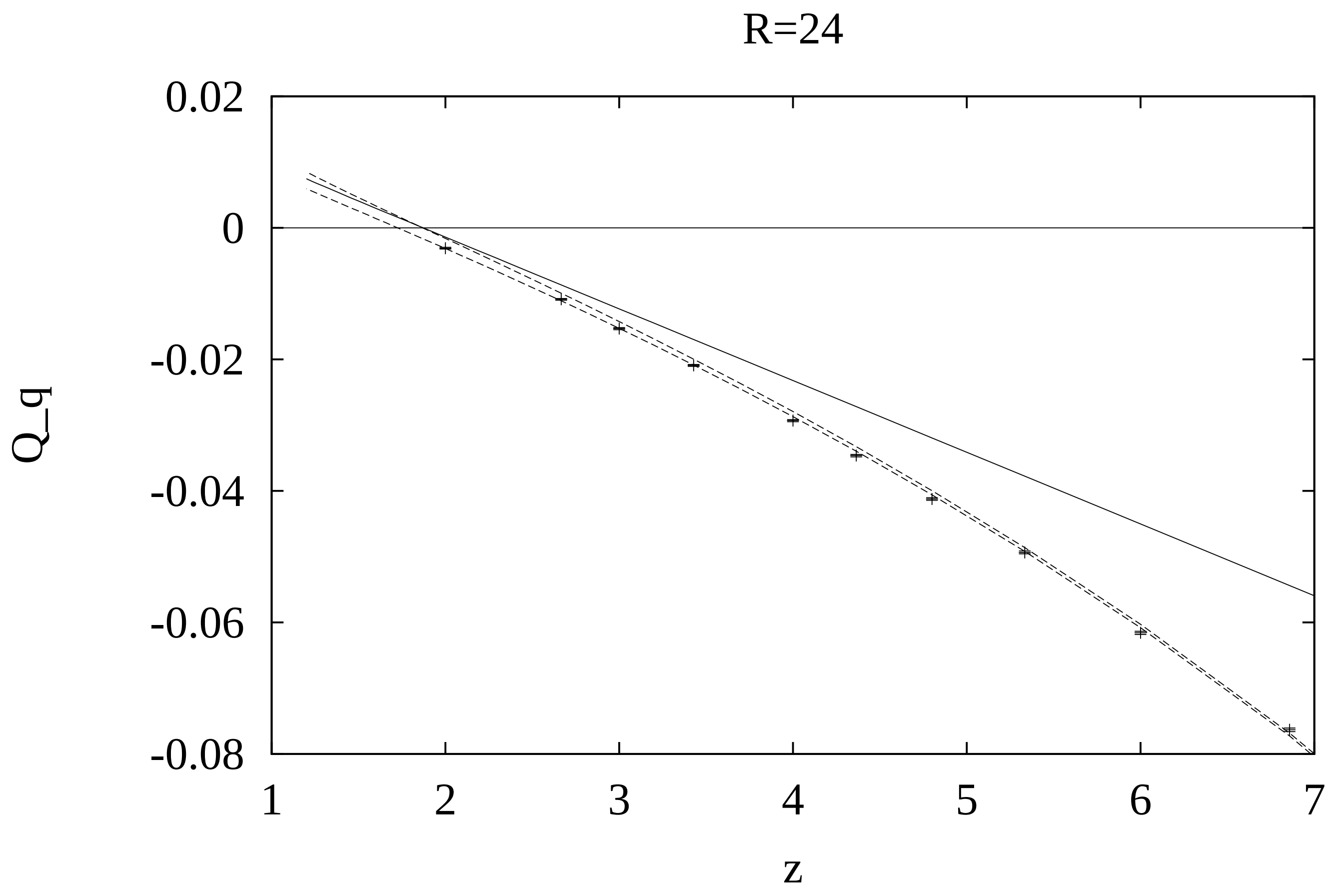


Figure 7: Same as above, but for a fixed $R = 24$

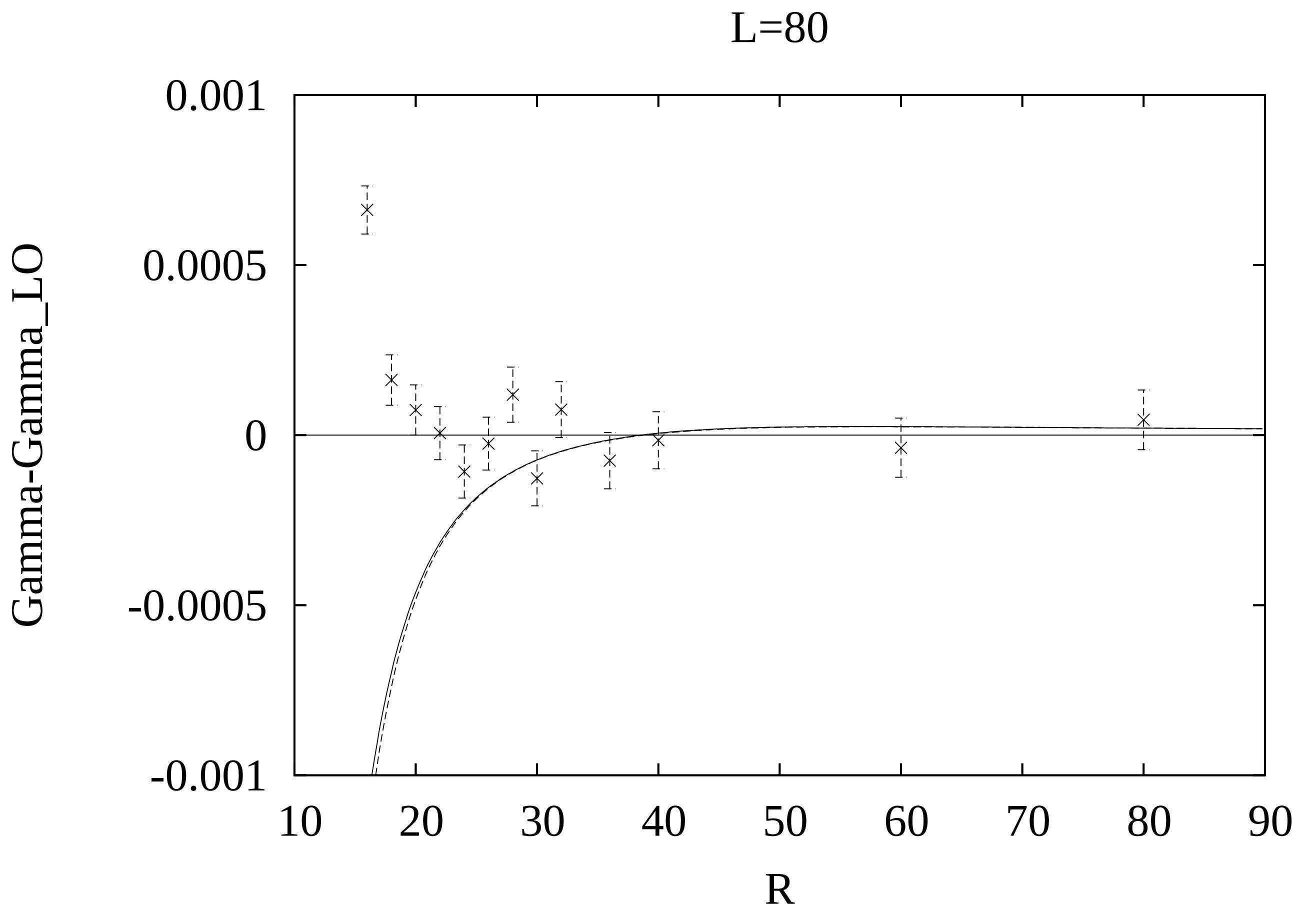


Figure 8: Differences between the values of $\Gamma(R)$ for the full Nambu–Goto action (dashed line), the NLO approximation (solid line) and the Monte Carlo results (crosses) with respect to the LO approximation for the sample at $\beta = 0.75180$, $L = 80$ in the Ising gauge model. Notice that for this value of L the NLO and full Nambu–Goto results almost coincide and cannot be separated in the figure.

