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Abstract. In modal logics, *graded (worlds) modalities* have been deeply investigated as a useful framework for generalizing standard existential and universal modalities, in such a way they can express statements about a given number of immediately accessible worlds. These modalities have been recently investigated with respect to the propositional μ-calculus, which provide formulas exponentially more succinct, without affecting the computational complexity for the extended logic, i.e., the satisfiability problem remains solvable in EXPTIME.
A natural question that arises is how less powerful logics than μ-calculus or more complex graded modalities affect the decidability of the logic. In this paper, we investigate this question in the case of the branching-time temporal logic CTL by introducing *graded path modalities*. These modalities naturally extend to (minimal) paths the generalization induced to successor worlds by classical graded modalities, i.e., they allow to express properties such as "there exist at least n minimal paths satisfying a given formula".
As interesting results, we show that CTL extended with graded path modalities is more expressive than CTL, it retains the tree and the finite model properties, and its satisfiability problem remains decidable in EXPTIME. The latter result is obtained by exploiting an automata-theoretic approach. In particular, we introduce the class of *partitioning alternating Büchi tree automata* and show that the emptiness problems for them is EXPTIME-COMPLETE.

1 Introduction

Temporal logics are a special kind of *modal logics* that provide a formal framework for qualitatively describing and reasoning about how the truth values of a given assertion changes over time. First pointed out by Pnueli in 1977 [Pnu77], these logics turn out to be particularly suitable for reasoning about correctness of concurrent programs [Pnu81].

Depending on the view of the underlying nature of time, two types of temporal logics are mainly considered [Lam80]. In *linear-time temporal logics*, such as LTL [Pnu77], time is treated as if each moment in time has a unique possible future. Conversely, in *branching-time temporal logics*, such as CTL [CE81] and CTL^* [EH86], each moment in time may split into various possible futures and *existential* and *universal quantifiers* are used to express properties along one or all the possible futures. In modal logics, such as $\mathcal{ALC}$ [SSS91, Sch91] and μ-calculus [Koz83], this kind of quantifiers have been generalized by means of *graded (worlds) modalities* [Fin72, Tob01, vdHdR95], which allow to express properties such as "there exist at least n accessible worlds satisfying a certain formula" or "all but n accessible worlds satisfy a certain formula". For example, in a multiprocessor scheduling specification, we can express properties such as each time a computation is invoked, there are always at least two processors immediately next available for it, without naming which processors they exactly are. This generalization has been proved to be very powerful as it allows to express system specifications in a very succinct way. In some cases, the extension makes the logic much more complex. An example is the guarded fragment of first order logic, which becomes undecidable when extended with a very weak form of counting quantifiers [Grä99]. In some other cases, one can extend a logic with very strong forms of "counting quantifiers" without increasing the computational complexity of the obtained logic. For example, this is the case for the $\mu\mathcal{ALCQ}$ [GL94, GL97, BCM+03] and the graded μ-calculus [KSV02, BLMV06], for which the decidability problem is EXPTIME-COMPLETE.

A natural question that arises is how less powerful logics than μ-calculus or more complex graded modalities affect the decidability of the logic. In this paper, we investigate this question in the case of the branching-time temporal logic CTL and by introducing *graded path modalities*. These modalities naturally extend to (minimal) paths the generalization induced to successor worlds by classical graded modalities, i.e., they allow to express properties such as "there are at least n minimal paths satisfying a formula" or "all but less than n minimal paths satisfy a formula", for a suitable and well-found concept of minimality among paths. Although the extension of CTL with graded path modalities (GCTL) seems a trivial task (since μ-calculus subsumes CTL), it is not at all immediate. In fact, differently from modal logics, such as $\mathcal{ALC}$ and μ-calculus, the underlying objects of temporal logics are both states and paths. Therefore, the concept of graded can relapse on both of them, as we investigate here. Clearly, the graded path quantifiers we consider subsume the graded idea used in $\mathcal{ALC}$ and μ-calculus.

As interesting results about GCTL, we show that it is more expressive than CTL (as it becomes not invariant under bisimulation). Nevertheless, we show that this extension retains the tree and the finite model properties, as well as its satisfiability problem is in EXPTIME, thus not harder than the decidability problem for the classical CTL [EH85]. These properties make GCTL a very appealing formalism in system specification for the following reasons. Firstly, it allows to express computational system properties not expressible in CTL, in an elegant and succinct way, and still decidable in EXPTIME. For example, coming back to the above multiprocessor scheduling, we can express properties such as each time a computation is required, then there are at least two distinct (i.e., non completely equivalent) computational paths that can take care of it. Secondly, our interpretation of graded path quantifiers has some similarity with the concept of *cyclomatic complexity* defined by McCabe in a seminal work in software engineering [McC76]. He studied a way to measure the complexity of a program, identifying it in the number of independent instruction flows. With our concept of graded path quantifiers, we can specify how many minimal computational paths satisfying a given property reside in an given program. From an intuitive point of view, with our concept of graded path quantifiers, we can subsume the cyclomatic complexity introduced by McCabe, where for independent we replace minimal. Thirdly, the concept of graded path quantifiers can be extended to other logics such as dynamic logics [FL79] and hence we can obtain corresponding EXPTIME upper bounds also for them for free.

The complexity result of the satisfiability problem for GCTL is obtained by exploiting an automata-theoretic approach [VW86, SE89, KVW00]. To develop a decision procedure for a logic with the tree model property, one first develops an appropriate notion of tree automata and studies their emptiness problem, then the satisfiability problem for the logic is reduced to the emptiness problem of the automata. To this aim, we introduce a new automata model: *partitioning alternating tree automata* (PATA). While a nondeterministic automaton on visiting a node of the input tree sends exactly one copy of itself to each successor of the node, an alternating automaton can send several copies of itself to the same successor. In particular, in *symmetric alternating automata* [JW95, Wil99] it is not necessary to specify the direction of the tree on which a copy is sent. In [KSV02], *graded alternating tree automata* (GATA) are introduced as a generalization of symmetric alternating tree automata, in such a way that the automaton can send copies of itself to a given number n of state successors, either in existential or universal way, without specifying which successors these exactly are. PATA further extend GATA in such a way that the automaton can send copies of itself to a given number n of paths. As we show later, for each GCTL formula φ, it is always possible to build in linear time a PATA along with a Büchi condition (PABT) $\mathcal{A}_\varphi$ that accepts all and only the tree models of φ. The major difficulty here is that whenever φ contains graded modalities such as "there exist at least n minimal paths satisfying a path property ψ", $\mathcal{A}_\varphi$ must accept trees in which there are at least n distinct paths satisfying ψ, where some groups of those paths can arbitrarily share the same (proper) prefixes, and we have to ensure this by opportunely constraining the transition relation of the automaton. We present an EXPTIME decision procedure for the emptiness of PABT through an exponential translation into non-deterministic Büchi tree automata (NBT). In more detail, we use a technical variation of the Miyano and Hayashi technique [MH84] for tree automata [Mos84], which has been deeply used in the literature for translating alternating Büchi automata to nondeterministic ones. Then, the result follows from the fact that the emptiness problem for NBT is solvable in polynomial time [VW86].

In this paper we left to investigate more complex logics, such as CTL^*, along with graded path quantifiers, i.e., GCTL^*. We believe that also for CTL^* the extension should gain expressiveness without paying any extra cost on deciding its satisfiability, i.e., we conjecture that GCTL^* has a 2EXPTIME-COMPLETE satisfiability problem, as we motivate at the end of the paper. However, we deserve this part for future works.

Due to space limitations, most of the proofs are omitted and reported in appendix.

2 Preliminaries

Given a *set* X of *objects* (numbers, words, sets, etc.), we denote by $|\mathrm{X}|$ the number of its elements, called *size* of X, and by 2^{X} the *powerset* of X itself. In addition, by X^n we denote the set of all n-tuples of elements from X, by $\mathrm{X}^* = \bigcup_{n=0}^{\omega} \mathrm{X}^n$ the set of *finite words* on the *alphabet* X, and by X^+ the set $\mathrm{X}^* \setminus \{\varepsilon\}$, where, as it is usual, ω is the *numerable infinity* and ε is the *empty word*. With $|x|$ we indicate the *length* of a word $x \in \mathrm{X}^*$ and with $\{x_i\}_i^n$ we denote the *ordered sequence* $(x_1, \ldots, x_n) \in \mathrm{X}^+$ of objects varying on the index i. As special sets, we also consider $\mathbb{N}$ and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ as, respectively, the sets of *natural numbers* and *positive natural numbers*. Furthermore, with $\mathbb{N}_{(n)}$ and $\mathbb{N}_{(n)+}$ we denote the subsets $\{k \in \mathbb{N} \mid k \leq n\}$ of $\mathbb{N}$ and $\{k \in \mathbb{N}_+ \mid k \leq n\}$ of $\mathbb{N}_+$, where $n \in \mathbb{N} \cup \{\omega\}$.

A *structure* $\mathcal{S}$ is an ordered tuple $\langle \mathrm{X}, \mathrm{R} \rangle$, where *(i)* $\mathrm{X} = \mathsf{dom}(\mathcal{S})$ is a non-empty set of objects, called *domain* of $\mathcal{S}$, and *(ii)* $\mathrm{R} \subseteq \mathrm{X} \times \mathrm{X}$ is a *binary relation* between objects. We denote the size $|\mathcal{S}|$ of $\mathcal{S}$ as the number $|\mathrm{X}|$ of objects of its domain. An infinite structure is a structure of infinite size. When the relation R is clear from the contex, to refer to a structure we only use its domain. A *tree* is a structure $\langle \mathrm{X}, \mathrm{R} \rangle$ in which the domain X, in the following also referred as set of *nodes*, is a subset of $\mathbb{N}^*$ such that *(i)* if $x \cdot a \in \mathrm{X}$, with $x \in \mathbb{N}^*$ and $a \in \mathbb{N}$, then also $x \in \mathrm{X}$ and *(ii)* $(x, x') \in \mathrm{R}$ iff $x' = x \cdot a$, for some $a \in \mathbb{N}$. The empty word ε is the *root* of the tree. A tree is *full* iff $x \cdot a \in \mathrm{X}$, with $a \in \mathbb{N}$, implies $x \cdot b \in \mathrm{X}$, for all $b \in \mathbb{N}_{(a)}$. A *path* is a tree $\langle \mathrm{X}, \mathrm{R} \rangle$ in which for all nodes $x \in \mathrm{X}$ there is at most one $a \in \mathbb{N}$ such that $x \cdot a \in \mathrm{X}$, i.e., the transitive closure of the relation R is a linear (total) order on X. A Σ-*labeled* structure $\mathcal{S} = \langle \Sigma, \mathrm{X}, \mathrm{R}, \mathrm{L} \rangle$ is a tuple in which *(i)* Σ is a finite set of *labels*, *(ii)* $\langle \mathrm{X}, \mathrm{R} \rangle$ is a structure, and *(iii)* $\mathrm{L} : \mathrm{X} \mapsto \Sigma$ is a *labeling function* that colors each object with a label. When both Σ and R are clear from the context, we indicate a labeled structure $\langle \Sigma, \mathrm{X}, \mathrm{R}, \mathrm{L} \rangle$ with the shorter tuple $\langle \mathrm{X}, \mathrm{L} \rangle$.

Let $\mathcal{S} = \langle \mathrm{X}, \mathrm{R} \rangle$ and $\mathcal{S}' = \langle \mathrm{X}', \mathrm{R}' \rangle$ be two structures. We say that $\mathcal{S}'$ is a *substructure* of $\mathcal{S}$, in symbols $\mathcal{S}' \preccurlyeq \mathcal{S}$, iff *(i)* $\mathrm{X}' \subseteq \mathrm{X}$ and *(ii)* $\mathrm{R}' = \mathrm{R} \cap (\mathrm{X}' \times \mathrm{X}')$ hold. Moreover, we say that $\mathcal{S}$ and $\mathcal{S}'$ are *comparable* iff *(i)* $\mathcal{S} \preccurlyeq \mathcal{S}'$ or *(ii)* $\mathcal{S}' \preccurlyeq \mathcal{S}$ holds, otherwise they are *incomparable*. For a set of structures $\mathfrak{S}$, we define the set of *minimal substructures* $\mathsf{minstructs}(\mathfrak{S})$ as the set containing all and only the structures $\mathcal{S} \in \mathfrak{S}$ such that for all $\mathcal{S}' \in \mathfrak{S}$, it holds that *(i)* $\mathcal{S} \preccurlyeq \mathcal{S}'$, or *(ii)* $\mathcal{S}'$ is not in relation with $\mathcal{S}$. Note that all structures in $\mathsf{minstructs}(\mathfrak{S})$ are incomparable among them. A structure $\mathcal{S}$ is *minimal* w.r.t. a set $\mathfrak{S}$ (or simply minimal, when the context clarify the set $\mathfrak{S}$) iff $\mathcal{S} \in \mathsf{minstructs}(\mathfrak{S})$. A set of structures $\mathfrak{S}$ is minimal iff $\mathfrak{S} = \mathsf{minstructs}(\mathfrak{S})$.

A *Kripke structure* $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ is a 2^{AP}-labeled structure, where AP is a set of *atomic propositions*, $\mathrm{W} = \mathsf{dom}(\mathcal{K})$ is a set of *worlds* domain of the structure, R is a relation on W, and $\mathrm{L} : \mathrm{W} \mapsto 2^{\mathrm{AP}}$ is the labeling function that maps each world to a set of atomic propositions true in that world. Given a Kripke structure $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ and a world $w \in \mathrm{W}$, we define the *unwinding* of the structure $\mathcal{K}$ starting from w as the full and possibly infinite 2^{AP}-labeled (Kripke) tree $\mathcal{U}_w^{\mathcal{K}} = \langle \mathrm{AP}, \mathrm{W}', \mathrm{R}', \mathrm{L}' \rangle$ such that there is a function $\mathsf{uf} : \mathrm{W}' \mapsto \mathrm{W}$, called *unwinding function*, satisfying the following properties: *(i)* $\mathsf{uf}(\varepsilon) = w$ and, for all $w', v' \in \mathrm{W}'$ and $u \in \mathrm{W}$, it holds that *(ii)* $\mathrm{L}'(w') = \mathrm{L}(\mathsf{uf}(w'))$, *(iii)* if $(w', v') \in \mathrm{R}'$, then $(\mathsf{uf}(w'), \mathsf{uf}(v')) \in \mathrm{R}$, and, *(iv)* if $(\mathsf{uf}(v'), u) \in \mathrm{R}$, then there is one and only one $u' \in \mathrm{W}'$ such that $\mathsf{uf}(u') = u$ and $(v', u') \in \mathrm{R}'$. Note that the unwinding function, and so the unwinding structure, is unique up to *isomorphisms*. Given a Kripke structure $\mathcal{K}$ and a world $w \in \mathrm{W} = \mathsf{dom}(\mathcal{K})$, we define $\mathsf{paths}(\mathcal{K}, w)$ as the set of paths of $\mathcal{K}$ starting from w. Formally, a path π is in $\mathsf{paths}(\mathcal{K}, w)$ iff $\pi \preccurlyeq \mathcal{U}_w^{\mathcal{K}}$. In addition, we set $\mathsf{paths}(\mathcal{K}) = \bigcup_{w \in \mathrm{W}} \mathsf{paths}(\mathcal{K}, w)$. With $\pi(\cdot)$ we denote the function $\pi : \mathbb{N}_{(|\pi|-1)} \mapsto \mathrm{W}$ that maps each number $k \in \mathbb{N}_{(|\pi|-1)}$ with the world $\pi(k) = \mathsf{uf}(w')$ of $\mathcal{K}$, which corresponds to the $(k+1)$-st position on the path π, where uf is the unwinding function relative to $\mathcal{U}_w^{\mathcal{K}}$, $w' \in \mathsf{dom}(\pi)$, and $|w'| = k$. Note that $\pi(0) = \mathsf{uf}(\varepsilon) = w$.

Finally, let $n \in \mathbb{N}_+$, we define the following two sets: $\mathsf{P}(n)$, as the set of all *solutions* $\{p_i\}_i^n$ to the *linear Diophantine equation* $1 * p_1 + 2 * p_2 + \ldots + n * p_n = n$ and $\mathsf{CP}(n)$ as the set of the *cumulative solutions* $\{cp_i\}_i^n$ obtained by summing increasing sets of elements from $\{p_i\}_i^n$. Formally, $\mathsf{P}(n) = \{\{p_i\}_i^n \in \mathbb{N}^n \mid \sum_{i=1}^n i * p_i = n\}$ and $\mathsf{CP}(n) = \{\{cp_i\}_i^n \in \mathbb{N}^n \mid \exists \{p_i\}_i^n \in \mathsf{P}(n) \, \forall i \in \mathbb{N}_{(n)+} : cp_i = \sum_{j=i}^n p_j\}$. Note that $|\mathsf{CP}(n)| = |\mathsf{P}(n)|$

and, since for each solution $\{p_i\}_i^n$ of the above Diophantine equation there is exactly one partition of n, it holds that $|\mathsf{CP}(n)| = p(n)$, where $p(n)$ is the number of partitions of n. Now, by a classical estimation of $p(n)$ due to Hardy and Ramanujan [Apo76, SP95], we know that, for a constants α, $p(n) = \Theta(\frac{1}{n}2^{\alpha\sqrt{n}})$, so it follows that $|\mathsf{CP}(n)| = \Theta(\frac{1}{n}2^{\alpha\sqrt{n}})$.

3 The Graded CTL temporal logic

In this section, we introduce an extension of the classical branching-time temporal logics CTL with graded path quantifiers. We show that this extension allows to gain expressiveness without paying any extra cost on deciding its satisfiability. For technical convenience, we introduce this logic through the state and path framework of CTL^*.

