Global Theory of Boundary Conditions and Topology Change

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[Alberto Ibort and Giuseppe Marmo]

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Quantum Boundary Conditions

Motivation:

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- Hearing the shape of a quantum drum
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- Topology change
- Holographic principle, Topological Field Theories, strings , D-branes and all that

Quantum Boundary data

- Riemannian manifold (M, g) with boundary $\partial M = \Gamma$
- Vector bundle $E(M, \mathbb{C}^N)$
- Hilbert product

$$\langle \psi_1, \psi_2 \rangle = \int_M (\psi_1(x), \psi_2(x))_x \, d\mu_g(x), \qquad d\mu_g(x) = \sqrt{g}\, d^n x$$

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- Laplace-Beltrami operator

$$\Delta_A = d_A^\dagger d_A$$

- Δ_A is a symmetric operator on $C_0^\infty(M, E)$

$$\langle \psi_1, \Delta_A \psi_2 \rangle = \langle \Delta_A \psi_1, \psi_2 \rangle$$

but not selfadjoint $\Delta_A \neq \Delta_A^\dagger$

Selfadjoint extensions:

[von Neumann theory]

Deficiency spaces

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- Not based on boundary data
- One needs to know $\mathcal{N}_+$ and $\mathcal{N}_-$ explicitly [not operative]

Boundary data approach

M orientable and Γ regular

Balance defect

$$\langle \psi_1, \Delta_A \psi_2 \rangle = \langle \Delta_A \psi_1, \psi_2 \rangle] - \int_M d \left[\left(*d_A \psi_1, \psi_2 \right) - \left(\psi_1, *d_A \psi_2 \right) \right]$$

Boundary flux term

$$\Sigma \left(\psi_1, \psi_2 \right) = i \int_\Gamma j^* \left[\left(*d_A \psi_1, \psi_2 \right) - \left(\psi_1, *d_A \psi_2 \right) \right]$$

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Theorem [Asorey-Ibort-Marmo]: The set $\mathcal{M}$ of self-adjoint extensions of Δ_A is in one-to-one correspondence with the group of unitary operators of $L^2(\Gamma, \mathbb{C}^N)$.

Boundary data approach

$$\Sigma\left(\psi_1, \psi_2\right) = i \int_{\Gamma} \left[(\dot{\varphi}_1, \varphi_2) - (\varphi_1, \dot{\varphi}_2)\right] d\mu_{\Gamma}$$

$$\varphi_i = j^* \psi_i = \psi_i|_{\Gamma} \qquad j^*[* d_A \psi_i] = \dot{\varphi}_i \, d\mu_{\Gamma} \qquad (i = 1, 2)$$

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$$\boxed{(\varphi - i\dot{\varphi}) = U(\varphi + i\dot{\varphi})}$$

- Equivalence to von Neumann theory
- Most general type of boundary condition
- Global theory of boundary conditions

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Examples

One-dimension $\Delta = \frac{-d^2}{dx^2}$ $M = [0, 1] \in \mathbb{R}$

1. Dirichlet boundary conditions

$$U = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \varphi(0) = \varphi(1) = 0$$

2. Neumann boundary conditions

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \varphi'(0) = \varphi'(1) = 0$$