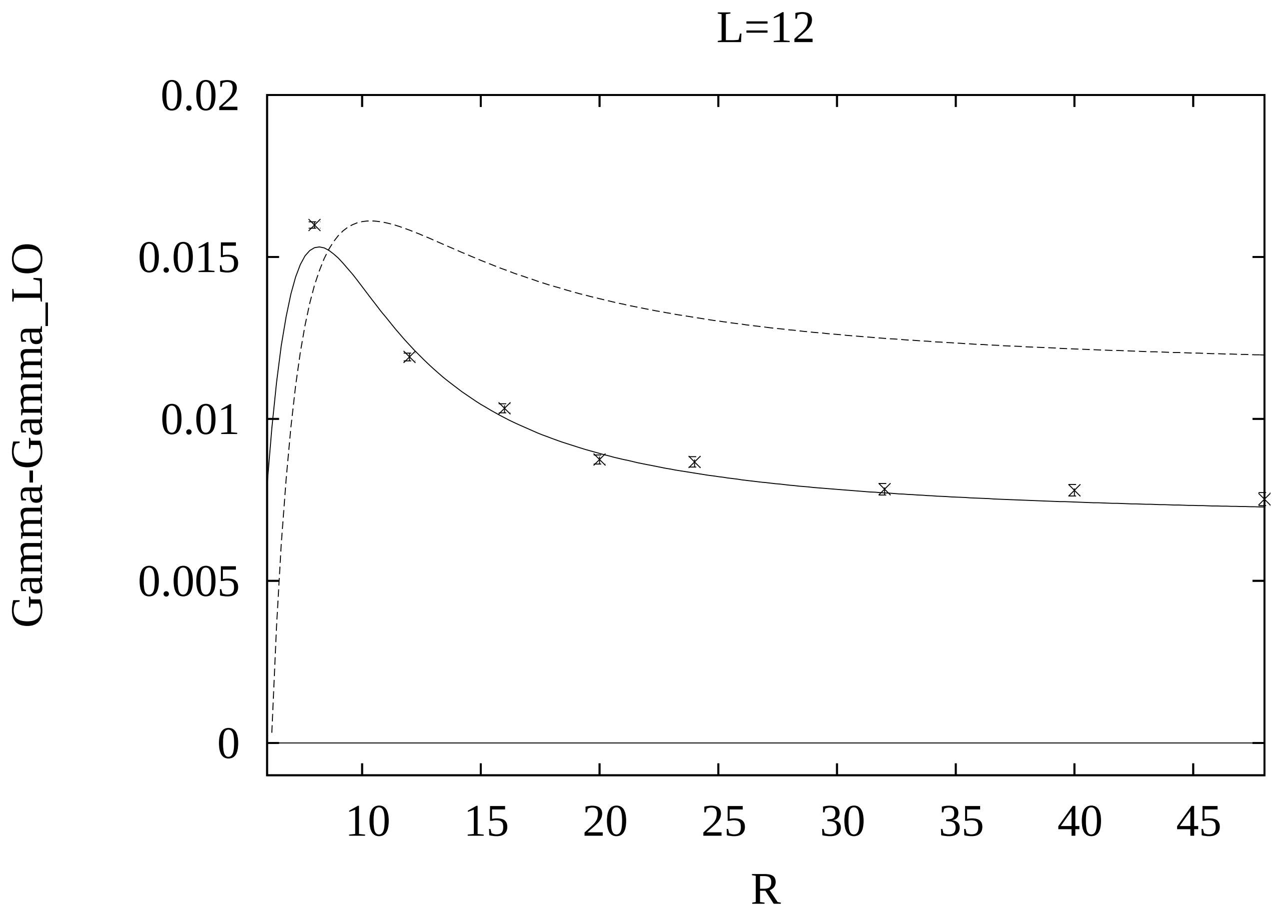


Figure 9: Same as the previous figure but for the data at $L = 12$. In this case the difference between NLO and full Nambu–Goto predictions is perfectly detectable.