The *graded computation tree logic* (GCTL^*, for short) extends CTL^* by using two special path quantifiers, the universal $\mathsf{A}^{<g}$ and the existential $\mathsf{E}^{\geq g}$, where $g \in \mathbb{N}$ denotes the corresponding *degree*. As in CTL^*, these path quantifiers can prefix a linear time formula composed by an arbitrary combination and nesting of the four linear temporal operators X (*"effective next"*), $\tilde{\mathsf{X}}$ (*"hypothetical next"*), U (*"until"*), and R (*"release"*). The quantifiers $\mathsf{A}^{<g}$ and $\mathsf{E}^{\geq g}$ can be respectively read as *"all but less than g minimal paths"* and *"there exist at least g minimal paths"*. In accordance with this reading, a syntactic phrase $\mathsf{A}^{<g}\psi$ is named *allbut-formula* and $\mathsf{E}^{\geq g}\psi$ is named *atleast-formula*. The syntax of GCTL^* is formally defined as follows.

Definition 1. **(Syntax)** *GCTL^** state *(φ) and* path *(ψ)* formulas *are built inductively from* AP *using the following context-free grammar, where $p \in$ AP and $g \in \mathbb{N}$:*

1. $\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{A}^{<g}\psi \mid \mathsf{E}^{\geq g}\psi,$
2. $\psi ::= \varphi \mid \neg\psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \mathsf{X}\,\psi \mid \tilde{\mathsf{X}}\,\psi \mid \psi\,\mathsf{U}\,\psi \mid \psi\,\mathsf{R}\,\psi.$

The class of GCTL^* formulas is the set of state formulas generated by the above grammar. In addition, the simpler class of GCTL formulas is obtained by forcing each temporal operator, occurring into a formula, to be coupled with a path quantifier.

For a state formula φ, we define the *degree* $\mathsf{deg}(\varphi)$ of φ as the maximum natural number g occurring among the degrees of all its path quantifiers. We assume that all such degrees are coded in unary. Accordingly, the *length* of a formula φ, denoted by $|\varphi|$, is defined inductively on the structure of φ itself in a classical way, and by also considering $|\mathsf{A}^{<g}\psi|$ and $|\mathsf{E}^{\geq g}\psi|$ to be equal to $g+1+|\psi|$. It is obvious that $\mathsf{deg}(\varphi) = O(|\varphi|)$.

We now define the semantics of GCTL^* w.r.t. a Kripke structure $\mathcal{K}$. For a world $w \in \mathsf{dom}(\mathcal{K})$, we write $\mathcal{K}, w \models \varphi$ to indicate that a state formula φ holds at w, and, for a path $\pi \in \mathsf{paths}(\mathcal{K})$, we write $\mathcal{K}, \pi, k \models \psi$ to indicate that a path formula ψ holds on π at position $k \in \mathbb{N}_{(|\pi|-1)}$. Note that, the relation $\mathcal{K}, \pi, k \models \psi$ does not hold for any point $k \in \mathbb{N}$, with $k \geq |\pi|$. For a better readability, in the semantics definition of GCTL^* we use the special set $\mathfrak{P}_A(\mathcal{K}, w, \psi)$ and its dual $\mathfrak{P}_E(\mathcal{K}, w, \psi)$, with the following meaning: $\mathfrak{P}_A(\mathcal{K}, w, \psi)$ is the set of all paths π starting in w such that all its extensions π' (including π) satisfy the path formula ψ. The semantics of GCTL^* is formally defined as follows.

Definition 2. **(Semantics)** *Given a Kripke structure $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ and $w \in \mathrm{W}$, for all GCTL^* state formulas φ, the relation $\mathcal{K}, w \models \varphi$, is inductively defined as follows.*

1. *$\mathcal{K}, w \models p$, with $p \in$ AP, iff $p \in \mathrm{L}(w)$.*

2. *For all state formulas* φ, φ_1, *and* φ_2, *it holds:*
   (a) $\mathcal{K}, w \models \neg\varphi$ *iff not* $\mathcal{K}, w \models \varphi$, *that is* $\mathcal{K}, w \not\models \varphi$*;*
   (b) $\mathcal{K}, w \models \varphi_1 \wedge \varphi_2$ *iff* $\mathcal{K}, w \models \varphi_1$ *and* $\mathcal{K}, w \models \varphi_2$*;*
   (c) $\mathcal{K}, w \models \varphi_1 \vee \varphi_2$ *iff* $\mathcal{K}, w \models \varphi_1$ *or* $\mathcal{K}, w \models \varphi_2$.
3. *For a path formula* ψ *and a natural number g, it holds:*
   (a) $\mathcal{K}, w \models \mathsf{A}^{<g}\psi$ *iff* $|\mathsf{minstructs}(\mathsf{paths}(\mathcal{K}, w) \setminus \mathfrak{P}_E(\mathcal{K}, w, \psi))| < g$*;*
   (b) $\mathcal{K}, w \models \mathsf{E}^{\geq g}\psi$ *iff* $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \psi))| \geq g$*;*
   where it is set that $\mathfrak{P}_A(\mathcal{K}, w, \psi) = \{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \forall \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi'$ *implies* $\mathcal{K}, \pi', 0 \models \psi\}$ *and* $\mathfrak{P}_E(\mathcal{K}, w, \psi) = \{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \exists \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi'$ *and* $\mathcal{K}, \pi', 0 \models \psi\}$.

For all GCTL path formulas* ψ*, paths* $\pi \in \mathsf{paths}(\mathcal{K})$*, and natural numbers* $k < |\pi|$*, the relation* $\mathcal{K}, \pi, k \models \psi$ *is inductively defined as follows.*

4. $\mathcal{K}, \pi, k \models \varphi$*, with* φ *state formula, iff* $\mathcal{K}, \pi(k) \models \varphi$.
5. *Where* ψ, ψ_1, *and* ψ_2 *are path formulas, we have:*
   (a) $\mathcal{K}, \pi, k \models \neg\psi$ *iff not* $\mathcal{K}, \pi, k \models \psi$, *that is* $\mathcal{K}, \pi, k \not\models \psi$*;*
   (b) $\mathcal{K}, \pi, k \models \psi_1 \wedge \psi_2$ *iff* $\mathcal{K}, \pi, k \models \psi_1$ *and* $\mathcal{K}, \pi, k \models \psi_2$*;*
   (c) $\mathcal{K}, \pi, k \models \psi_1 \vee \psi_2$ *iff* $\mathcal{K}, \pi, k \models \psi_1$ *or* $\mathcal{K}, \pi, k \models \psi_2$.
6. *Where* ψ, ψ_1, *and* ψ_2 *path formulas, we have:*
   (a) $\mathcal{K}, \pi, k \models \mathsf{X}\,\psi$ *iff* $k < |\pi| - 1$ *and* $\mathcal{K}, \pi, (k+1) \models \psi$*;*
   (b) $\mathcal{K}, \pi, k \models \tilde{\mathsf{X}}\,\psi$ *iff* $k = |\pi| - 1$ *or* $\mathcal{K}, \pi, (k+1) \models \psi$*;*
   (c) $\mathcal{K}, \pi, k \models \psi_1 \mathsf{U}\, \psi_2$ *iff there exists an index i, with* $k \leq i < |\pi|$*, such that* $\mathcal{K}, \pi, i \models \psi_2$ *and, for all indexes j with* $k \leq j < i$*, it holds* $\mathcal{K}, \pi, j \models \psi_1$*;*
   (d) $\mathcal{K}, \pi, k \models \psi_1 \mathsf{R}\, \psi_2$ *iff for all indexes i, with* $k \leq i < |\pi|$*, it holds* $\mathcal{K}, \pi, i \models \psi_2$ *or there exists an index j with* $k \leq j < i$*, such that* $\mathcal{K}, \pi, j \models \psi_1$.

Remark 1. G*CTL** (resp., GCTL) formulas with degrees 1 are *CTL** (resp., CTL) formulas.

Remark 2. The inner definition of $\mathfrak{P}_A(\mathcal{K}, w, \psi)$ and $\mathfrak{P}_E(\mathcal{K}, w, \psi)$, formally stated that they are dual of each other, i.e., $\mathfrak{P}_A(\mathcal{K}, w, \psi) = \mathsf{paths}(\mathcal{K}, w) \setminus \mathfrak{P}_E(\mathcal{K}, w, \neg\psi)$.

For all state formulas φ_1 and φ_2 (resp., path formulas ψ_1 and ψ_2), we say that φ_1 is *equivalent* to φ_2, formally $\varphi_1 \equiv \varphi_2$, (resp., ψ_1 is *equivalent* to ψ_2, formally $\psi_1 \equiv \psi_2$) iff for all Kripke structures $\mathcal{K}$ and worlds $w \in \mathsf{dom}(\mathcal{K})$ it holds that $\mathcal{K}, w \models \varphi_1$ iff $\mathcal{K}, w \models \varphi_2$ (resp., $\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \psi_1)) = \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \psi_2))$).

In the rest of the paper, we only consider formulas in *existential normal form* or in *positive normal form*, i.e., formulas in which only existential quantifiers occur or negation is applied only to atomic propositions, respectively. In fact, it is to this aim that we have considered in the syntax of G*CTL** both the connectives $\wedge$ and $\vee$, the quantifiers $\mathsf{A}^{<g}$ and $\mathsf{E}^{\geq g}$, and the dual operators $\tilde{\mathsf{X}}$ and R. Indeed, all formulas can be converted in existential or positive normal form by using De Morgan's laws and the following equivalences, which directly follow from the semantics of the logic. Let ψ, ψ_1, and ψ_2 be path formulas and $g \in \mathbb{N}$, it holds that $\neg\mathsf{A}^{<g}\psi \equiv \mathsf{E}^{\geq g}\neg\psi$, $\neg\mathsf{X}\,\psi \equiv \tilde{\mathsf{X}}\,\neg\psi$, and $\neg(\psi_1 \mathsf{U}\, \psi_2) \equiv \neg\psi_1 \mathsf{R}\, \neg\psi_2$. In order to abbreviate writing formulas we also use the boolean values $\mathfrak{t}$ (*"true"*) and $\mathfrak{f}$ (*"false"*) and the path quantifiers $\mathsf{E}\psi \equiv \mathsf{E}^{\geq 1}\psi$ (*"there is a minimal path"*) and $\mathsf{E}^{>g}\psi \equiv \mathsf{E}^{\geq g+1}\psi$ (*"there are more than one minimal path"*).

The following lemma shows some interesting equivalences among GCTL formulas that will be useful to prove important properties of this introduced logic. In particular, we show fixed point equivalences that extend to "graded" formulas the well known analogous ones for "ungraded" formulas.

Lemma 1. *For all state formulas* φ_1 *and* φ_2 *and degrees* $g > 1$*, it holds that:*

$$i \begin{cases} \mathsf{E}(\varphi_1 \mathsf{U} \varphi_2) & \equiv \varphi_2 \lor \varphi_1 \land \mathsf{ex}(\varphi_1 \mathsf{U} \varphi_2, 1) \\ \mathsf{E}^{\geq g}(\varphi_1 \mathsf{U} \varphi_2) & \equiv \neg\varphi_2 \land \varphi_1 \land \mathsf{ex}(\varphi_1 \mathsf{U} \varphi_2, g) \end{cases}$$

$$ii \begin{cases} \mathsf{E}(\varphi_1 \mathsf{R} \varphi_2) & \equiv \varphi_2 \land (\varphi_1 \lor \mathsf{E}\tilde{\mathsf{X}}\mathfrak{f} \lor \mathsf{ex}(\varphi_1 \mathsf{R} \varphi_2, 1)) \\ \mathsf{E}^{\geq g}(\varphi_1 \mathsf{R} \varphi_2) & \equiv \varphi_2 \land \neg\varphi_1 \land \mathsf{EX}\,\mathsf{E}\neg(\varphi_1 \mathsf{R} \varphi_2) \land \mathsf{ex}(\varphi_1 \mathsf{R} \varphi_2, g) \end{cases}$$

where $\mathsf{ex}(\psi, g) = \bigvee_{\{h_i\}_i^g \in \mathsf{CP}(g)} \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$.

Remark 3. The function $\mathsf{ex}(\psi, g)$ used in the above lemma allows to partition g paths trough h_1 successor worlds, for a given sequence $\{h_i\}_i^g \in \mathsf{CP}(g)$. Indeed, h_i is the number of successor worlds from which at least i paths satisfying ψ start. Therefore, h_1 is a right bound on the number of successor worlds we have to consider to ensure the satisfiability of the formula. By a simple calculation, it follows also that $|\mathsf{ex}(\psi, g)| = g * (|\psi| + \frac{g+11}{2}) * |\mathsf{CP}(g)| - 1 = \Theta((|\psi| + \frac{g}{2}) * 2^{\alpha\sqrt{g}})$, for a constant α.

Remark 4. For $g = 1$, Lemma 1 gives the two classical fixed point expansions for CTL: $\mathsf{E}(\varphi_1 \mathsf{U} \varphi_2) \equiv \varphi_2 \lor \varphi_1 \land \mathsf{EX}\,\mathsf{E}(\varphi_1 \mathsf{U} \varphi_2)$ and $\mathsf{E}(\varphi_1 \mathsf{R} \varphi_2) \equiv \varphi_2 \land (\varphi_1 \lor \mathsf{E}\tilde{\mathsf{X}}\mathfrak{f} \lor \mathsf{EX}\,\mathsf{E}(\varphi_1 \mathsf{R} \varphi_2))$.

Let $\mathcal{K}$ be a Kripke structure and φ a $GCTL^*$ formula. Then, $\mathcal{K}$ is a *model* for φ, denoting this by $\mathcal{K} \models \varphi$, iff there is $w \in \mathsf{dom}(\mathcal{K})$ such that $\mathcal{K}, w \models \varphi$. In this case, we also say that $\mathcal{K}$ is a model for φ on w. A $GCTL^*$ formula φ is said *satisfiable* iff there exists a model for it, moreover it is *invariant* on the two Kripke structures $\mathcal{K}$ and $\mathcal{K}'$ iff either $\mathcal{K} \models \varphi$ and $\mathcal{K}' \models \varphi$ or $\mathcal{K} \not\models \varphi$ and $\mathcal{K}' \not\models \varphi$.

By showing an exponential reduction of GCTL to the graded μ-calculus[1] and by using the fact that for the latter the satisfiability problem is solvable in EXPTIME [KSV02], we immediately get that the satisfiability problem for GCTL is decidable and solvable in 2EXPTIME. This result is reported in Theorem 1. However, in the next section we improve this result by showing that the satisfiability problem for GCTL is solvable in EXPTIME, by exploiting an automata-theoretic approach that deeply makes use of the idea behind the above function $\mathsf{ex}(\psi, g)$, without using it explicitly.

Theorem 1. *The satisfiability problem for the GCTL logic is decidable and in particular solvable in 2EXPTIME.*

We conclude this section by showing some intersting properties about GCTL. First of all, by using a proof by induction we can show that this logic is invariant under the unwinding of a model. Directly from this, we get that it also enjoys the tree model property. Moreover, by extending a technique introduced in [EH85] along with Lemma 1, for each GCTL formula φ it is possible to build an Hintikka structure from which we can get a finite model for φ. By means a counterexample, we can also show that GCTL

[1] The μ-calculus is a well-known modal logic augmented with fixed point operators [Koz83]. The graded μ-calculus extends the μ-calculus with graded state quantifiers [KSV02, BLMV06].

is not invariant under bisimulation among models, so directly from this, we obtain that it is more expressive than CTL, since the latter is invariant under bisimulation. All these properties are reported in the next theorem.

Theorem 2. *For GCTL it holds that it* (i) *is invariant under unwinding;* (ii) *has the tree model property;* (iii) *has the finite model property;* (iv) *is not invariant under bisimulation; and* (v) *is more expressive than CTL.*

4 Partitioning Büchi Tree Automata

Nondeterministic automata on infinite trees are an extension of nondeterministic automata on infinite words and finite trees (see [Tho90] for an introduction). *Alternating automata* [MS87] are a generalization of nondeterministic automata that embody the same concept of alternation of Turing machines [CKS81]. Intuitively, while a nondeterministic automaton that visits a node of the input tree sends exactly one copy of itself to each of the successors of the node, an alternating automaton can send several copies of itself to the same successor. *Symmetric automata* [JW95, Wil99] are a variation of classical (asymmetric) alternating automata in which it is not necessary to specify the direction of the tree on which a copy is sent. In fact, through three generalized directions (ε-moves, existential moves, and universal moves), it is possible to send a copy of the automaton on a node of the input tree to the same node, to some of its successors, or to all its successors, so the automaton cannot distinguish between directions. As a generalization of symmetric automata *graded alternating tree automata* (*GATA*, for short) have also been introduced [KSV02]. In this framework, the automaton can send copies of itself to a given number n of successors, either in existential or universal way, without specifying which successors these exactly are. Moreover, a GATA can also send a copy of itself to the reading node by pursuing an ε-move.

Here, we consider *partitioning alternating tree automata* (*PATA*, for short) as a generalization of GATA in such a way that the automaton can send copies of itself to a given number n of paths starting from the current node. As we show later, for each GCTL formula φ, it is possible to built a PATA that accepts all and only the tree models of φ. The key idea is to extend GATA's runs by also labeling their nodes with a natural number, with the aim of collecting "graded path information". We give an idea on how a PATA $\mathcal{A}$ works w.r.t. the logic GCTL through an example.

First, consider that $\mathcal{A}$ uses as states all possible subformulas of the considered formula[2]. Now, suppose that the automaton is in the node x of an input tree T and in state $\mathsf{E}^{\geq g}\psi$, where ψ is also a GCTL path formula, then in a state corresponding to ψ, the automaton sends $n \leq g$ copies of itself to n successors of x with degrees $\{g_1, \ldots, g_n\}$ that sum to g. One can note that this sequence of n degrees is a partition of the number g. The degree g_i associated to a successor x_i of x denotes that at least g_i paths starting from x_i have to satisfy ψ and the automaton take care of it through the transition function. In more details, we individuate the set of n directions relative to successors of x w.r.t. the degrees $\{g_1, \ldots, g_n\}$ by means of a decreasing chain $\{M_1, \ldots, M_{n+1}\}$, such that for each

[2] More precisely, the automaton uses as states an extended definition of the Fischer-Ladner. See proof of Theorem 3 for a formal definition.

i, it holds that $M_i \setminus M_{i+1}$ contains all directions of x that are associated with a degree i. Clearly, there could be different possible chains satisfying such a property and each one induces a different run of $\mathcal{A}$ on T. As a particular case, $\mathcal{A}$ sends g copies of itself to g distinct successors of x on choosing $|M_1| = g$ and, for each $i > 1$, $M_i = \emptyset$.