3. Periodic boundary conditions

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \varphi(0) = \varphi(1)$$

$$M = \cup_{i=1}^{N} [a_i, b_i] \in \mathbb{R}$$

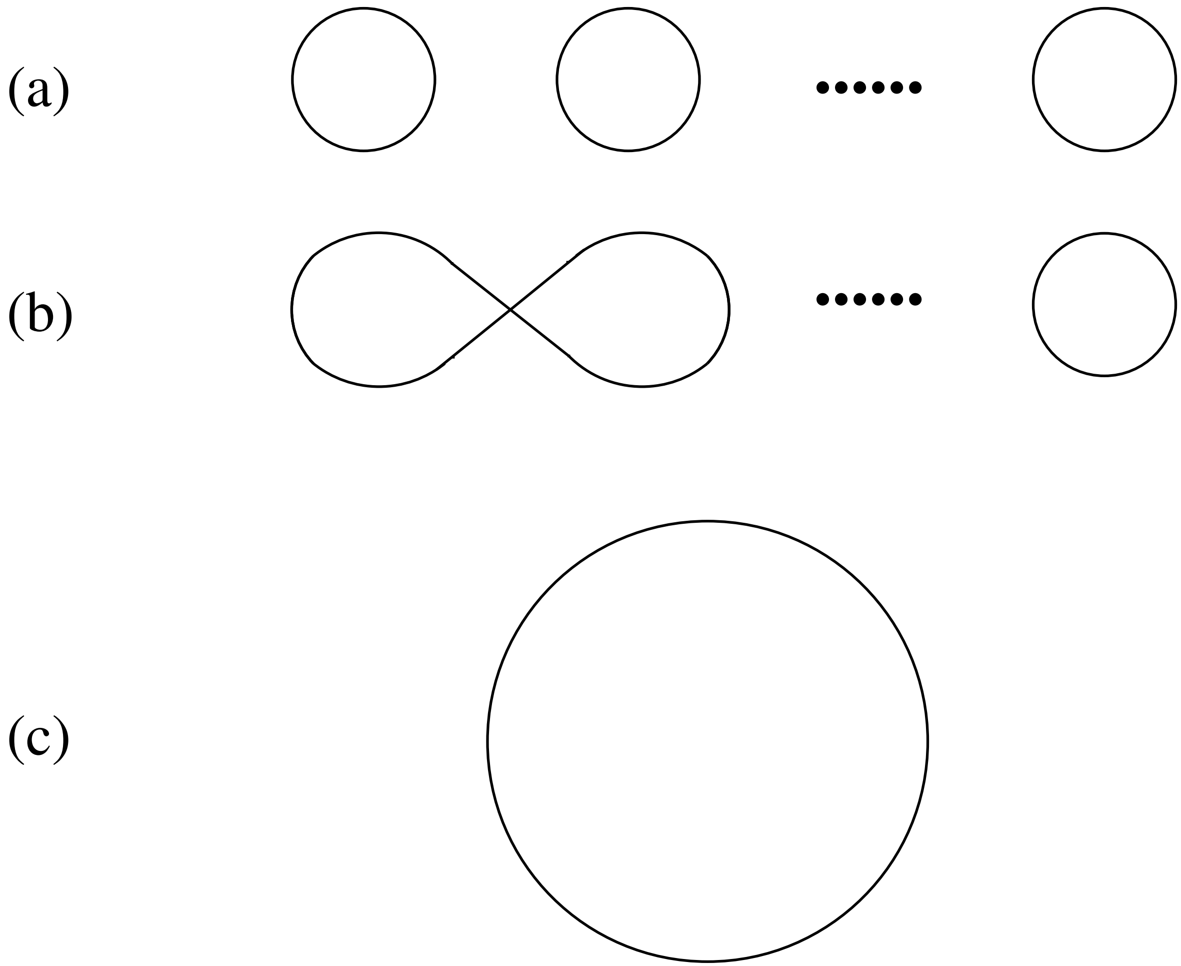
4. One single circle

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ . & . & . & . & . & \cdots & . & . \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

5. N Disconected circles

$$U_N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ . & . & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

TOPOLOGY CHANGE



Cayley Transform

1. If $-1 \notin \mathrm{Sp}U$ the boundary condition reduces to

$$\dot{\varphi} = -i\frac{\mathbb{1} - U}{\mathbb{1} + U}\varphi$$

21. If $1 \notin \mathrm{Sp}U$ the boundary condition reduces to

$$\varphi = i\frac{\mathbb{1} + U}{\mathbb{1} - U}\dot{\varphi}$$

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Cayley transform

$$A = -i \frac{\mathbb{1} - U}{\mathbb{1} + U}$$

Inverse Cayley transform

$$U = \frac{\mathbb{1} - iA}{\mathbb{1} + iA}$$

Cayley submanifolds. Maslov index

Cayley submanifolds:

$$\mathcal{C}_{\pm} = \left\{ U \in \mathcal{U}\left(L^2(\Gamma, \mathbb{C}^N)\right) \,\middle|\, \pm 1 \in \mathrm{Sp}(U) \right\}$$

The topology of the space of selfadjoint extensions is non-trivial

$$\pi_1 \left[\mathcal{U}(L^2(\Gamma, \mathbb{C}^N)\right] = \mathbb{Z}$$

Maslov index :

If $U = I + K$ with $\mathrm{Tr}\ \mathrm{K}^{\dagger}\mathrm{K} < \infty$ (K Hilbert-Schmidt) $\Rightarrow$ the determinant is finite

$$\log\ \det{}' U = \mathrm{Tr}\ \log\ \frac{1+K}{e^K},$$

Cayley submanifolds. Maslov index

The Maslov index of a closed path $\gamma\colon S^1 \to \mathcal{U}(L^2(\Gamma, \mathbb{C}^N)_G$ of selfadjoint extensions is

$$\nu_M(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \partial_\vartheta \log\ \det{}'(\gamma(\vartheta)) d\vartheta$$

Theorem: The Maslov index of a closed path γ is equal to the indexed sum of crossing of γ throught the Cayley submanifold $\mathcal{C}_-$

$$\nu_M(\gamma) = \int_0^{2\pi} \partial_\vartheta n(\gamma(\vartheta)) d\vartheta$$

Topology change and edge states

The selfadjoint extensions of Δ_A may not be positive operators:

$$(\Psi_1, \Delta_A \Psi_2) = (d\ \Psi_1, d\ \Psi_2) - \left(\varphi_1, A\varphi_2\right)$$

where A is the Cayley transform of U

$$\nu_M(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \partial_\vartheta \log\ \det{}'(\gamma(\vartheta)) d\vartheta$$

Theorem: For any selfadjoint extension Δ_A^U of Δ_A with $-1 \in \mathrm{Sp}U$ and smooth eigenfunction, the family of selfadjoint extensions $\Delta_A^{U_t}$ with $U_t = U e^{it}$ and $0 < t << 2\pi$ has one negative energy level E_- which corresponds to an edge state. $E_- \to -\infty$ as $t \to 0$

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- Global theory of boundary conditions. Non trivial topology$\Rightarrow$ Cayley submanifolds
- Topology change involves an infinite amount of energy
- Edge states are associated to boundary conditions in Cayley submanifolds
- Application to Topological Field Theories and string theory: D-branes, M-branes, . . .
- Extension for Dirac operators (non-elliptic extensions)