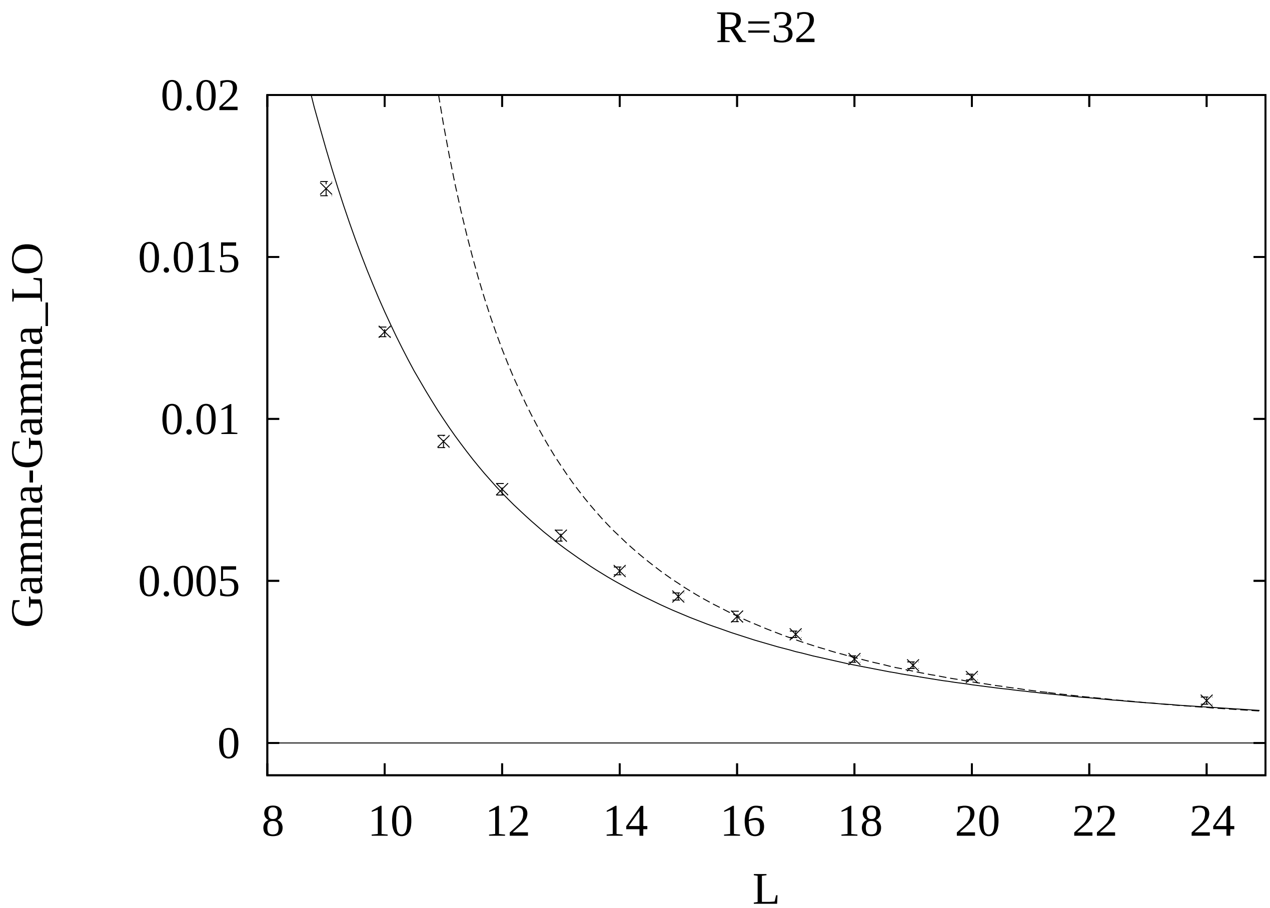


Figure 10: Same as previous figures, but for the data at $R = 32$. The Monte Carlo results interpolate between the full Nambu–Goto behaviour at low temperature (dashed line) and the NLO one (solid line) at high temperature.