The formal definition of a PATA along with the Büchi acceptance condition follows. In particular, we give a definition without any constraint on the use of its labeling degrees, which allows to introduce a more general class of automata, independently from the logic we consider here. Note that by the definition we give, the automaton at its own cannot constraint that multiple successors in which it is sent are all distinct. However, we can force this by means of the transition function. First, we introduce some extra notation. With $\mathtt{B}^+(\mathrm{X})$ we denote the sets of *positive Boolean formulas* over X (i.e., Boolean formulas built from elements in X using $\wedge$ and $\vee$) where we also allow the formulas $\mathfrak{t}$ (true) and $\mathfrak{f}$ (false). For a set $\mathrm{X}' \subseteq \mathrm{X}$ and a formula $\phi \in \mathtt{B}^+(\mathrm{X})$, we say that X' satisfies ϕ, $\mathrm{X}' \models \phi$, iff the assigning of true to elements in X' and false to elements in $\mathrm{X} \setminus \mathrm{X}'$ makes ϕ true. With D_b and D_b^ε we denote the sets $\{\Diamond, \Box\} \times \mathbb{N}_{(b)+}$ and $D_b \cup \{\varepsilon\}$, respectively. Intuitively, these two sets represent the generalized directions that one can use, through the transition function, to define the behavior of the automaton.

Definition 3. (PABT) *A* partitioning alternating Büchi tree automaton *is a tuple* $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F \rangle$*, where* Q *is a finite set of states,* Σ *is a finite input alphabet,* $b \in \mathbb{N}$ *is a counting branching bound,* $\delta : Q \times \mathbb{N}_{(b)} \times \Sigma \mapsto \mathtt{B}^+(D_b^\varepsilon \times Q)$ *is a transition function,* $q_0 \in Q$ *is an initial state,* $g_0 \in \mathbb{N}$ *is an initial branching degree, and* $F \subseteq Q \times \mathbb{N}_{(b)}$ *is a Büchi acceptance condition, which will be defined later.*

The behavior of a PABT is described by means of a run. As for classical alternating automata, given a PABT $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F \rangle$ and a Σ-labeled tree $\langle \mathsf{T}, \mathsf{inp} \rangle$ in input, a run $\langle \mathsf{T_r}, \mathsf{run} \rangle$ of $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp} \rangle$ is induced by the sets of pairs $S \subseteq D_b^\varepsilon \times Q$ satisfying its transition function δ. Here, we first give an intuition of such a run through an example. Suppose that $\mathcal{A}$, while reading a node x of T labeled with σ is in a state q with degree g, at the node y of the run, and $\delta(q, g, \sigma) = (\varepsilon, q_1) \wedge ((\langle 3 \rangle, q_2) \vee ([2], q_3)$. Also, suppose that x has three successors $\{x \cdot 0, x \cdot 1, x \cdot 2\}$. Consider now $S = \{(\varepsilon, q_1), (\langle 3 \rangle, q_2)\}$ satisfying $\delta(q, g, \sigma)$. Accordingly, the automaton can send a copy of itself to node x in the state q_1 (by performing an ε-move) and three copies of itself in the state q_2 to three paths through either one, two, or all successors of x. Now, suppose that we want to send two copies of $\mathcal{A}$ through one successor and one through another. This can be characterized by taking $M_1 = \{0, 1\}$, $M_2 = \{1\}$, and $M_3 = M_4 = \emptyset$. Consequently, the run must have three successors $\{y \cdot 0, y \cdot 1, y \cdot 2\}$, one labeled with $(x, q_1, 0)$ (for the ε-move), another labeled with $(x \cdot 0, q_2, 1)$, and the last one labeled with $(x \cdot 1, q_2, 2)$.

We now give the formal definition of a run. To this aim, we first formally define the sets $\{M_i\}_i^{g+1}$ introduced above, through a function spart. Then, we introduce a function exec that makes us able to construct all the possible execution steps. For brevity, we often write $\langle g \rangle$ and $[g]$ instead of $(\Diamond, g)$ and $(\Box, g)$, respectively.

Definition 4. (Splitting partition function) *A* splitting partition function $\mathsf{spart} : (D, d) \in 2^{\mathbb{N}} \times D_b \mapsto \mathsf{spart}(D, d) \in 2^{(2^{\mathbb{N}})^+}$ *maps a set* D *and a direction* d *into a set of decreasing chains* $\{M_i\}_i$ *of subset of* D *(*$M_i \subseteq D$ *and* $M_i \supseteq M_{i+1}$*) such that:*

1. *if $d = \langle g \rangle$, then for all $\{M_i\}_i^{g+1} \in \mathsf{spart}(D,d) \subseteq (2^D)^{g+1}$, it holds that $M_{g+1} = \emptyset$ and there is a sequence $\{h_i\}_i^g \in \mathsf{CP}(g)$ such that $|M_j| = h_j$, for all $j \in \mathbb{N}_{(g)+}$;*
2. *if $d = [g]$, then for all $\{M_i\}_i^{g+1} \in \mathsf{spart}(D,d) \subseteq (2^D)^{g+1}$, it holds that $M_1 = D$ and for all sequences $\{h_i\}_i^g \in \mathsf{CP}(g)$ there is $j \in \mathbb{N}_{(g)+}$ such that $|M_{j+1}| < h_j$.*

Differently form the GABT case, one can see that in general the sets $\mathsf{spart}(D, \langle g \rangle)$ and $\mathsf{spart}(D, [g])$ are not the dual of each other. This is due to the fact that in PABT, for a considered node x, we may want to check properties along paths starting in x, instead of just looking at the successors of x, as it is done in GABT. This induces, in the $d = \langle g \rangle$ case, to take care of just g paths (on which we check that a certain property holds), while in the $d = [g]$ case we have to take care of all paths (i.e., that in less than g paths the property may or may not hold, while in all the remaining ones it must hold).

We now give the formal definition of the function exec. $\mathbb{N}_\varepsilon$ denotes the set $\mathbb{N} \cup \{\varepsilon\}$.

Definition 5. (Execution function) *An* execution function $\mathsf{exec} : (S,D) \in 2^{D_b^\varepsilon \times Q} \times 2^{\mathbb{N}} \mapsto \mathsf{exec}(S,D) \in 2^{2^{\mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}}}$ *maps the two sets S and D into the set of all possible subset of $\mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}$, called* configurations *of the execution, such that, for all sets $E \in 2^{\mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}}$ we have $E \in \mathsf{exec}(S,D)$ iff for all pairs $(d,q) \in S$ it holds that:*

1. *if $d = \varepsilon$ then $(\varepsilon, q, 0) \in E$;*
2. *if either $d = \langle g \rangle$ or $d = [g]$ then there exists a sequence $\{M_i\}_i^{g+1} \in \mathsf{spart}(D,d)$ such that for all indexes $i \in \mathbb{N}_{(g)+}$ and direction $x \in M_i \setminus M_{i+1}$, it holds that $(x,q,i) \in E$.*

The above function exec allows us to give the following definition of PABT's run in a very concise and elegant way. First, we introduce the following extra notation. Let $\mathrm{X}' \subseteq \mathrm{X}^*$ be a set of words on X and $x \in \mathrm{X}^*$. Then, we denote by $\mathsf{succ}_{\mathrm{X}'}(x)$ the set of *successor words* of x in X', i.e., $\mathsf{succ}_{\mathrm{X}'}(x) = \{x \cdot a \in \mathrm{X}' \mid a \in \mathbb{N}\}$ and by $\mathsf{dir}_{\mathrm{X}'}(x)$ the set of *direction* of x in X', i.e., $\mathsf{dir}_{\mathrm{X}'}(x) = \{a \in \mathbb{N} \mid x \cdot a \in \mathrm{X}'\}$. Now, let $f : \mathrm{X}' \mapsto \mathrm{X}''$. We use $\mathsf{inf}(f)$ to refer to the set $\{x' \in \mathrm{X}' \mid |f^{-1}(x')| = \omega\}$, i.e., the set of elements of X' that f uses infinitely often as labels for elements in X, and $f_{|\mathrm{X}'''}$ to indicate the restriction of f to X''', i.e., $f_{|\mathrm{X}'''} : \mathrm{X}''' \mapsto \mathrm{X}''$, where $\mathrm{X}''' \subseteq \mathrm{X}'$. In the following we also write $S \models \delta(q,g,\sigma)$ to denote that S is a set of tuples $(d,q) \in D_b^\varepsilon \times Q$ that satisfies $\delta(q,g,\sigma)$.

Definition 6. (Run of a PABT) *A* run *of a PABT $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F \rangle$ on a Σ-labeled tree $\langle \mathsf{T}, \mathsf{inp} \rangle$ is a $(\mathsf{T} \times Q \times \mathbb{N}_{(b)})$-labeled full tree $\langle \mathsf{T}_\mathsf{r}, \mathsf{run} \rangle$ satisfying the following conditions:*

1. *$\mathsf{run}(\varepsilon) = (\varepsilon, q_0, g_0)$;*
2. *for all $y \in \mathsf{T}_\mathsf{r}$ with $\mathsf{run}(y) = (x,q,g)$, there exist a set $S \subseteq D_b^\varepsilon \times Q$, where $S \models \delta(q,g,\mathsf{inp}(x))$, and a set $E \in \mathsf{exec}(S, \mathsf{dir}_\mathsf{T}(x))$ such that for all configurations $(d,q', g') \in E$ there is a node $y' \in \mathsf{succ}_{\mathsf{T}_\mathsf{r}}(y)$ such that $\mathsf{run}(y') = (x \cdot d, q', g')$.*

The run $\langle \mathsf{T}_\mathsf{r}, \mathsf{run} \rangle$ is accepting *iff all its infinite paths satisfy the acceptance condition, i.e., for all paths $\pi \preccurlyeq \mathsf{T}_\mathsf{r}$, with $|\pi| = \omega$, it holds that $\mathsf{inf}(\mathsf{run}_{|\pi}) \cap \mathsf{T} \times F \neq \emptyset$. A tree $\langle \mathsf{T}, \mathsf{inp} \rangle$ is accepted by $\mathcal{A}$ iff there is an accepting run of $\mathcal{A}$ on it. With $\mathcal{L}(\mathcal{A})$ we denote the language accepted by the automaton $\mathcal{A}$, i.e., the set of all input trees that $\mathcal{A}$ accepts.*

By extending a construction given in [KVW00], we obtain the following result.

Theorem 3. *Given a GCTL formulas φ with degree b, we can construct in time $O(|\varphi|)$ a PABT $\mathcal{A}_\varphi$, with $O(|\varphi|)$ states and counting branching bound b, such that $\mathcal{L}(\mathcal{A}_\varphi)$ is exactly the set of all of tree models of φ.*

Proof. (Sketch.) The automaton $\mathcal{A}_\varphi$ accepts trees at whose root φ holds. As set of states for $\mathcal{A}_\varphi$ we use an extended *Fisher-Ladner* closure $\mathsf{ecl}(\varphi)$ of φ defined as follows. First, we recall the classical definition of *Fischer-Ladner closure* $\mathsf{cl}(\varphi)$ of φ [FL79], i.e., the set of all state formulas contained in φ (including φ). Let $g \in \mathbb{N}_+$, $\mathtt{Qnt} \in \{\mathsf{E}^{\geq g}, \mathsf{A}^{<g}\}$, $\mathtt{Op} \in \{\wedge, \vee\}$, $\mathtt{Op}' \in \{\mathsf{X}, \tilde{\mathsf{X}}\}$ and $\mathtt{Op}'' \in \{\mathsf{U}, \mathsf{R}\}$ we have: *(i)* $\varphi \in \mathsf{cl}(\varphi)$, *(ii)* if $\varphi_1 \mathtt{Op} \varphi_2 \in \mathsf{cl}(\varphi)$ then $\varphi_1, \varphi_2 \in \mathsf{cl}(\varphi)$, *(iii)* if $\mathtt{Qnt}\,\mathtt{Op}'\varphi' \in \mathsf{cl}(\varphi)$ then $\varphi' \in \mathsf{cl}(\varphi)$, and *(iv)* if $\mathtt{Qnt}\,(\varphi_1 \mathtt{Op}''\varphi_2) \in \mathsf{cl}(\varphi)$ then $\varphi_1, \varphi_2 \in \mathsf{cl}(\varphi)$. Let $\natural\varphi'$ denote the GCTL formula in positive normal form equivalent to $\neg\varphi'$. The extended closure $\mathsf{ecl}(\varphi)$ satisfies all the above properties of $\mathsf{cl}(\varphi)$ and additionally it satisfies the following: for all $g \in \mathbb{N}_+$, $\mathtt{Op} \in \{\mathsf{X}, \tilde{\mathsf{X}}\}$, and ψ until or release GCTL path formula, it holds that *(i)* if $\mathsf{E}^{\geq g}\mathtt{Op}\,\varphi' \in \mathsf{ecl}(\varphi)$ then $\langle\varphi'\rangle, \langle\natural\varphi'\rangle \in \mathsf{ecl}(\varphi)$, *(ii)* if $\mathsf{A}^{<g}\mathtt{Op}\,\varphi' \in \mathsf{ecl}(\varphi)$ then $[\varphi'], [\natural\varphi'] \in \mathsf{ecl}(\varphi)$, *(iii)* if $\mathsf{E}^{\geq g}\psi \in \mathsf{ecl}(\varphi)$ then $\langle\psi\rangle, \langle\natural\psi\rangle \in \mathsf{ecl}(\varphi)$, *(iv)* if $\mathsf{A}^{<g}\psi \in \mathsf{ecl}(\varphi)$ then $[\psi], [\natural\psi] \in \mathsf{ecl}(\varphi)$, *(v)* if $\langle\varphi_1 \mathsf{U} \varphi_2\rangle$ or $[\varphi_1 \mathsf{R} \varphi_2]$ are in $\mathsf{ecl}(\varphi)$ then $\natural\varphi_2 \in \mathsf{ecl}(\varphi)$, and *(iv)* if $\langle\varphi_1 \mathsf{R} \varphi_2\rangle$ or $[\varphi_1 \mathsf{U} \varphi_2]$ are in $\mathsf{ecl}(\varphi)$ then $\natural\varphi_1 \in \mathsf{ecl}(\varphi)$. It is obvious that $|\mathsf{ecl}(\varphi)| = O(|\varphi|)$.

We define $\mathcal{A}_\varphi$ as $\langle \mathsf{ecl}(\varphi), 2^{\mathrm{AP}}, \mathsf{deg}(\varphi), \delta, \varphi, 0, F\rangle$, where the acceptance condition F is the set of all pairs $(\langle\varphi_1 \mathsf{R} \varphi_2\rangle, 1)$ and $([\varphi_1 \mathsf{R} \varphi_2], 1)$ in $\mathsf{ecl}(\varphi) \times \mathbb{N}_{(b)}$. It remains to define the transition function δ. Mainly, it extends the transition function introduced in [KVW00] for *CTL* along with the extra graded path modalities. Before giving the formal definition, we show an intuition of the δ through a couple of examples.

First, recall that δ is a function from $\mathsf{ecl}(\varphi) \times \mathbb{N}_{(b)} \times 2^{\mathrm{AP}}$ into $\mathtt{B}^+(D_b^\varepsilon \times \mathsf{ecl}(\varphi))$. Consider the state formula $\varphi = \mathsf{E}^{\geq g}\mathsf{X}\,\varphi'$. This formula is true on a tree model rooted at a node x having at least g distinct successors of x satisfying φ'. This is ensured through the δ in two successive steps. First, starting form the state $\mathsf{E}^{\geq g}\mathsf{X}\,\varphi'$, the δ gives the formula $(\langle g\rangle, \langle\varphi'\rangle)$, which intends to send to g successors (not necessarily distinct) the check of the satisfiability of φ'. Then from state $\langle\varphi'\rangle$ we have to ensure that each of such successor nodes, say it y, contributes to the satisfiability of exactly one φ' (intuitively one degree of φ). Therefore, on reading y, if the degree associated with the state $\langle\varphi'\rangle$ is greater then 1, the δ returns false, otherwise, with an ε-move, we move to state φ'. Accordingly, in the δ we use as counting branching positive numbers to indicate formulas' degrees which have to be accomplished along paths and use as a convention 0 if we have none to accomplish. In particular, ε-moves always give 0 as counting branching.

As another example, consider the state formula $\varphi = \mathsf{E}^{\geq g}(\varphi_1 \mathsf{U} \varphi_2)$. This formula is true on a tree model rooted at a node x having at least g distinct minimal paths satisfying $\varphi_1 \mathsf{U} \varphi_2$. As in classical temporal logics, the until path formula $\varphi_1 \mathsf{U} \varphi_2$ is true on a path if φ_2 is immediately true, or φ_1 is immediately true and the until is satisfied on the successor node. Moreover, in the considered graded path case of φ we have to ensure that all the minimal paths that satisfy φ are at least g. Therefore, if $g = 1$ the δ proceeds as in *CTL*. Conversely, if $g > 1$ we have to force that φ_2 is not immediately true (otherwise we have less than g paths satisfying the formula). Therefore, we use the δ to ensure that φ_1 is immediately true and that g successive paths (but not necessarily all distinct) satisfy $\varphi_1 \mathsf{U} \varphi_2$. Iteratively, the δ keeps using the above idea up to all states corresponding to the until formula are sent to next nodes with counting branching 1.

This ensures that the considered tree model has at least g minimal paths satisfying the until formula $\varphi_1 \mathsf{U} \varphi_2$. Note that if less then g of such paths exist in the tree model, then the automaton keeps regenerating infinitely often the state corresponding to the until formula. Such a tree is then not accepting as this state is not in F. It is worth noticing that, the above iteration upon the until states inherits the fixed point idea of the function $\mathsf{ex}(\psi, g)$ introduced in Lemma 1. In particular we formally inglobe it into the δ through the formula $(\langle 1 \rangle, \langle \varphi_1 \mathsf{U} \varphi_2 \rangle)$ (see below for more details). This is a key step in our construction, since it allows to treat the exponential blow-up induced by the mentioned function by only using a constant rule into the δ. The formal definition of the δ follows. For all $\sigma \in 2^{\mathrm{AP}}$ and $g, h \in \mathbb{N}_{(b)+}$, with $h \neq 1$, we set:

- $\delta(\mathfrak{t}, g, \sigma) = \mathfrak{t}$
- $\delta(p, 0, \sigma) = (p \in \sigma)$
- $\delta(\varphi_1 \wedge \varphi_2, 0, \sigma) = (\varepsilon, \varphi_1) \wedge (\varepsilon, \varphi_2)$
- $\delta(\mathsf{E}^{\geq g} \mathsf{X} \varphi, 0, \sigma) = (\langle g \rangle, \langle \varphi \rangle)$
- $\delta(\mathsf{E} \tilde{\mathsf{X}} \varphi, 0, \sigma) = ([1], \mathfrak{f}) \vee (\langle 1 \rangle, \langle \varphi \rangle)$
- $\delta(\mathsf{E}^{\geq h} \tilde{\mathsf{X}} \varphi, 0, \sigma) = (\langle 1 \rangle, \langle \natural \varphi \rangle) \wedge (\langle h \rangle, \langle \varphi \rangle)$
- $\delta(\langle \varphi \rangle, 1, \sigma) = (\varepsilon, \varphi)$
- $\delta(\langle \varphi \rangle, h, \sigma) = \mathfrak{f}$
- $\delta(\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U} \varphi_2), 0, \sigma) = \delta(\langle \varphi_1 \mathsf{U} \varphi_2 \rangle, g, \sigma)$
- $\delta(\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R} \varphi_2), 0, \sigma) = \delta(\langle \varphi_1 \mathsf{R} \varphi_2 \rangle, g, \sigma)$
- $\delta(\mathfrak{f}, g, \sigma) = \mathfrak{f}$
- $\delta(\neg p, 0, \sigma) = (p \notin \sigma)$
- $\delta(\varphi_1 \vee \varphi_2, 0, \sigma) = (\varepsilon, \varphi_1) \vee (\varepsilon, \varphi_2)$
- $\delta(\mathsf{A}^{<g} \tilde{\mathsf{X}} \varphi, 0, \sigma) = ([g], [\varphi])$
- $\delta(\mathsf{A} \mathsf{X} \varphi, 0, \sigma) = (\langle 1 \rangle, \mathfrak{t}) \wedge ([1], [\varphi])$
- $\delta(\mathsf{A}^{<h} \mathsf{X} \varphi, 0, \sigma) = ([1], [\natural \varphi]) \vee ([h], [\varphi])$
- $\delta([\varphi], 1, \sigma) = (\varepsilon, \varphi)$
- $\delta([\varphi], h, \sigma) = \mathfrak{t}$
- $\delta(\mathsf{A}^{<g}(\varphi_1 \mathsf{U} \varphi_2), 0, \sigma) = \delta([\varphi_1 \mathsf{U} \varphi_2], g, \sigma)$
- $\delta(\mathsf{A}^{<g}(\varphi_1 \mathsf{R} \varphi_2), 0, \sigma) = \delta([\varphi_1 \mathsf{R} \varphi_2], g, \sigma)$
- $\delta(\langle \varphi_1 \mathsf{U} \varphi_2 \rangle, 1, \sigma) = (\varepsilon, \varphi_2) \vee (\varepsilon, \varphi_1) \wedge (\langle 1 \rangle, \langle \varphi_1 \mathsf{U} \varphi_2 \rangle)$
- $\delta(\langle \varphi_1 \mathsf{U} \varphi_2 \rangle, h, \sigma) = (\varepsilon, \natural \varphi_2) \wedge (\varepsilon, \varphi_1) \wedge (\langle h \rangle, \langle \varphi_1 \mathsf{U} \varphi_2 \rangle)$
- $\delta([\varphi_1 \mathsf{U} \varphi_2], 1, \sigma) = (\varepsilon, \varphi_2) \vee (\varepsilon, \varphi_1) \wedge (\langle 1 \rangle, \mathfrak{t}) \wedge ([1], [\varphi_1 \mathsf{U} \varphi_2])$
- $\delta([\varphi_1 \mathsf{U} \varphi_2], h, \sigma) = (\varepsilon, \varphi_2) \vee (\varepsilon, \natural \varphi_1) \vee ([1], [\natural(\varphi_1 \mathsf{U} \varphi_2)]) \vee ([h], [\varphi_1 \mathsf{U} \varphi_2])$
- $\delta(\langle \varphi_1 \mathsf{R} \varphi_2 \rangle, 1, \sigma) = (\varepsilon, \varphi_2) \wedge ((\varepsilon, \varphi_1) \vee ([1], \mathfrak{f}) \vee (\langle 1 \rangle, \langle \varphi_1 \mathsf{R} \varphi_2 \rangle))$
- $\delta(\langle \varphi_1 \mathsf{R} \varphi_2 \rangle, h, \sigma) = (\varepsilon, \varphi_2) \wedge (\varepsilon, \natural \varphi_1) \wedge (\langle 1 \rangle, \langle \natural(\varphi_1 \mathsf{R} \varphi_2) \rangle) \wedge (\langle h \rangle, \langle \varphi_1 \mathsf{R} \varphi_2 \rangle)$
- $\delta([\varphi_1 \mathsf{R} \varphi_2], 1, \sigma) = (\varepsilon, \varphi_2) \wedge ((\varepsilon, \varphi_1) \vee ([1], [\varphi_1 \mathsf{R} \varphi_2]))$
- $\delta([\varphi_1 \mathsf{R} \varphi_2], h, \sigma) = (\varepsilon, \natural \varphi_2) \vee (\varepsilon, \varphi_1) \vee ([h], [\varphi_1 \mathsf{R} \varphi_2])$

To prove soundness and completeness of the above construction we use a proof by induction on the structure of the formula φ. Due to its complexity and length, the interested reader can find it in Appendix D.

In the remaining part of this section, we show that the emptiness problem for PABT is solvable in EXPTIME. To gain this result, we use a technical variation of the Miyano and Hayashi technique [MH84] for tree automata [Mos84], which has been deeply used in the literature for translating asymmetric alternating Büchi automata to nondeterministic ones. Here, we use this technique to translate in exponential-time PABT into nondeterministic Büchi tree automata (NBT, for short). The fact that PABT are symmetric requires further non-trivial work. Indeed, while for symmetric automata there is bijective correspondence between direction of both the input and output automaton, in our case we have to build this correspondence by looking at the δ of the input automaton. We solve this by extending the core of the classical Miyano-Hayashi technique through a "pair develop" function showed below. To formally define this function, we make use of two intermediate functions given in the next two definitions.

Definition 7. (Satisfiability function) *A* satisfiability function $\mathsf{sat} : (H,\sigma) \in 2^{Q\times\mathbb{N}_{(b)}} \times \Sigma \mapsto \mathsf{sat}(H,\sigma) \in 2^{2^{D_b^\varepsilon \times Q}}$ *maps a set H and a label σ into a set of subset of $D_b^\varepsilon \times Q$ such that for all $S \subseteq D_b^\varepsilon \times Q$ it holds that $S \in \mathsf{sat}(H,\sigma)$ iff $S \models \bigwedge_{(q,g)\in H} \delta(q,g,\sigma)$.*

Definition 8. (Develop function) *A* develop function $\mathsf{dev} : (H,\sigma,d) \in 2^{Q\times\mathbb{N}_{(b)}} \times \Sigma \times \mathbb{N} \mapsto \mathsf{dev}(H,\sigma,d) \in 2^{2^{\mathbb{N}_\varepsilon\times Q\times\mathbb{N}_{(b)}}}$ *maps a set H, a label σ, and a number d into a set of subset of $\mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}$ such that for all $E \subseteq \mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}$ it holds that $E \in \mathsf{dev}(H,\sigma,d)$ iff there exists $S \in \mathsf{sat}(H,\sigma)$ such that $E \in \mathsf{exec}(S,\mathbb{N}_{(d)})$.*

Definition 9. (Pair develop function) *A* pair develop function $\mathsf{pairdev} : (H,H',\sigma,d) \in (2^{Q\times\mathbb{N}_{(b)}})^2 \times \Sigma \times \mathbb{N} \mapsto \mathsf{pairdev}(H,H',\sigma,d) \in 2^{(2^{\mathbb{N}_\varepsilon\times Q\times\mathbb{N}_{(b)}})^2}$ *maps the two sets H and H', a label σ, and a number d into a pair of sets of subset of $\mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}$ such that for all $E, E' \subseteq \mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}$ it holds that $(E,E') \in \mathsf{pairdev}(H,H',\sigma,d)$ iff $E' \subseteq E$, $E \in \mathsf{dev}(H,\sigma,d)$, and if $H' = \emptyset$ then $E' = E$ otherwise $E' \in \mathsf{dev}(H',\sigma,d)$.*

We now show the translation from PABT to NBT.

Theorem 4. *Let $\mathcal{A}$ be a PABT with n states and counting branching bound b. Then, there exists a NBT $\mathcal{A}'$ with $2^{2n*(b+1)}$ states and direction degree $n*b(b+1)/2$ such that $\mathcal{A}$ accepts a tree iff $\mathcal{A}'$ accepts a tree as well.*

Proof. (Sketch.) The nondeterministic automaton $\mathcal{A}'$ guesses a subset construction applied to a run of $\mathcal{A}$. At a given node x of a run of $\mathcal{A}'$, it keeps in its memory the set of states in which the various copies of $\mathcal{A}$ visit the node x in the guessed run. In order to make sure that every infinite path visits states in F infinitely often, $\mathcal{A}'$ keeps track of states that "owe" a visit to F.

Let $\mathcal{A} = \langle Q,\Sigma,b,\delta,q_0,g_0,F\rangle$ be a PABT and $\mathcal{A}' = \langle Q',\Sigma,d',\delta',q_0',F'\rangle$ be an NBT, where $Q' = (2^{Q\times\mathbb{N}_{(b)}})^2$, $d' = n*b(b+1)/2$, $q_0' = (\{(q_0,g_0)\},\emptyset)$, $F' = 2^{Q\times\mathbb{N}_{(b)}} \times \{\emptyset\}$, and $\delta' : Q' \times \Sigma \mapsto 2^{Q'^{(d'+1)}}$ is such that for all $H \subseteq Q \times \mathbb{N}_{(b)}$, $H' \subseteq H$ and $\sigma \in \Sigma$, we have

$$\delta'((H,H'),\sigma) = \bigcup_{\substack{(E,E')\in \\ \mathsf{pairdev}(H,H',\sigma,d'-1)}} \{(\prod_{d=0}^{d'-1}(E_d, E_d' \setminus F)) \times (E_\varepsilon, E_\varepsilon' \setminus F)\}$$

with $E_d = \{(q,g) \in Q \times \mathbb{N}_{(b)} \mid (d,q,g) \in E\}$. By using a non trivial proof, it is possible to show that $\mathcal{L}(\mathcal{A}) \neq \emptyset$ iff $\mathcal{L}(\mathcal{A}') \neq \emptyset$. Here, we only give some intuition of soundness and completeness for the construction of the automaton $\mathcal{A}'$.

First note that, differently from the classical approach, we have to convert the symmetric automaton $\mathcal{A}$ into a nondeterministic one. This induces to deal with two extra problems: *(i)* $\mathcal{A}$ can perform a ε-moves and *(ii)* $\mathcal{A}$ does not have an upper bound on the number of directions it uses. The first problem is solved by allocating in $\mathcal{A}'$ an apposite direction (namely d') that collects all the states of the automaton $\mathcal{A}$ sent through ε-moves during a given execution. We face the second problem thanks to the following property due to the splitting partition function we use: if $\mathcal{A}$ accepts a tree T, it must accepts also a tree T' with branching degree at most equal to $d' = n*b(b+1)/2$. This holds since, in each state q and degree g at a node x of the input tree, a set S that satisfies the $\delta(q,g,\mathsf{inp}(x))$ can contain at most $|Q \times \mathbb{N}_{(b)+}|$ pairs of the kind $(\langle g'\rangle, q')$, so the spart function can split each of such a pair in at most g' nodes of degree 1 and then for each state q' we can have at most $b(b+1)/2$ different successors of x. Therefore, it is possible to construct a relative run of $\mathcal{A}'$ by restricting our attention only to trees with degree at most d'.

We now give some explanation about the δ' through an example. Suppose $\mathcal{A} = \langle\{q_0, q_1\}, \{a\}, 2, \delta, q_0, 0, F\rangle$, where the δ contains $\delta(q_0, 0, a) = (\varepsilon, q_0) \wedge (\langle 2 \rangle, q_1)$. Hence, $d' = 6$. Also, suppose that $H = \{(q_0, 0)\}$ and $H' = \emptyset$. Accordingly to the satisfiability function, we have $\mathsf{sat}(H, \{a\}) = \{\{(\varepsilon, q_0), (\langle 2 \rangle, q_1)\}\}$ and accordingly to the develop function, the set E represents one of the following possibilities: either $\mathcal{A}'$ sends a copy of itself to one child with degree 2 or to two children with degree 1. In both cases $\mathcal{A}'$ also sends a witness of the ε-move to direction d' More formally, we have that E is equal to either $\{(\varepsilon, q_0, 0), (i, q_1, 2)\}$, for $0 \leq i < d'$ or $\{(\varepsilon, q_0, 0), (i, q_1, 1), (j, q_1, 1)\}$, for $0 \leq i, j < d'$ and $i \neq j$. Since $H' = \emptyset$ we also have $E' = E$. Finally, by using the pair develop function, we get twenty one corresponding transition rules in δ'.

Recall that for the NBT $\mathcal{A}'$ the emptiness problem is solvable in PTIME [VW86], in particular in $O(|Q'|^{2d'})$ (we directly consider the one-letter automaton associated to $\mathcal{A}'$). Then, by Theorem 4, the following result follows.

Corollary 1. *The emptiness problem for a PABT $\mathcal{A}$ with n states and counting branching bound b can be decided in time* $2^{O(n^2 * b^3)}$.

By Theorem 3 and Corollary 1, since $n = |\mathsf{ecl}(\varphi)| = O(|\varphi|)$ and $b = \mathsf{deg}(\varphi) = O(|\varphi|)$, we get that the satisfiability problem for GCTL is solvable in time $2^{O(|\varphi|^5)}$, i.e., in EXPTIME, thus not harder than CTL. Moreover, by Remark 1, CTL is subsumed by GCTL, so the following holds.

Corollary 2. *The satisfiability problem for the GCTL logic is* EXPTIME-COMPLETE.

5 Discussion

In this paper, we have investigate the logic GCTL as the extension of the branching-time temporal logics CTL with *graded path modalities*. We have shown that GCTL allows to gain expressiveness, as it becomes invariant under bisimulation, while it retains the tree and finite model properties. Moreover, we have shown that its satisfiability problem is EXPTIME-COMPLETE, thus not harder than that for the classical CTL.

As natural future work, it could be interesting to investigate graded path modalities along with more complex logics, such as CTL^*, i.e., to investigate GCTL^*. We believe that is not hard to extend to this logic the properties showed for GCTL in Theorem 2. On the contrary, to evaluate the complexity of the satisfiability problem for GCTL^* is rather than immediate as the automata model we have considered in this paper for GCTL is not appropriate for dealing with GCTL^*. Indeed, by using a theoretic-automata approach similar to the one used for GCTL, we can reduce the satisfiability problem for GCTL^* to the emptiness problem of PATA, but with an acceptance condition stronger than Büchi, such as the parity one [Mos84, SE89, Tho97]. Unfortunately, the technique we have shown to translate PABT into NBT is not appropriate for parity automata. However, by using a technique based on promises and strategies, as it was done in [KSV02], we conjecture that also PATA along with a parity condition can be translated in exponential-time into a nondeterministic parity tree automata. Then, by using the fact that for the latter the emptiness problem is solvable in exponential-time, we get that the satisfiability problem for GCTL^* is 2EXPTIME-COMPLETE, thus not harder than that for the classical CTL^*.

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A Proof of Lemma 1

For all state formulas φ_1 and φ_2 (resp., path formulas ψ_1 and ψ_2), we say that φ_1 *implies* φ_2, formally $\varphi_1 \Rightarrow \varphi_2$, (resp., ψ_1 *implies* ψ_2, formally $\psi_1 \Rightarrow \psi_2$) iff for all Kripke structures $\mathcal{K}$ and worlds $w \in \mathsf{dom}(\mathcal{K})$ it holds that if $\mathcal{K}, w \models \varphi_1$ then $\mathcal{K}, w \models \varphi_2$ (resp., $\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \psi_1)) \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \psi_2))$). It is obvious that, φ_1 is *equivalent* to φ_2, (resp., ψ_1 is *equivalent* to ψ_2) iff $\varphi_1 \Rightarrow \varphi_2$ and $\varphi_2 \Rightarrow \varphi_1$ (resp., $\psi_1 \Rightarrow \psi_2$ and $\psi_2 \Rightarrow \psi_1$).

The following two propositions are immediately derived from the semantics of GCTL^*.