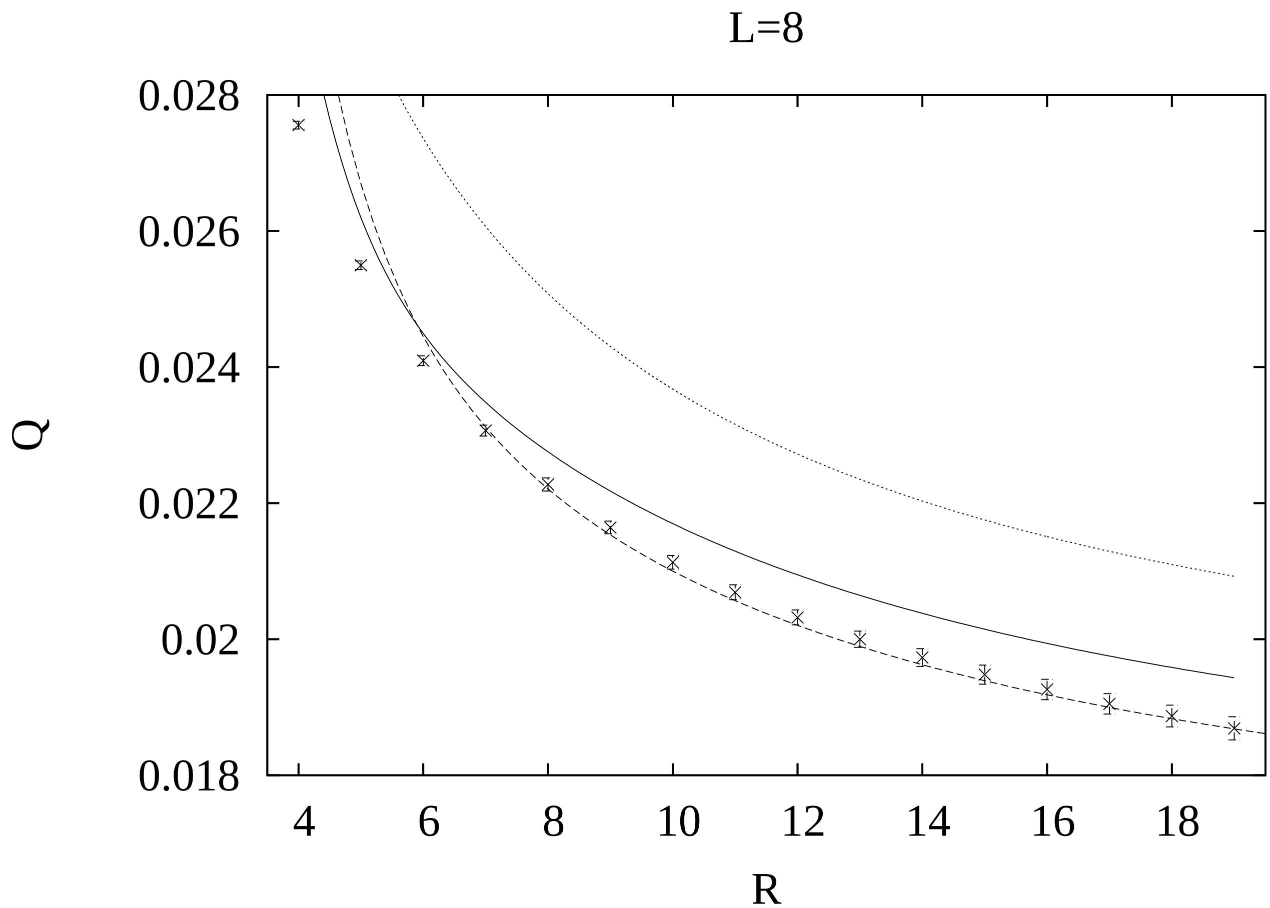


Figure 11: Values of $Q(R)$ for the full Nambu–Goto action (dashed line), the NLO approximation (solid line), the LO approximation (dotted line) and the Monte Carlo results (crosses) for the sample at $\beta = 9.0$, $L = 8$ in the $SU(2)$ model.

Conclusions

- MC simulations strongly support the conjecture that the effective string theory which describes confinement in the infrared regime of various LGT's (3d gauge Ising model, SU(2), SU(3) in d=3 and d=4) at large enough distances and low enough temperatures is a simple free bosonic string theory.

- At smaller distances and/or higher temperatures the effective string picture still holds, but corrections, presumably due to the self-interaction terms in the string action, appear.

- The peculiar geometry of the Polyakov loop correlators implies that they are perfect tools to explore this regime. In particular at intermediate temperatures (say, $T \geq T_c/3$) higher order effects (which encode the self-interaction of the bosonic fields) become important and cannot be neglected.

- This fact can be used to better understand the nature of the underlying effective string. In principle there could be various different classes of effective string theories. All of them with the same large distance limit (the free bosonic string)

As a first step in this direction we studied the "Nambu-Goto" action and compared it with high precision MC data for the Z_2, $SU(2)$ and $SU(3)$ gauge models in $d = 3$ with the following results:

- The Nambu-Goto action describes well the Montecarlo data for large values of L and R.

- For large values of L, as R decreases we observe a remarkably universal pattern: all the three models that we studied show a clear discrepancy with respect to the N–G expectation. This discrepancy increases as R decreases and is worsened by the addition of higher perturbative orders (i.e. moving from the truncated approximation to the whole N-G estimates)