Proposition 1. *For all state formulas* φ*, path formulas* ψ*, finite sequences of path formulas* $\{\psi_i\}_i^n$*, and degree* $g \in \mathbb{N}_+$ *it holds that:* (i) $\mathsf{E}^{\geq 0}\psi \equiv \mathfrak{t}$, (ii) $\mathsf{E}^{>g}\psi \Rightarrow \mathsf{E}^{\geq g}\psi$, (iii) $\mathsf{E}\varphi \equiv \varphi$, (iv) $\mathsf{E}^{>g}\varphi \equiv \mathfrak{f}$, (v) $\mathsf{E}^{\geq g}(\varphi \wedge \psi) \equiv \varphi \wedge \mathsf{E}^{\geq g}\psi$, (vi) $\mathsf{E}(\varphi \vee \psi) \equiv \varphi \vee \mathsf{E}\psi$, (vii) $\mathsf{E}^{>g}(\varphi \vee \psi) \equiv \neg\varphi \wedge \mathsf{E}^{>g}\psi$, (viii) $\mathsf{E}\bigwedge_i \psi_i \Rightarrow \bigwedge_i \mathsf{E}\psi_i$, (ix) $\mathsf{E}\bigvee_i \psi_i \equiv \bigvee_i \mathsf{E}\psi_i$, (x) $\mathsf{E}^{>g}\bigvee_i \psi_i \Rightarrow \bigvee_i \mathsf{E}^{<g}\psi_i$, (xi) $\mathsf{E}\tilde{\mathsf{X}}\,\psi \equiv \mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{E}\mathsf{X}\,\psi$*, and* (xii) $\mathsf{E}^{>g}\tilde{\mathsf{X}}\,\psi \equiv \mathsf{E}^{>g}\mathsf{X}\,\psi \wedge \mathsf{E}\mathsf{X}\,\neg\psi$.

Proposition 2. *For all path formulas* ψ*,* ψ_1*, and* ψ_2*, it holds that:* (i) $\tilde{\mathsf{X}}\,\psi \equiv \tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{X}\,\psi$, (ii) $\psi_1 \mathsf{U}\, \psi_2 \equiv \psi_2 \vee \psi_1 \wedge \mathsf{X}\,(\psi_1 \mathsf{U}\, \psi_2)$*, and* (iii) $\psi_1 \mathsf{R}\, \psi_2 \equiv \psi_2 \wedge (\psi_1 \vee \tilde{\mathsf{X}}\,(\psi_1 \mathsf{R}\, \psi_2))$.

Let X be a set of objects and $\mathrm{R} \subseteq \mathrm{X} \times \mathrm{X}$ be an equivalence relations on X, i.e., R is reflexive, symmetric, and transitive. Then, it is possible to split the set X into a partition of equivalence classes induced by the relation R. With $\mathcal{E}_{\mathrm{R}}(\mathrm{X})$ we denote the set of all these equivalence classes, i.e., for all $C_1, C_2 \in \mathcal{E}_{\mathrm{R}}(\mathrm{X})$, with $C_1 \neq C_2$, it holds that *(i)* $\emptyset \neq C_1 \subseteq \mathrm{X}$ and *(ii)* for all elements $x, y \in C_1$ and $z \in C_2$ it holds that $(x, y) \in \mathrm{R}$ and $(x, z) \notin \mathrm{R}$. It is important to remind that for a partition of a set X the following two properties hold: *(i)* $\bigcup_{C \in \mathcal{E}_{\mathrm{R}}(\mathrm{X})} C = \mathrm{X}$ and *(ii)* for all $C_1, C_2 \in \mathcal{E}_{\mathrm{R}}(\mathrm{X})$, with $C_1 \neq C_2$, it holds that $C_1 \cap C_2 = \emptyset$.

Definition 10. **(i-step congruence relation)** *Let* $\mathcal{K}$ *be a Kripke structure and* $\mathfrak{P} \subseteq \mathsf{paths}(\mathcal{K})$ *be a set of paths such that there is* $i \in \mathbb{N}$ *for which* $\pi \in \mathfrak{P}$ *implies* $|\pi| > i$*. Then, for all paths* $\pi, \pi' \in \mathfrak{P}$ *we say that* π *is* i-step congruent *to* π'*, denoting this with* $\pi \asymp_i \pi'$*, iff for all* $j \in \mathbb{N}_{(i)}$ *it holds that* $\pi(j) = \pi'(j)$*, i.e., the two paths are identical up to the i-th position.*

Definition 11. **(n-size 1-step classes set)** *Let* $\mathcal{E}_{\asymp_1}(\mathfrak{P})$ *be the set of* 1*-step congruence classes on* $\mathfrak{P}$*. Then, with* $I_n(\mathfrak{P})$ *we denote the set of all paths in* $\mathfrak{P}$ *that are in a congruence class of* $\mathfrak{P}$ *itself with cardinality n, i.e.,* $I_n(\mathfrak{P}) = \{\pi \in \mathfrak{P} \mid \exists C \in \mathcal{E}_{\asymp_1}(\mathfrak{P}), |C| = n : \pi \in C\}$.

Lemma 2. *For all finite sets* $\mathfrak{P}$ *it holds that* $\{\frac{|I_n(\mathfrak{P})|}{n}\}_n^{|\mathfrak{P}|} \in \mathsf{P}(|\mathfrak{P}|)$.

Proof. Since $\mathfrak{P}$ is finite, it holds that $\mathcal{E}_{\asymp_1}(\mathfrak{P})$ is finite as well. Consequently, the sets of equivalence classes given by $Q_n = \{C \in \mathcal{E}_{\asymp_1}(\mathfrak{P}) \mid |C| = n\}$ satisfy $|Q_n| < \omega$, i.e., there exists a number $k_n \in \mathbb{N}$ such that $|Q_n| = k_n$. Then, since $I_n(\mathfrak{P}) = \bigcup_{C \in Q_n} C$, it is obvious that $|I_n(\mathfrak{P})| = k_n * n$. By Definition 11, it follows that $\{I_n(\mathfrak{P})\}_n^{|\mathfrak{P}|}$ is a partition of $\mathfrak{P}$, so we have that $\sum_{n=1}^{|\mathfrak{P}|} |I_n(\mathfrak{P})| = |\mathfrak{P}|$, and then $\sum_{n=1}^{|\mathfrak{P}|} n * \frac{|I_n(\mathfrak{P})|}{n} = |\mathfrak{P}|$. Now, by the previous observation, we have that for all numbers $n \in \mathbb{N}$ it holds $\frac{|I_n(\mathfrak{P})|}{n} = k_n \in \mathbb{N}$, so the sequence $\{\frac{|I_n(\mathfrak{P})|}{n}\}_n^{|\mathfrak{P}|}$ is a solution of the Diophantine equation $1 * p_1 + 2 * p_2 + \ldots + |\mathfrak{P}| * p_{|\mathfrak{P}|} = |\mathfrak{P}|$ and then $\{\frac{|I_n(\mathfrak{P})|}{n}\}_n^{|\mathfrak{P}|} \in \mathsf{P}(|\mathfrak{P}|)$.

Let $\pi \in \mathsf{paths}(\mathcal{K})$ and $n \in \mathbb{N}_{(|\pi|-1)}$. With $\pi_{\geq n}$ we denote the *suffix* of π starting at position n. Formally, *(i)* $|\pi_{\geq n}| = |\pi| - n$ and *(ii)* for all indexes $i \in \mathbb{N}_{(|\pi|-n-1)}$, it holds that $\pi(n+i) = \pi_{\geq n}(i)$.

Lemma 3. *Let $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ be a Kripke structure, $w, w' \in \mathrm{W}$ be two worlds such that $(w, w') \in \mathrm{R}$, and ψ be a GCTL* path formula. Then, it holds that $\mathfrak{P}_A(\mathcal{K}, w', \psi) = \{\pi_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w') \mid \pi \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)\}$.*

Proof. By definition, we have that $\pi \in \mathfrak{P}_A(\mathcal{K}, w', \psi)$ iff for all paths $\pi' \in \mathsf{paths}(\mathcal{K}, w')$ such that $\pi \preccurlyeq \pi'$ it holds that $\mathcal{K}, \pi', 0 \models \psi$. Since $(w, w') \in \mathrm{R}$, for all $\pi, \pi' \in \mathsf{paths}(\mathcal{K}, w')$ there exist $\pi'', \pi''' \in \mathsf{paths}(\mathcal{K}, w)$ such that $\pi = \pi''_{\geq 1}$ and $\pi' = \pi'''_{\geq 1}$, so we have that $\pi \in \mathfrak{P}_A(\mathcal{K}, w', \psi)$ iff for all paths $\pi''' \in \mathsf{paths}(\mathcal{K}, w)$ such that $\pi''_{\geq 1} \preccurlyeq \pi'''_{\geq 1}$ it holds that $\mathcal{K}, \pi'''_{\geq 1}, 0 \models \psi$, thus $\mathcal{K}, \pi''', 1 \models \psi$ and then $\mathcal{K}, \pi''', 0 \models \mathsf{X}\,\psi$. Now, we can observe that, since $\pi'', \pi''' \in \mathsf{paths}(\mathcal{K}, w)$, it holds that $\pi'' \preccurlyeq \pi'''$ iff $\pi''_{\geq 1} \preccurlyeq \pi'''_{\geq 1}$, thus we obtain that $\pi \in \mathfrak{P}_A(\mathcal{K}, w', \psi)$ iff for all paths $\pi''' \in \mathsf{paths}(\mathcal{K}, w)$ such that $\pi'' \preccurlyeq \pi'''$ it holds that $\mathcal{K}, \pi''', 0 \models \mathsf{X}\,\psi$, i.e., $\pi'' \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)$, where $\pi = \pi''_{\geq 1}$. Finally, $\pi \in \mathfrak{P}_A(\mathcal{K}, w', \psi)$ iff $\pi''_{\geq 1} = \pi \in \mathsf{paths}(\mathcal{K}, w')$, with $\pi'' \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)$, i.e., $\pi \in \{\pi''_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w') \mid \pi'' \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)\}$.

Lemma 4. *For all GCTL*path formulas ψ it holds that:*

i) $\mathsf{E}^{\geq g}\mathsf{X}\,\psi \equiv \bigvee_{\{h_i\}_i^g \in \mathsf{CP}(g)} \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$;

ii) $\mathsf{E}^{\geq g}\tilde{\mathsf{X}}\,\psi \equiv \begin{cases} \mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{E}\mathsf{X}\,\mathsf{E}\psi, & \text{if } g = 1; \\ \mathsf{E}\mathsf{X}\,\mathsf{E}\neg\psi \wedge \bigvee_{\{h_i\}_i^g \in \mathsf{CP}(g)} \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi, & \text{otherwise.} \end{cases}$

Proof. Item (i)*, ($\Rightarrow$).* First, assume that $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ is a model for $\mathsf{E}^{\geq g}\mathsf{X}\,\psi$ in $w \in \mathrm{W}$. Then, by definition of the semantic for existential quantifiers, there exists a subset $\mathfrak{P}$ of $\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi))$, with $|\mathfrak{P}| = g$. We want to show that, let $h_i = \sum_{n=i}^{g} \frac{|I_n(\mathfrak{P})|}{n}$, it holds that $\mathcal{K}, w \models \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$. For each number $n \in \mathbb{N}_{(g)+}$, consider the partition $Q_n = \mathcal{E}_{\asymp_1}(I_n(\mathfrak{P})) = \{C \in \mathcal{E}_{\asymp_1}(\mathfrak{P}) \mid |C| = n\}$ of $I_n(\mathfrak{P})$ in $k_n = \frac{|I_n(\mathfrak{P})|}{n}$ sets. For a fixed $n \in \mathbb{N}_+$, we indicate all these classes with the sequence $\{C_{n,k}\}_k^{k_n}$. Since $C_{n,k} \subseteq \mathfrak{P} \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi))$, it is obvious that all its elements are incomparable minimal paths. Moreover, it is possible to associate a world $w_{n,k}$ to each class $C_{n,k}$ such that for all $\pi \in C_{n,k}$ it holds that $\pi(1) = w_{n,k}$. By Lemma 3, since $(w, w_{n,k}) \in \mathrm{R}$, we have that $\mathfrak{P}_A(\mathcal{K}, w_{n,k}, \psi) = \{\pi'_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w_{n,k}) \mid \pi' \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)\}$, so, for all $\pi \in C_{n,k}$, it holds that $\pi_{\geq 1} \in \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w_{n,k}, \psi))$. Indeed, $\pi \in \mathfrak{P} \subseteq \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)$ and $\pi_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w_{n,k})$, thus $\pi_{\geq 1} \in \mathfrak{P}_A(\mathcal{K}, w_{n,k}, \psi)$. Moreover, since π is minimal in $\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)$, also $\pi_{\geq 1}$ is minimal in $\mathfrak{P}_A(\mathcal{K}, w_{n,k}, \psi)$, because otherwise if there is $\pi' \in \mathsf{paths}(\mathcal{K}, w)$, $\pi' \neq \pi$, such that $\pi'_{\geq 1} \preccurlyeq \pi_{\geq 1}$ we have $\pi' \preccurlyeq \pi$, which contradicts the fact that π is minimal. Now, $|C_{n,k}| = n$ and $\{\pi_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w_{n,k}) \mid \pi \in C_{n,k}\} \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w_{n,k}, \psi))$, thus $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w_{n,k}, \psi))| \geq n$. Then, for each $i, n \in \mathbb{N}_{(g)+}$, with $i \leq n$, and for all $k \in \mathbb{N}_{(k_n)+}$, it holds that $\mathcal{K}, w_{n,k} \models \mathsf{E}^{\geq i}\psi$, so for all $\pi \in Q'_n = \{\pi' \in \mathsf{paths}(\mathcal{K}, w) \mid |\pi'| = 2, \exists k \in \mathbb{N}_{(k_n)+} : \pi(1) = w_{n,k}\}$ we have $\mathcal{K}, \pi(1) \models \mathsf{E}^{\geq i}\psi$ that is $\mathcal{K}, \pi, 1 \models \mathsf{E}^{\geq i}\psi$ and then $\mathcal{K}, \pi, 0 \models \mathsf{X}\,\mathsf{E}^{\geq i}\psi$. This means that for all $\pi \in \bigcup_{n=i}^{g} Q'_n$ we have $\mathcal{K}, \pi, 0 \models \mathsf{X}\,\mathsf{E}^{\geq i}\psi$. Observe now that, since each world $w_{n,k}$ is the characteristic world for the equivalence class $C_{n,k} \in \mathcal{E}_{\asymp_1}(\mathfrak{P})$, there is a different world $w_{n,k}$ for each class $C_{n,k}$, so we have that all the sets in $\{Q'_n\}_n^g$ are disjoint and $|Q'_n| = k_n$. It is obvious then that $\bigcup_{n=i}^{g} Q'_n \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\mathsf{E}^{\geq i}\psi))$ so $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\mathsf{E}^{\geq i}\psi))| \geq |\bigcup_{n=i}^{g} Q'_n| = \sum_{n=i}^{g} |Q'_n| = \sum_{n=i}^{g} k_n = \sum_{n=i}^{g} \frac{|I_n(\mathfrak{P})|}{n} = h_i$. Trivially, it follows that $\mathcal{K}, w \models \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$ and then $\mathcal{K}, w \models \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$. Now, by Lemma 2, we have $\{\frac{|I_n(\mathfrak{P})|}{n}\}_n^g \in \mathsf{P}(g)$, and then, by the definition of the set $\mathsf{CP}(g)$, it holds $\{h_i\}_i^g \in \mathsf{CP}(g)$. Hence, we get the thesis for this direction.