- For lower values of L (i.e. as we approach the deconfinement temperature) the situation drastically changes: The N-G action seems to describe better and better the Montecarlo data as T approaches T_c.

From the Nambu-Goto action to the free bosonic string.

The Nambu string action is given by the area of the world–sheet:

$$S = \sigma \int_0^T d\tau \int_0^R d\varsigma \sqrt{g} \quad ,$$

where g is the determinant of the two–dimensional metric induced on the world–sheet by the embedding in R^d:

$$\begin{aligned} g = \det(g_{\alpha\beta}) &= \det\ \partial_\alpha X^\mu \partial_\beta X^\mu \quad . \\ & (\alpha, \beta = \tau, \varsigma,\ \mu = 1, \ldots, d) \end{aligned}$$

and σ is the string tension.

The reparametrization and Weyl invariances of the action require a gauge choice for quantization.

We choose the "physical gauge"

$$\begin{aligned} X^1 &= \tau \\ X^2 &= \varsigma \end{aligned}$$

so that g is expressed as a function of the transverse degrees of freedom only:

$$\begin{aligned} g &= 1 + \partial_\tau X^i \partial_\tau X^i + \partial_\varsigma X^i \partial_\varsigma X^i \\ &\quad + \partial_\tau X^i \partial_\tau X^i \partial_\varsigma X^j \partial_\varsigma X^j - (\partial_\tau X^i \partial_\varsigma X^i)^2 \\ &\quad (i = 3, \ldots, d) \ . \end{aligned}$$

The fields $X^i(\tau, \varsigma)$ satisfy Dirichlet boundary conditions on M:

$$X^i(0, \varsigma) = X^i(T, \varsigma) = X^i(\tau, 0) = X^i(\tau, R) = 0 \quad .$$

Due to the Weyl anomaly this gauge choice can be performed at the quantum level only in the critical dimension $d = 26$. However, the effect of the anomaly is known to disappear at large distances, which is the region we are interested in.