Item (i), *(⇐).* Assume now that $\mathcal{K}$ is a model for $\bigvee_{\{h_i\}_i^g \in \mathsf{CP}(g)} \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$ in $w \in \mathrm{W}$. Then, there is a sequence $\{h_i\}_i^g \in \mathsf{CP}(g)$ such that $\mathcal{K}, w \models \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$. Thus, for all indexes $i \in \mathbb{N}_{(g)+}$, it holds that $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\mathsf{E}^{\geq i}\psi))| \geq h_i$, since $\mathcal{K}, w \models \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$. Let $k_i = h_i - h_{i+1}$, for $i \in \mathbb{N}_{(g-1)+}$, and $k_g = h_g$. Since $\{h_i\}_i^g \in \mathsf{CP}(g)$, it is obvious that $\{k_i\}_i^g \in \mathsf{P}(g)$. Now, since $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\mathsf{E}^{\geq g}\psi)))| \geq h_g = k_g$, we can construct a set $\mathfrak{P}_g \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\mathsf{E}^{\geq g}\psi))$, with $|\mathfrak{P}_g| = k_g$. Moreover, for all $i \in \mathbb{N}_{(g-1)+}$, let $\mathfrak{P}_i \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\mathsf{E}^{\geq i}\psi)) \setminus \bigcup_{j=i+1}^{g} \mathfrak{P}_j$, with $|\mathfrak{P}_i| = h_i - |\bigcup_{j=i+1}^{g} \mathfrak{P}_j| \leq |(\mathsf{minstructs}(\mathfrak{P}\mathcal{K}w\mathsf{X}\,\mathsf{E}^{\geq i}\psi) \setminus \bigcup_{j=i+1}^{g} \mathfrak{P}_j)|$. It is evident that all the sets $\mathfrak{P}_i$ are disjoint. Furthermore, each of them has just k_i elements. Indeed, by construction we have that $|\mathfrak{P}_g| = k_g$, and, if all sets $\mathfrak{P}_j$, with $j > i$, have cardinality k_i, it holds that $|\mathfrak{P}_i| = h_i - |\bigcup_{j=i+1}^{g} \mathfrak{P}_j| = h_i - \sum_{j=i+1}^{g} |\mathfrak{P}_j| = h_i - \sum_{j=i+1}^{g} k_j = h_i - h_{i+1} = k_i$. Since for all $i \in \mathbb{N}$ it holds that $\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\mathsf{E}^{\geq i}\psi)) \supseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\mathsf{E}^{\geq i+1}\psi))$, we have $\mathfrak{P}' = \bigcup_{i=1}^{g} \mathfrak{P}_i \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\mathsf{E}^{\geq 1}\psi))$, so all paths in $\mathfrak{P}'$ are incomparable, i.e. $\mathfrak{P}' = \mathsf{minstructs}(\mathfrak{P}')$. For simplicity, for all $i \in \mathbb{N}_{(g)+}$, we denote with the sequence $\{\pi_{i,j}\}_j^{k_i}$ all the paths into the set $\mathfrak{P}_i$. Note that all paths $\pi_{i,j}$ have length 2. Indeed by definition, $\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\mathsf{E}^{\geq 1}\psi)$ is equal to $\{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \forall \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi' \text{ implies } \mathcal{K}, \pi', 0 \models \mathsf{X}\,\mathsf{E}^{\geq 1}\psi\}$, so, since $\mathcal{K}, \pi', 0 \models \mathsf{X}\,\mathsf{E}^{\geq 1}\psi$ implies $\mathcal{K}, \pi'(1) \models \mathsf{E}^{\geq 1}\psi$ and $\pi(1) = \pi'(1)$, we have $\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\mathsf{E}^{\geq 1}\psi) = \{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \mathcal{K}, \pi(1) \models \mathsf{E}^{\geq 1}\psi\}$. Then, applying the minimal structures function to the above sets, we obtain that $\mathfrak{P}' \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\mathsf{E}^{\geq 1}\psi)) = \mathsf{minstructs}(\{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \mathcal{K}, \pi(1) \models \mathsf{E}^{\geq 1}\psi\}) = \{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid |\pi| = 2, \mathcal{K}, \pi(1) \models \mathsf{E}^{\geq 1}\psi\}$. Now, for all indexes $i \in \mathbb{N}_{(g)+}$, $j \in \mathbb{N}_{(k_i)+}$, set $w_{i,j} = \pi_{i,j}(1)$. Since all the paths $\pi_{i,j}$ are incomparable paths of length 2 and $\pi_{i,j}(0) = w$, we derive that all the worlds $w_{i,j}$ are different. Moreover, since $\mathcal{K}, \pi_{i,j}(1) \models \mathsf{E}^{\geq i}\psi$ it holds also that $\mathcal{K}, w_{i,j} \models \mathsf{E}^{\geq i}\psi$ and then $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w_{i,j}, \psi))| \geq i$. Thus, since $(w, w_{i,j}) \in \mathrm{R}$, by Lemma 3 we obtain that $|\mathsf{minstructs}(\{\pi_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w_{i,j}) \mid \pi \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)\})| \geq i$. At this point, $\pi'_{\geq 1} \in \mathsf{minstructs}(\{\pi_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w_{i,j}) \mid \pi \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)\})$ implies that π' is minimal, i.e., $\pi' \in \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi))$. Indeed, if this is not the case, there is $\pi'' \in \mathsf{paths}(\mathcal{K}, w)$, $\pi'' \neq \pi'$, such that $\pi'' \preccurlyeq \pi'$, and then $\pi''_{\geq 1} \preccurlyeq \pi'_{\geq 1}$ that contradicts the fact that $\pi'_{\geq 1}$ is minimal. Then, let $\mathfrak{P}_{i,j} = \{\pi' \in \mathsf{paths}(\mathcal{K}, w) \mid \pi'_{\geq 1} \in \mathsf{minstructs}(\{\pi_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w_{i,j}) \mid \pi \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)\})\}$, we have $\mathfrak{P}_{i,j} \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi))$. Furtermore, $|\mathfrak{P}_{i,j}| = i$. Let now $\mathfrak{P} = \bigcup_{i=1}^{g} \bigcup_{j=1}^{k_i} \mathfrak{P}_{i,j}$. It is evident that $\mathfrak{P} \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi))$ and then $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi))| \geq |\mathfrak{P}|$. Moreover, $|\mathfrak{P}| = \sum_{i=1}^{g} \sum_{j=1}^{k_i} |\mathfrak{P}_{i,j}| =$

$\sum_{i=1}^{g}\sum_{j=1}^{k_i} i = \sum_{i=1}^{g} i * k_i$. Since, as we have previously noted, $\{k_i\}_i^g \in \mathsf{P}(g)$, it holds that $|\mathfrak{P}| = g$, so $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi))| \geq g$. The thesis for the other direction follows immediately.

Item (ii). At the formula $\mathsf{E}^{\geq g}\tilde{\mathsf{X}}\,\psi$ we can apply in sequence either the equivalence *(xi)* of Proposition 1, if $g = 1$, or the equivalence *(xii)* of the same proposition and then the item *(i)* of this lemma, obtaining the thesis.

Now, we are able to prove Lemma 1.

Proof. To show the equivalence *(i)*, it is possible to apply, at the formula $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2)$, the following sequence of equivalences: item *(ii)* of Proposition 2, either item *(vi)*, if $g = 1$, or item *(vii)* of Proposition 1 otherwise, item *(v)* of Proposition 1, and, finally, item *(i)* of Lemma 4.

At the same way, to show the equivalence *(ii)*, it is possible to apply, at the formula $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2)$, the following sequence of equivalences: item *(iii)* of Proposition 2, item *(v)* of Proposition 1, either item *(vi)*, if $g = 1$, or item *(vii)* of Proposition 1 otherwise, and, finally, item *(ii)* of Lemma 4.

B Theorem 1

Proof. Given a GCTL formula φ, we proceed as follows. First we show a fixed point form of the formula derived by the previous equivalences and then we propose a translation which allow us to obtain an equivalent graded μ-calculus formula.

By Lemma 1, we notice that $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2)$ and $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2)$ formulas are definable in a fixed point form. This can be obtained by putting the equivalences written in lemma in terms of functions, that is in a more formal way, we can write $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2) \equiv \mathsf{eu}(\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2), \varphi_1, \varphi_2, g)$ and $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2) \equiv \mathsf{er}(\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2), \varphi_1, \varphi_2, g)$, for two suitable fixed point functions $\mathsf{eu}(\cdot,\cdot,\cdot,\cdot)$ and $\mathsf{er}(\cdot,\cdot,\cdot,\cdot)$ such that until formulas with degree g do not occur into $\mathsf{eu}(\cdot,\cdot,\cdot,g)$ nor the release ones with degree g into $\mathsf{er}(\cdot,\cdot,\cdot,g)$. For example, when $g > 1$, we have that $\mathsf{eu}(X, \varphi_1, \varphi_2, g) = \neg\varphi_2 \wedge \varphi_1 \wedge \mathsf{ex}'(\varphi_1 \mathsf{U}\, \varphi_2, g) \wedge \mathsf{EX}X$ and $\mathsf{er}(X, \varphi_1, \varphi_2, g) = \varphi_2 \wedge \neg\varphi_1 \wedge \mathsf{EX}\,\mathsf{E}\neg(\varphi_1 \mathsf{R}\, \varphi_2) \wedge \mathsf{ex}'(\varphi_1 \mathsf{R}\, \varphi_2, g) \wedge \mathsf{EX}X$, where $\mathsf{ex}'(\psi, g) = \bigvee_{\{h_i\}_i^g \in \mathsf{CP}'(g)} \bigwedge_{i=1}^{g-1} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$, with $\mathsf{CP}'(g) = \mathsf{CP}(g) \setminus \{\{h_i\}_i^g \mid h_g = 1\}$. Note that $|\mathsf{ex}'(\psi, g)| = \Theta((|\psi| + \frac{g}{2}) * 2^{\alpha\sqrt{g}})$, so we have $|\mathsf{eu}(X, \varphi_1, \varphi_2, g)| = |\mathsf{er}(X, \varphi_1, \varphi_2, g)| = \Theta(|\varphi_1| + |\varphi_2| + g * 2^{\alpha\sqrt{g}})$.

Now, W.l.o.g we assume that φ is in existential normal form (we recall that any GCTL formula can be linearly translated in this form). Thanks to the above fixed point functions, we can now conclude the proof by showing a translation function "$\mathsf{trn}(\cdot)$: GCTL $\mapsto$ graded μ-calculus" which allow to get the desired graded μ-calculus formula $\varphi' = \mathsf{trn}(\varphi)$ equivalent to φ. The function $\mathsf{trn}(\cdot)$ is inductively defined as follows: *(i)* $\mathsf{trn}(p) = p$ with $p \in \mathrm{AP}$; *(ii)* $\mathsf{trn}(\neg\varphi) = \neg\,\mathsf{trn}(\varphi)$; *(iii)* $\mathsf{trn}(\mathsf{E}^{\geq 0}\psi) = \mathsf{t}$; *(iv)* $\mathsf{trn}(\mathsf{E}^{\geq g}\mathsf{X}\,\varphi) = \neg end \wedge \langle g-1 \rangle\, \mathsf{trn}(\varphi)$; *(v)* $\mathsf{trn}(\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\varphi) = end \vee \neg end \wedge \langle 0 \rangle\, \mathsf{trn}(\varphi)$; *(vi)* $\mathsf{trn}(\mathsf{E}^{> g}\tilde{\mathsf{X}}\,\varphi) = \neg end \wedge \langle 0 \rangle\, \mathsf{trn}(\neg\varphi) \wedge \langle g \rangle\, \mathsf{trn}(\varphi)$; *(vii)* $\mathsf{trn}(\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2)) = \mu X.\mathsf{trn}(\mathsf{eu}(X, \varphi_1, \varphi_2, g))$; *(viii)* $\mathsf{trn}(\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2)) = \nu X.\mathsf{trn}(\mathsf{er}(X, \varphi_1, \varphi_2, g))$, where $g \in \mathbb{N}_+$.

By induction on the structure of the formula, it is not hard to see that, for each model $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ of φ the structure $\mathcal{K}' = \langle \mathrm{AP} \cup \{end\}, \mathrm{W}, \mathrm{R}, \mathrm{L}' \rangle$ is a model of φ', where, let $W' = \{w \in W \mid \nexists w' \in W : (w, w') \in R\}$, for all $w \in W \setminus W'$ and $w' \in W'$ it holds that $L'(w) = L(w)$ and $L'(w') = L(w') \cup \{end\}$. Moreover, from a model $\mathcal{K} = \langle AP, W, R, L\rangle$ of φ' it is possible to extract one of φ simply substituting the relation R with a new relation R' defined as follows: for all $w, w' \in W$, it holds that $(w, w') \in R'$ iff $(w, w') \in R$ and $end \notin L(w)$.

C Theorem 2

Consider two Kripke structures $\mathcal{K} = \langle AP, W, R, L\rangle$ and $\mathcal{K}' = \langle AP', W', R', L'\rangle$. We say that $\mathcal{K}$ is *bisimilar* to $\mathcal{K}'$, denoting this by $\mathcal{K} \sim \mathcal{K}'$ iff there exists a non-empty relation $\mathsf{B} \subseteq W \times W'$, called *relation of bisimulation*, such that for all pairs of worlds $(w, w') \in \mathsf{B}$ it holds that: *(i)* $L(w) = L'(w')$; *(ii)* $(w, v) \in R$ implies that there exists a world $v' \in W'$ such that $(v, v') \in \mathsf{B}$ and $(w', v') \in R'$; *(iii)* $(w', v') \in R'$ implies that there exists a world $v \in W$ such that $(v, v') \in \mathsf{B}$ and $(w, v) \in R$. Note that also $\mathsf{B}^{-1} = \{(w', w) \in W' \times W \mid (w, w') \in \mathsf{B}\}$ is a relation of bisimulation.

It is easy to see that an unwinding function is a particular relation of bisimulation.

Proof. Item (i) Let $\mathcal{K} = \langle AP, W, R, L\rangle$ be a Kripke structure. We show that for each GCTL formula φ and world $w \in W$, it holds that $\mathcal{K}, w \models \varphi$ if and only if $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \varphi$. The proof procedes by mutual induction on the structure of the formula φ (external induction) and on the structure of all path formulas it contains (internal induction). Let us start with the external induction. The base step for atomic propositions and the boolean combination cases are easy and left to the reader. Therefore, let us consider the case where φ is of the form $\mathsf{E}^{\geq g}\psi$, for $g \in \mathbb{N}$. The proof proceeds by internal induction on the path formula ψ. As base case, ψ does not contain any quantifier (i.e., ψ is a temporal operators defined on combinations of atomic propositions). First, note that $\mathcal{U}_w^{\mathcal{K}}$ is also an unwinding of itself, so for the construction of $\mathsf{paths}(\mathcal{K}, w)$ and $\mathsf{paths}(\mathcal{U}_w^{\mathcal{K}}, \varepsilon)$ we can choose the same unwinding, obtaining that for all worlds $w \in W$, it holds $\mathsf{paths}(\mathcal{K}, w) = \mathsf{paths}(\mathcal{U}_w^{\mathcal{K}}, \varepsilon)$. Now we show that for all paths $\pi \in \mathsf{paths}(\mathcal{K}, w)$ it holds that $\mathcal{K}, \pi, 0 \models \psi$ if and only if $\mathcal{U}_w^{\mathcal{K}}, \pi, 0 \models \psi$. Indeed, if ψ is a state formula, by the external inductive hypothesis, we obtain the above statement. Then, by induction on the structure of ψ, it is easy to show that the above statement holds for all path formulas. By definition of the satisfiability path set, it follows that $\mathfrak{P}_A(\mathcal{K}, w, \psi) = \mathfrak{P}_A(\mathcal{U}_w^{\mathcal{K}}, \varepsilon, \psi)$. Therefore, by the semantics of the existential quantifiers, we have that $\mathcal{K}, w \models \mathsf{E}^{\geq g}\psi$ if and only if $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \mathsf{E}^{\geq g}\psi$. Now, let us consider the case where ψ contains $n > 0$ nested quantifiers. For the internal inductive step, we have $\mathcal{K}, w \models \mathsf{E}^{\geq g}\psi'$ if and only if $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \mathsf{E}^{\geq g}\psi'$, where ψ' contains $n-1$ nested quantifiers. For reasoning analogous to the internal base case, we obtain that $\mathfrak{P}_A(\mathcal{K}, w, \psi) = \mathfrak{P}_A(\mathcal{U}_w^{\mathcal{K}}, \varepsilon, \psi)$, where we recall that ψ contains n nested quantifiers, and then $\mathcal{K}, w \models \mathsf{E}^{\geq g}\psi$ if and only if $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \mathsf{E}^{\geq g}\psi$. So we have done with the proof.

Item (ii) Consider a GCTL formula φ and suppose that it is satisfiable. Then, there is a model $\mathcal{K}$ for φ in a world $w \in \mathsf{dom}(\mathcal{K})$. By item *(i)*, φ is satisfied at the root of the unwinding $\mathcal{U}_w^{\mathcal{K}}$. Thus, since $\mathcal{U}_w^{\mathcal{K}}$ is a tree, immediately follows that φ is satisfied on a tree model.

Item (iii) Extending, by Lemma 1, the Definition 3.1 of Hintikka structure in [EH85] it is derivable an assertion equivalent to that of Theorem 4.1 in [EH85] itself.

Thus we have that, if φ is satisfiable, it has a "small model", i.e., a model of finite size function of the length of φ.

Item (iv) We show that GCTL distinguishes between bisimilar models. Consider the two trees $\mathcal{T}$ and $\mathcal{T}'$ such as $\mathcal{T}$ contains only the root node and one successor, while $\mathcal{T}'$ contains also another successor. Formally, $\mathcal{T} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$, with $\mathrm{AP} = \emptyset$, $\mathrm{W} = \{\varepsilon, 0\}$, and $\mathrm{R} = \{(\varepsilon, 0)\}$, and $\mathcal{T}' = \langle \mathrm{AP}, \mathrm{W}', \mathrm{R}', \mathrm{L} \rangle$, with $\mathrm{W}' = \mathrm{W} \cup \{1\}$, and $\mathrm{R}' = \mathrm{R} \cup \{(\varepsilon, 1)\}$. From the definition of bisimulation, it immediately follows that $\mathcal{K} \sim \mathcal{K}'$. Now, consider the *GCTL* formula $\varphi = \mathsf{E}^{>1} \mathsf{X}\, \mathfrak{t}$. Then, we have that $\mathfrak{P}_A(\mathcal{T}, \varepsilon, \mathsf{X}\, \mathfrak{t}) = \{\pi\}$, with $\pi(1) = 0$, and $\mathfrak{P}_A(\mathcal{T}', \varepsilon, \mathsf{X}\, \mathfrak{t}) = \{\pi, \pi'\}$, with $\pi'(1) = 1$. Since π and π' are incomparable, it holds that $\{\pi\} = \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{T}, \varepsilon, \mathsf{X}\, \mathfrak{t})) \neq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{T}', \varepsilon, \mathsf{X}\, \mathfrak{t})) = \{\pi, \pi'\}$, so $\mathcal{T}, \varepsilon \not\models \varphi$, but $\mathcal{T}', \varepsilon \models \varphi$, and then $\mathcal{T} \not\models \varphi$ but $\mathcal{T}' \models \varphi$, i.e., φ is not an invariant on $\mathcal{K}$ and $\mathcal{K}'$.

Item (v) Consider the above two bisimilar tree models $\mathcal{T}$ and $\mathcal{T}'$. Since CTL is invariant under bisimulation, it cannot distinguish between them. Moreover, CTL is a sublogic of GCTL, as observer in Remark 1, so we have that the latter can characterize more models than those characterizable by the former logic. Then, it follows that GCTL is more expressive than CTL.

D Theorem 3 (Soundness and completeness)

Lemma 5. *For all sequences $\{h_i\}_i^g \in \mathsf{CP}(g)$ and $\{h'_i\}_i^{g-1} \in \mathsf{CP}(g-1)$, with $h_g = 0$, there exists an index $j \in \mathbb{N}_{(g-1)+}$ such that $h'_j < h_j$.*

Proof. If for contradiction for all $j \in \mathbb{N}_{(g-1)+}$ we have $h'_j \geq h_j$ then, since $h_g = 0$, we would find out that $g - 1 = \sum_{j=1}^{g-1} h'_j \geq \sum_{j=1}^{g-1} h_j = \sum_{j=1}^{g} h_j = g$, that is impossible.

Lemma 6. *For all sequences $\{M_i\}_i^{g+1} \in \mathsf{spart}(D, [g])$ it holds that $|M_g| \leq 1$.*

Proof. If $g = 2$, there are only two sequences $\{h_i\}_i^2 \in \mathsf{CP}(2)$ and those are $h_1 = 1$ and $h_2 = 1$ or $h_1 = 2$ and $h_2 = 0$. Since in both the cases we have $|M_2| < h_1$ it holds that $|M_2| < 2$. Consider now $g > 2$. Then, in $\mathsf{CP}(g)$ there exists the sequence $\{h_i\}_i^2$ such that $h_1 = 2$, $h_2 = \ldots = h_{g-1} = 1$, and $h_g = 0$. By Definition 4, there exists a j such that $|M_{j+1}| < h_j$. It cannot be $j = g$ since $|M_{g+1}| < h_g = 0$ is false. If $j = g - 1$ it holds $|M_g| < h_{g-1} = 1$ and then we have done with the proof. Finally, for $j < g - 1$, it holds $|M_{j+1}| < h_j \leq 2$. It is known that $M_g \subseteq \ldots \subseteq M_{j+1}$ so we have $|M_g| \leq |M_{j+1}| < 2$.

Lemma 7. *Let $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ be a Kripke structure and $w \in \mathrm{W}$ be a world. Then, $\mathcal{K}, w \models \mathsf{E}^{\geq 1} \tilde{\mathsf{X}}\, \mathfrak{f}$ (resp., $\mathcal{K}, w \models \mathsf{A}^{<1} \mathsf{X}\, \mathfrak{t}$) iff $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) = \emptyset$, (resp., $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \neq \emptyset$).*

Proof. Then, $\mathcal{K}, w \models \mathsf{E}^{\geq 1} \tilde{\mathsf{X}}\, \mathfrak{f}$ iff $\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \tilde{\mathsf{X}}\, \mathfrak{f})) \neq \emptyset$, that is $\{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \forall \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi' \text{ implies } \mathcal{K}, \pi', 0 \models \tilde{\mathsf{X}}\, \mathfrak{f}\} \neq \emptyset$. Now, since for all $\pi \in \mathsf{paths}(\mathcal{K}, w)$ we have $\mathcal{K}, \pi, 0 \models \tilde{\mathsf{X}}\, \mathfrak{f}$ iff $|\pi| = 1$, it holds that $\mathcal{K}, w \models \mathsf{E}^{\geq 1} \tilde{\mathsf{X}}\, \mathfrak{f}$ iff $\{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \forall \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi' \text{ implies } |\pi| = 1\} \neq \emptyset$, that is $\mathcal{K}, w \models \mathsf{E}^{\geq 1} \tilde{\mathsf{X}}\, \mathfrak{f}$ iff for all $\pi \in \mathsf{paths}(\mathcal{K}, w)$ it holds $|\pi| = 1$, and then there is no world $w' \in \mathsf{dom}(\mathcal{K})$ such that $(w, w') \in \mathrm{R}$, hence $\mathcal{K}, w \models \mathsf{E}^{\geq 1} \tilde{\mathsf{X}}\, \mathfrak{f}$ iff $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) = \emptyset$. Now, since $\mathcal{K}, w \models \mathsf{A}^{<1} \mathsf{X}\, \mathfrak{t}$ iff $\mathcal{K}, w \models \neg \mathsf{E}^{\geq 1} \tilde{\mathsf{X}}\, \mathfrak{f}$, we have that $\mathcal{K}, w \models \mathsf{A}^{<1} \mathsf{X}\, \mathfrak{t}$ iff $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \neq \emptyset$.

Lemma 8. *Let $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ be a Kripke structure, $w \in \mathrm{W}$ be a world, and φ be a GCTL state formula. Then, it holds that (i) $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\varphi))| = |\{w' \in \mathrm{W} \mid (w, w') \in \mathrm{R},\ \mathcal{K}, w' \models \varphi\}|$ and (ii) $|\mathsf{minstructs}(\mathsf{paths}(\mathcal{K}, w) \setminus \mathfrak{P}_E(\mathcal{K}, w, \tilde{\mathsf{X}}\,\varphi))| = |\{w' \in \mathrm{W} \mid (w, w') \in \mathrm{R},\ \mathcal{K}, w' \models \neg\varphi\}|$.*

Proof. Now, we prove the equality *(i)*. The equality *(ii)* easily follows by the duality equality $\mathfrak{P}_A(\mathcal{K}, w, \neg\varphi) = \mathsf{paths}(\mathcal{K}, w) \setminus \mathfrak{P}_E(\mathcal{K}, w, \varphi)$.

By definition, it holds that $\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\varphi)$ is equal to $\{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \forall \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi'$ implies $\mathcal{K}, \pi', 0 \models \mathsf{X}\,\varphi\}$, so it is equal to $\{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \forall \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi'$ implies $\mathcal{K}, \pi', 1 \models \varphi\}$. It is evident then that for each path $\pi \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\varphi)$ it holds that $\mathcal{K}, \pi, 1 \models \varphi$, so, since φ is a state formula, we have that $\mathcal{K}, \pi(1) \models \varphi$ and then $\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\varphi) = \{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid |\pi| > 1,\ \mathcal{K}, \pi(1) \models \varphi\}$. Now, it is obvious that minimal paths in the set $\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\varphi)$ are all the paths of length 2, starting in w, and which satisfy φ on the second world, so we obtain that the set $\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\varphi))$ is equal to $\{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid |\pi| = 2,\ \mathcal{K}, \pi(1) \models \varphi\}$. Since the paths of length two, which have w as their first world, are as many as the successors of w itself, because such paths are made by w and by one of its successors that is $(\pi(0), \pi(1)) = (w, \pi(1)) \in \mathrm{R}$, we have that $|\{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid |\pi| = 2,\ \mathcal{K}, \pi(1) \models \varphi\}|$ is equal to $|\{w' \in \mathrm{W} \mid (w, w') \in \mathrm{R},\ \mathcal{K}, w' \models \varphi\}|$, thus the thesis follows.

Definition 12. (Partial run of a PABT) *A* partial run *of a PABT $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F \rangle$ on a Σ-labeled tree $\langle \mathsf{T}, \mathsf{inp} \rangle$ is a $(\mathsf{T} \times Q \times \mathbb{N}_{(b)} \times \mathbb{N}_{(1)})$-labeled full tree $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ satisfying the following conditions:*

1. $\mathsf{prun}(\varepsilon) = (\varepsilon, q_0, g_0, l_0)$, *for some* $l_0 \in \mathbb{N}_{(1)}$;
2. *for all* $y \in \mathsf{T}_{\mathsf{pr}}$ *with* $\mathsf{prun}(y) = (x, q, g, 0)$, *it holds that* $\mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(y) = \emptyset$, *i.e., y have no successors;*
3. *for all* $y \in \mathsf{T}_{\mathsf{pr}}$ *with* $\mathsf{prun}(y) = (x, q, g, 1)$, *there exists a set* $S \subseteq D_b^{\varepsilon} \times Q$, *where* $S \models \delta(q, g, \mathsf{inp}(x))$, *and a set* $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathsf{T}}(x))$ *such that for all configurations* $(d, q', g') \in E$ *there is a node* $y' \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(y)$ *such that* $\mathsf{prun}(y') = (x \cdot d, q', g', l)$, *for some* $l \in \mathbb{N}_{(1)}$.

A 0-labeled (resp., 1-labeled) node is a node with label that ends with 0 (resp., 1). The partial run with all 1-labeled nodes is called a 1-labeled partial run. Finally, the partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is accepting *iff all its infinite paths satisfy the acceptance condition, i.e., for all paths $\pi \preccurlyeq \mathsf{T}_{\mathsf{pr}}$, with $|\pi| = \omega$, it holds that $\mathsf{inf}(\mathsf{prun}_{|\pi}) \cap \mathsf{T} \times F \times \mathbb{N}_{(1)} \neq \emptyset$.*

It's evident that, the projection of a 1-labeled partial run on $\mathsf{T} \times Q \times \mathbb{N}_{(b)}$ is a run, moreover, if such a partial run is accepting the corresponding run is accepting too. In addition, if there exists a run, we can build a corresponding partial run by adding to all labels a 1 at the end of the labels.

Let $a \in \mathbb{N}^*$ and $\mathrm{X} \subseteq \mathbb{N}^*$. Then, with $a \triangleleft \mathrm{X}$ and $a \triangleright \mathrm{X}$ we denote, respectively, the two sets $\{x \in \mathbb{N}^* \mid a \cdot x \in \mathrm{X}\}$ and $\{a \cdot x \in \mathbb{N}^* \mid x \in \mathrm{X}\}$. Moreover, let $\langle \mathrm{X}, \mathrm{L} \rangle$ be a labeled tree. Then, with $a \triangleleft \langle \mathrm{X}, \mathrm{L} \rangle$ we denote the labeled tree $\langle \mathrm{X}', \mathrm{L}' \rangle$, where it is set $\mathrm{X}' = a \triangleleft \mathrm{X}$ and, for all $x \in \mathrm{X}'$, $\mathrm{L}'(x) = \mathrm{L}(a \cdot x)$.

Definition 13. (Extention of a partial run) *Let us consider a partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ of a PABT $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F \rangle$ on an input $\langle \mathsf{T}, \mathsf{inp} \rangle$, with a sequence of nodes*

$\{y_i\}_i^n \subseteq \mathsf{T}_{\mathsf{pr}}$ *such that* $\mathsf{prun}(y_i) = (x_i, q_i, g_i, 0)$, *for all indexes* $i \in \mathbb{N}_{(n)+}$. *Consider also a sequence of partial runs* $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^n$ *of a sequence of PABTs* $\{\mathcal{A}_i\}_i^n$, $\mathcal{A}_i = \langle Q, \Sigma, b, \delta, q_i, g_i, F \rangle$, *on the inputs* $\{x_i \triangleleft \langle \mathsf{T}, \mathsf{inp} \rangle\}_i^n$. *Then, we call an extension of* $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ *with respect to* $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^n$ *on the nodes* $\{y_i\}_i^n$, *a* $(\mathsf{T} \times Q \times \mathbb{N}_{(b)} \times \mathbb{N}_{(1)})$*-labeled tree* $\langle \mathsf{T}_{\mathsf{pr}}', \mathsf{prun}' \rangle$ *obtained by substituting each node* y_i *with the tree* $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$. *More formally, we construct* $\langle \mathsf{T}_{\mathsf{pr}}', \mathsf{prun}' \rangle$ *as follows:* (i) $\mathsf{T}_{\mathsf{pr}}' = \mathsf{T}_{\mathsf{pr}} \cup \bigcup_{i=1}^n (y_i \triangleright \mathsf{T}_{\mathsf{pr}_i})$*;* (ii) *for all* $z \in \mathsf{T}_{\mathsf{pr}} \setminus \bigcup_{i=1}^n \{y_i\}$ *it holds that* $\mathsf{prun}'(z) = \mathsf{prun}(z)$*;* (iii) *for all* $i \in \mathbb{N}_{(n)+}$ *and* $z \in \mathsf{T}_{\mathsf{pr}}'$, *with* $z = y_i \cdot y'$ *and* $\mathsf{prun}_i(y') = (x', q', g', l')$, *it holds that* $\mathsf{prun}'(z) = (x_i \cdot x', q', g', l')$.

Lemma 9. *Let* $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ *be a partial run of a PABT* $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F \rangle$ *on an input* $\langle \mathsf{T}, \mathsf{inp} \rangle$, *with a sequence of nodes* $\{y_i\}_i^n \subseteq \mathsf{T}_{\mathsf{pr}}$ *such that* $\mathsf{prun}(y_i) = (x_i, q_i, g_i, 0)$ *for all indexes* $i \in \mathbb{N}_{(n)+}$, *and* $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^n$ *be a a sequence of partial run of a sequence of PABTs* $\{\mathcal{A}_i\}_i^n$, $\mathcal{A}_i = \langle Q, \Sigma, b, \delta, q_i, g_i, F \rangle$, *on the inputs* $\{\langle \mathsf{T}_i, \mathsf{inp}_i \rangle\}_i^n$, *where* $\langle \mathsf{T}_i, \mathsf{inp}_i \rangle = x_i \triangleleft \langle \mathsf{T}, \mathsf{inp} \rangle$. *Then, we have that:*

1. *the extension* $\langle \mathsf{T}_{\mathsf{pr}}', \mathsf{prun}' \rangle$ *of* $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ *with respect to* $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^n$ *on the nodes* $\{y_i\}_i^n$ *is a partial run of* $\mathcal{A}$ *on* $\langle \mathsf{T}, \mathsf{inp} \rangle$*;*
2. *if the partial runs* $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ *and* $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^n$ *are accepting then* $\langle \mathsf{T}_{\mathsf{pr}}', \mathsf{prun}' \rangle$ *is accepting as well;*
3. *if* $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^n$ *are* 1*-labeled and* $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ *has only* $\{y_i\}_i^n$ *as* 0*-labeled nodes then* $\langle \mathsf{T}_{\mathsf{pr}}', \mathsf{prun}' \rangle$ *is* 1*-labeled as well.*

Proof. Item (i) We show that the three properties in Definition 12 of partial run holds.

1. Property of the root.
   (a) If it is labeled by $(\varepsilon, q, h, 1)$ then its label in $\langle \mathsf{T}_{\mathsf{pr}}', \mathsf{prun}' \rangle$ remains the same.
   (b) If it is labeled by $(\varepsilon, q, h, 0)$ it has no successor, so there exists a unique node $y_1 \in \mathsf{T}_{\mathsf{pr}}$ such that $\mathsf{prun}(y_1) = (x_1, q_1, g_1, 0)$. The corresponding partial run $\langle \mathsf{T}_{\mathsf{pr}1}, \mathsf{prun}_1 \rangle$ has a root labeled by (ε, q, h, l), with $l \in\in \mathbb{N}_{(1)}$, thus, by substitution, the root has the same label in $\langle \mathsf{T}_{\mathsf{pr}}', \mathsf{prun}' \rangle$.
2. Property of 0-labeled nodes.
   (a) For all $z \in \mathsf{T}_{\mathsf{pr}}'$, such that for all $i \in \mathbb{N}_{(n)+}$ it holds that y_i is not a prefix of z, we have that $\mathsf{prun}'(z) = \mathsf{prun}(z) = (x, q, g, 0)$. Moreover, since $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is a partial run, z has no successor in it, so it holds that $\mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}'}(z) = \emptyset$.
   (b) For all $z \in \mathsf{T}_{\mathsf{pr}}'$, such that there exists $i \in \mathbb{N}_{(n)+}$ for which it holds that $z = y_i \cdot y$, we have that $\mathsf{prun}'(z) = (x_i \cdot x, q, g, 0)$, since $\mathsf{prun}_i(y) = (x, q, g, 0)$. Moreover, since $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ is a partial run, y has no successor in it, so it holds that $\mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}'}(z) = \emptyset$.
3. Property of 1-labeled nodes.
   (a) For all $z \in \mathsf{T}_{\mathsf{pr}}'$, such that for all $i \in \mathbb{N}_{(n)+}$ it holds that y_i is not a prefix of z, we have that $\mathsf{prun}'(z) = \mathsf{prun}(z) = (x, q, g, 1)$. Moreover, for all successors $z' \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}'}(z)$ with $\mathsf{prun}(z') = (x', q', h', l')$, it holds that $\mathsf{prun}'(z') = (x', q', h', l'')$, where l' may be not equal to l'' only if $z' = y$. Now, since the property expressed in item 3 only depends on the first three components of the labels and $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is a partial run, it necessarily holds that item 3 also holds between z and its successors in $\langle \mathsf{T}_{\mathsf{pr}}', \mathsf{prun}' \rangle$.

(b) For all $z \in \mathsf{T_{pr}}'$, such that there exists $i \in \mathbb{N}_{(n)+}$ for which it holds that $z = y_i \cdot y$ with $\mathsf{prun}'(z) = (x_i \cdot x, q, g, 1)$ and $\mathsf{prun}_i(y) = (x, q, g, 1)$, we have that, since $\langle \mathsf{T_{pr}}_i, \mathsf{prun}_i \rangle$ is a partial run, there exists a set $S \subseteq D_b^{\varepsilon} \times Q$, where $S \models \delta(q, g, \mathsf{inp}_i(x))$, and a set $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathsf{T}_i}(x))$ such that for all configurations $(d, q', g') \in E$ there is a node $y' \in \mathsf{succ}_{\mathsf{T_{pr}}_i}(y)$ such that $\mathsf{prun}_i(y') = (x \cdot d, q', g', l)$. Now, since $\mathsf{inp}'(x_i \cdot x) = \mathsf{inp}_i(x)$ and $\mathsf{dir}_{\mathsf{T}'}(x_i \cdot x) = \mathsf{dir}_{\mathsf{T}_i}(x)$ we have that there exists a set $S \subseteq D_b^{\varepsilon} \times Q$, where $S \models \delta(q, g, \mathsf{inp}'(x_i \cdot x))$, and a set $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathsf{T}'}(x_i \cdot x))$ such that for all configurations $(d, q', g') \in E$ there is a node $z' = y_i \cdot y' \in \mathsf{succ}_{\mathsf{T_{pr}}'}(z)$ such that $\mathsf{prun}'(z') = (x_i \cdot x \cdot d, q', g', l)$.

Item (ii) Let us consider an infinite path $\pi \preccurlyeq \mathsf{T_{pr}}'$. Then, two situations can arise.