Expanding the square root in the action we obtain, discarding terms of order X^4 and higher

$$
\begin{aligned}
S &= \sigma R T + \frac{\sigma}{2} \int d^2\xi X^i (-\partial^2) X^i \\
\partial^2 &= \partial_\tau^2 + \partial_\varsigma^2 \ .
\end{aligned}
$$

The partition function of the free bosonic string.

The partition function for the free bosonic action is

$$Z_q(R,T) \propto \left[\det(-\partial^2)\right]^{-\frac{d-2}{2}} \ .$$

The determinant must be evaluated with Dirichlet boundary conditions.

The spectrum of $-\partial^2$ with Dirichlet boundary conditions is given by the eigenvalues

$$\lambda_{mn} = \pi^2 \left(\frac{m^2}{T^2} + \frac{n^2}{R^2} \right)$$

corresponding to the normalized eigenfunctions

$$\psi_{mn}(\xi) = \frac{2}{\sqrt{RT}} \sin \frac{m\pi\tau}{T} \sin \frac{n\pi\varsigma}{R} \ .$$

The determinant can be regularized with the ζ-function technique: defining

$$\zeta_{-\partial^2}(s) \equiv \sum_{mn=1}^{\infty} \lambda_{mn}^{-s}$$

the regularized determinant is defined through the analytic continuation of $\zeta'_{-\partial^2}(s)$ to $s = 0$:

$$\det(-\partial^2) = \exp\left[-\zeta'_{-\partial^2}(0)\right] \quad .$$

The series can be transformed, using the Poisson summation formula which states that:

$$\sum_{n=-\infty}^{\infty} f(nr) = \frac{1}{r} \sum_{m=-\infty}^{\infty} \tilde{f}\left(\frac{m}{r}\right)$$

where the Fourier transform $\tilde{f}$ is defined as

$$\tilde{f}(y) = \int_{-\infty}^{\infty} dx\ e^{-2\pi i x y} f(x)$$

we obtain:

$$\zeta_{-\partial^2}(s) = -\frac{1}{2}\left(\frac{R^2}{\pi^2}\right)^s \zeta_R(2s) +$$
$$\frac{\sqrt{\pi} Im\tau \Gamma(s-1/2)}{2\Gamma(s)}\left(\frac{R^2}{\pi^2}\right)^s \zeta_R(2s-1) +$$
$$\frac{2\sqrt{\pi}}{\Gamma(s)}\left(\frac{T^2}{\pi^2}\right)^s \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\left(\frac{\pi p}{n Im\tau}\right)^{s-1/2} K_{s-1/2}(2\pi p n Im\tau)$$

where $\tau = iT/R$, $\zeta_R(s)$ is the Riemann ζ function and $K_\nu(x)$ is a modified Bessel function. The derivative $\zeta'_{-\partial^2}(s)$ can be analytically continued to $s=0$ where it is given by

$$\zeta'_{-\partial^2}(0) = \log(\sqrt{2R}) - \frac{i\pi\tau}{12} - \sum_{n=1}^{\infty}\log(1-q^n)$$

where we have defined

$$q \equiv e^{2\pi i\tau} \ .$$

Introducing the Dedekind η-function

$$\eta(\tau) = q^{1/24}\Pi_{n=1}^{\infty}(1-q^n)$$

we obtain finally

$$\det(-\partial^2) = \exp[-\zeta'_{-\partial^2}(0)] = \frac{\eta(\tau)}{\sqrt{2R}}$$

and

$$Z_q(R,T) \propto \left[\frac{\eta(\tau)}{\sqrt{R}}\right]^{-\frac{d-2}{2}} \quad .$$