1. If $\pi \preccurlyeq \mathsf{T_{pr}}$, we have that $\mathsf{inf}(\mathsf{prun}'_{|\pi}) \cap \mathsf{T} \times F \neq \emptyset$, since $\langle \mathsf{T_{pr}}, \mathsf{prun} \rangle$ is accepting and $\mathsf{prun}'_{|\pi} = \mathsf{prun}_{|\pi}$.
2. If $\pi \not\preccurlyeq \mathsf{T_{pr}}$ then there exists an index $i \in \mathbb{N}_{(n)+}$ and a path $\pi' \preccurlyeq \mathsf{T_{pr}}_i$ such that $\pi_{\geq |y_i|} = \pi'$. Since $\langle \mathsf{T_{pr}}_i, \mathsf{prun}_i \rangle$ is accepting and $\mathsf{inf}(\mathsf{prun}'_{|\pi}) = \mathsf{inf}(\mathsf{prun}_{i|\pi'})$, we have that $\mathsf{inf}(\mathsf{prun}'_{|\pi}) \cap \mathsf{T} \times F \neq \emptyset$.

Item (iii) Finally, let us consider a node $z \in \mathsf{T_{pr}}'$. Then, two situations can arise.

1. If $z \in \mathsf{T_{pr}}$ and $z \neq y_i$, for all indexes $i \in \mathbb{N}_{(n)+}$, it holds that $\mathsf{prun}'(z) = \mathsf{prun}(z)$, so z is 1-labeled.
2. If there is an index $i \in \mathbb{N}_{(n)+}$ such that $z = y_i \cdot y$, it holds that $\mathsf{prun}'(z) = (x_i \cdot x, q, g, 1)$, since $\mathsf{prun}_i(y) = (x, q, g, 1)$, so z is also 1-labeled.

Definition 14. (Extraction of a partial run) *Let us consider a partial run $\langle \mathsf{T_{pr}}, \mathsf{prun} \rangle$ of a PABT $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F \rangle$ on an input $\langle \mathsf{T}, \mathsf{inp} \rangle$, with a sequence of nodes $\{y_i\}_i^n \subseteq \mathsf{T_{pr}}$ such that $\mathsf{prun}(y_i) = (x_i, q_i, g_i, l_i)$ for all indexes $i \in \mathbb{N}_{(n)+}$. Then, we call an extraction of $\langle \mathsf{T_{pr}}, \mathsf{prun} \rangle$ on the nodes $\{y_i\}_i^n$ a sequence of $(\mathsf{T} \times Q \times \mathbb{N}_{(b)} \times \mathbb{N}_{(1)})$-labeled trees $\{\langle \mathsf{T_{pr}}_i, \mathsf{prun}_i \rangle\}_i^n$, where $\langle \mathsf{T_{pr}}_i, \mathsf{prun}_i \rangle$ is given by the subtree rooted at node y_i for all indexes $i \in \mathbb{N}_{(n)+}$. More formally, we construct a $\langle \mathsf{T_{pr}}_i, \mathsf{prun}_i \rangle$ as follows:* (i) $\mathsf{T_{pr}}_i = y_i \triangleleft \mathsf{T_{pr}}$*;* (ii) *for all $z \in \mathsf{T_{pr}}_i$, with $\mathsf{prun}(y_i \cdot z) = (x_i \cdot x', q', g', l')$, it holds that $\mathsf{prun}_i(z) = (x', q', g', l')$.*

Lemma 10. *Let $\langle \mathsf{T_{pr}}, \mathsf{prun} \rangle$ be a partial run of a PABT $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F \rangle$ on an input $\langle \mathsf{T}, \mathsf{inp} \rangle$, with a sequence of nodes $\{y_i\}_i^n \subseteq \mathsf{T_{pr}}$ such that $\mathsf{prun}(y_i) = (x_i, q_i, g_i, l_i)$ for all indexes $i \in \mathbb{N}_{(n)+}$. Then we have:*

1. *the extraction $\{\langle \mathsf{T_{pr}}_i, \mathsf{prun}_i \rangle\}_i^n$ of $\langle \mathsf{T_{pr}}, \mathsf{prun} \rangle$ on the nodes $\{y_i\}_i^n$ is a sequence of partial runs of the sequence of PABTs $\{\mathcal{A}_i\}_i^n$, $\mathcal{A}_i = \langle Q, \Sigma, b, \delta, q_i, g_i, F \rangle$, on the inputs $\{\langle \mathsf{T}_i, \mathsf{inp}_i \rangle\}_i^n$, where $\langle \mathsf{T}_i, \mathsf{inp}_i \rangle = x_i \triangleleft \langle \mathsf{T}, \mathsf{inp} \rangle$;*
2. *if the partial run $\langle \mathsf{T_{pr}}, \mathsf{prun} \rangle$ is accepting then all the partial runs $\{\langle \mathsf{T_{pr}}_i, \mathsf{prun}_i \rangle\}_i^n$ are accepting as well;*
3. *if $\langle \mathsf{T_{pr}}, \mathsf{prun} \rangle$ is 1-labeled then all the partial runs $\{\langle \mathsf{T_{pr}}_i, \mathsf{prun}_i \rangle\}_i^n$ are 1-labeled as well.*

Proof. Item (i) We show that the three properties in Definition 12 of partial run holds.

1. Property of the root.
   The root of $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ is labeled by $\mathsf{prun}_i(\varepsilon) = (\varepsilon, q_i, h_i, l_i)$, since $\mathsf{prun}(y_i) = (x_i, q_i, h_i, l_i)$.
2. Property of 0-labeled nodes.
   For all $y \in \mathsf{T}_{\mathsf{pr}_i}$, there exists $z \in \mathsf{T}_{\mathsf{pr}}$ such that $z = y_i \cdot y$, so we have that $\mathsf{prun}_i(y) = (x, q, g, 0)$, since $\mathsf{prun}(z) = (x_i \cdot x, q, g, 0)$. Moreover, since $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is a partial run, z has no successor in it, so it holds that $\mathsf{succ}_{\mathsf{T}_{\mathsf{pr}_i}}(y) = \emptyset$.
3. Property of 1-labeled nodes.
   For all $y \in \mathsf{T}_{\mathsf{pr}_i}$, there exists $z \in \mathsf{T}_{\mathsf{pr}}$ such that $z = y_i \cdot y$, so we have that $\mathsf{prun}_i(y) = (x, q, g, 1)$, since $\mathsf{prun}'(z) = (x_i \cdot x, q, g, 1)$. Moreover, $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is a partial run, so there exists a set $S \subseteq D_b^{\varepsilon} \times Q$, where $S \models \delta(q, g, \mathsf{inp}(x_i \cdot x))$, and a set $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathsf{T}}(x_i \cdot x))$ such that for all configurations $(d, q', g') \in E$ there is a node $z' = y_i \cdot y' \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(z)$ such that $\mathsf{prun}(z') = (x_i \cdot x \cdot d, q', g', l)$.
   Now, since $\mathsf{inp}_i(x) = \mathsf{inp}(x_i \cdot x)$ and $\mathsf{dir}_{\mathsf{T}_i}(x) = \mathsf{dir}_{\mathsf{T}}(x_i \cdot x)$ we have that there exists a set $S \subseteq D_b^{\varepsilon} \times Q$, where $S \models \delta(q, g, \mathsf{inp}_i(x))$, and a set $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathsf{T}_i}(x))$ such that for all configurations $(d, q', g') \in E$ there is a node $y' \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}_i}}(y)$ such that $\mathsf{prun}_i(y') = (x \cdot d, q', g', l)$.

Item (ii) Let us consider an infinite path $\pi \preccurlyeq \mathsf{T}_{\mathsf{pr}_i}$. Then there exists a path $\pi' \preccurlyeq \mathsf{T}_{\mathsf{pr}}$ such that $\pi = \pi'_{\geq |y_i|}$. Since $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is accepting and $\mathsf{inf}(\mathsf{prun}_{i|\pi}) = \mathsf{inf}(\mathsf{prun}_{|\pi'})$, we have that $\mathsf{inf}(\mathsf{prun}_{i|\pi}) \cap \mathsf{T} \times F \neq \emptyset$.

Item (iii) Finally, let us consider a node $y \in \mathsf{T}_{\mathsf{pr}_i}$. Then there is a node $z \in \mathsf{T}_{\mathsf{pr}}$ such that $z = y_i \cdot y$, so it holds that $\mathsf{prun}'(y) = (x, q, g, 1)$, since $\mathsf{prun}_i(z) = (x_i \cdot x, q, g, 1)$, i.e., y is also 1-labeled.

Lemma 11. *Let φ be a GCTL state formula and $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ be a Kripke structure. Then, for all worlds $w \in \mathrm{W}$ and subformula $\varphi' \in \mathsf{cl}(\varphi)$ it holds that $\mathcal{K}, w \models \varphi'$ iff the unwinding $\mathcal{U}_w^{\mathcal{K}} = \langle \mathrm{AP}', \mathrm{W}', \mathrm{R}', \mathrm{L}' \rangle$ of $\mathcal{K}$ starting from w is accepted by the automaton $\mathcal{A}'_{\varphi'} = \langle \mathsf{ecl}(\varphi), 2^{\mathrm{AP}}, \mathsf{deg}(\varphi), \delta, \varphi', 0, F \rangle$. Moreover, if $\varphi' = \mathsf{E}^{\geq g}(\varphi_1 \mathsf{U} \varphi_2)$ (resp., $\mathsf{A}^{<g}(\varphi_1 \mathsf{U} \varphi_2)$, $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R} \varphi_2)$, or $\mathsf{A}^{<g}(\varphi_1 \mathsf{R} \varphi_2)$), the same unwinding is accepted by the automaton $\mathcal{A}''_{\varphi'} = \langle \mathsf{ecl}(\varphi), 2^{\mathrm{AP}}, \mathsf{deg}(\varphi), \delta, \gamma, g, F \rangle$, with $\gamma = \langle \varphi_1 \mathsf{U} \varphi_2 \rangle \in \mathsf{ecl}(\varphi)$ (resp., $[\varphi_1 \mathsf{U} \varphi_2]$, $\langle \varphi_1 \mathsf{R} \varphi_2 \rangle$, or $[\varphi_1 \mathsf{R} \varphi_2]$).*

Proof. We will show the thesis by induction on the structure of the formula φ. Note that, $\mathcal{A}'_{\varphi'}$ (resp., $\mathcal{A}''_{\varphi'}$) accepts the unwinding $\mathcal{U}_w^{\mathcal{K}}$ iff it has a run on it and so a 1-labeled partial run on it.
Base case: Atomic propositions. $\varphi' = p$ (resp., $\varphi' = \neg p$), with $p \in \mathrm{AP} = \mathrm{AP}'$.

1. If $\mathcal{K}, w \models \varphi'$ then the run of $\mathcal{A}'_{\varphi'}$ consisting of the only root is accepting, indeed we have $\delta(\varphi', 0, \mathrm{L}'(\varepsilon)) = \mathfrak{t}$ since $\delta(p, 0, \mathrm{L}'(\varepsilon)) = (p \in \mathrm{L}'(\varepsilon))$ and $p \in \mathrm{L}'(\varepsilon) = \mathrm{L}(w)$ (resp., $\delta(\neg p, 0, \mathrm{L}'(\varepsilon)) = (p \notin \mathrm{L}'(\varepsilon))$ and $p \notin \mathrm{L}'(\varepsilon) = \mathrm{L}(w)$), thus we can choose an empty set S satisfying the delta, which implies that the corresponding run will not have successors of the root and then any infinite path, so it will be accepting for definition.
2. Let us suppose that there exists an accepting run for $\mathcal{A}'_{\varphi'}$ on the unwinding tree in input. Since $\delta(p, 0, \mathrm{L}'(\varepsilon)) = (p \in \mathrm{L}'(\varepsilon))$ (resp., $\delta(\neg p, 0, \mathrm{L}'(\varepsilon)) = (p \notin \mathrm{L}'(\varepsilon))$) the only way for the tree to be accepting is that $\delta(\varphi', 0, \mathrm{L}'(\varepsilon)) = \mathfrak{t}$, thus p must be (resp., must not be) in $\mathrm{L}'(\varepsilon) = \mathrm{L}(w)$ and then $\mathcal{K}, w \models \varphi'$.

Inductive case: And (resp., Or). $\varphi' = \varphi_1 \wedge \varphi_2$ (resp., $\varphi' = \varphi_1 \vee \varphi_2$).

1. If $\mathcal{K}, w \models \varphi'$ it holds that $\mathcal{K}, w \models \varphi_1$ and $\mathcal{K}, w \models \varphi_2$ (resp., $\mathcal{K}, w \models \varphi_1$ or $\mathcal{K}, w \models \varphi_2$). For inductive hypothesis we have that both $\mathcal{A}'_{\varphi_1}$ and $\mathcal{A}'_{\varphi_2}$ have (resp., at least one between $\mathcal{A}'_{\varphi_1}$ and $\mathcal{A}'_{\varphi_2}$ has) an accepting 1-labeled partial run on the unwinding tree $\mathcal{U}^{\mathcal{K}}_w$ in input. Let $\langle \mathsf{T}_{\mathsf{pr}_1}, \mathsf{prun}_1 \rangle$ and $\langle \mathsf{T}_{\mathsf{pr}_2}, \mathsf{prun}_2 \rangle$ be these two partial runs (resp., let $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ be this partial run).
   Since $\delta(\varphi_1 \wedge \varphi_2, 0, \sigma) = (\varepsilon, \varphi_1) \wedge (\varepsilon, \varphi_2)$ (resp., $\delta(\varphi_1 \vee \varphi_2, 0, \sigma) = (\varepsilon, \varphi_1) \vee (\varepsilon, \varphi_2)$), the transition function is satisfied by the set $S = \{(\varepsilon, \varphi_1), (\varepsilon, \varphi_2)\}$ (resp., $\{(\varepsilon, \varphi_i)\}$), so we construct the following accepting partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$: $\mathsf{T}_{\mathsf{pr}} = \{\varepsilon, 0, 1\}$, $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$, $\mathsf{prun}(0) = (\varepsilon, \varphi_1, 0, 0)$, $\mathsf{prun}(1) = (\varepsilon, \varphi_2, 0, 0)$ (resp., $\mathsf{T}_{\mathsf{pr}} = \{\varepsilon, 0\}$, $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$, $\mathsf{prun}(0) = (\varepsilon, \varphi_i, 0, 0)$).
   Now, extending $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ with $\langle \mathsf{T}_{\mathsf{pr}_1}, \mathsf{prun}_1 \rangle$ and $\langle \mathsf{T}_{\mathsf{pr}_2}, \mathsf{prun}_2 \rangle$ on 0 and 1 (resp., extending $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ with $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ on 0), by Lemma 5 we obtain an accepting 1-labeled partial run of $\mathcal{A}'_{\varphi'}$ on the unwinding.
2. Let us suppose that there exists an accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on the unwinding tree $\mathcal{U}^{\mathcal{K}}_w$ in input. Since $\delta(\varphi_1 \wedge \varphi_2, 0, \sigma) = (\varepsilon, \varphi_1) \wedge (\varepsilon, \varphi_2)$ (resp., $\delta(\varphi_1 \vee \varphi_2, 0, \sigma) = (\varepsilon, \varphi_1) \vee (\varepsilon, \varphi_2)$), the transition function is satisfied by the set $S = \{(\varepsilon, \varphi_1), (\varepsilon, \varphi_2)\}$ (resp., $\{(\varepsilon, \varphi_i)\}$), so the root of the partial run must have two successors 0 and 1 with labels $(\varepsilon, \varphi_1, 0, 1)$ and $(\varepsilon, \varphi_2, 0, 1)$ (resp., at least the successor 0 with label $(\varepsilon, \varphi_i, 0, 1)$).
   Now, consider the two trees $\langle \mathsf{T}_{\mathsf{pr}_1}, \mathsf{prun}_1 \rangle$ and $\langle \mathsf{T}_{\mathsf{pr}_2}, \mathsf{prun}_2 \rangle$ (resp., the tree $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$) extracted from $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ on the nodes 0 and 1 (resp., on the node 0). By Lemma 6, we obtain that these two trees are (resp., this tree is an) accepting 1-labeled partial runs (resp., run) of the automata $\mathcal{A}'_{\varphi_1}$ and $\mathcal{A}'_{\varphi_2}$ (resp., of the automaton $\mathcal{A}'_{\varphi_i}$) on the same tree in input, so by inductive hypothesis it holds that $\mathcal{K}, w \models \varphi_1$ and $\mathcal{K}, w \models \varphi_2$ (resp., $\mathcal{K}, w \models \varphi_1$ or $\mathcal{K}, w \models \varphi_2$) and then $\mathcal{K}, w \models \varphi'$.

Inductive case: Exists Effective Next. $\varphi' = \mathsf{E}^{\geq g}\mathsf{X}\,\varphi''$.

1. If $\mathcal{K}, w \models \mathsf{E}^{\geq g}\mathsf{X}\,\varphi''$ it holds that $\mathcal{U}^{\mathcal{K}}_w, \varepsilon \models \mathsf{E}^{\geq g}\mathsf{X}\,\varphi''$. Let $\mathrm{X} = \{x \in \mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon) \mid \mathcal{U}^{\mathcal{K}}_w, x \models \varphi''\}$. By Lemma 8, $|\mathrm{X}| = |\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{U}^{\mathcal{K}}_w, \varepsilon, \mathsf{X}\,\varphi''))| \geq g$, so it is possible to choose a set $\mathrm{X}' = \{x_1, \ldots, x_g\} \subseteq \mathbb{N}$ of g nodes in X. By inductive hypothesis, we have that $\mathcal{A}'_{\varphi''}$ has a 1-labeled accepting partial run $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ on $x_i \triangleleft \mathcal{U}^{\mathcal{K}}_w$, for each index $i \in \mathbb{N}_{(g)+}$.
   Since $\delta(\mathsf{E}^{\geq g}\mathsf{X}\,\varphi'', 0, \sigma) = (\langle g \rangle, \langle \varphi'' \rangle)$, the transition function is satisfied by the set $S = \{(\langle g \rangle, \langle \varphi'' \rangle)\}$. Now, there is a sequence of numbers $\{h_i\}_i^g \in \mathsf{CP}(g)$ with $h_1 = g$ and $h_2 = \ldots = h_g = 0$, so there is a sequence of sets $\{M_i\}_i^{g+1} \in \mathsf{spart}(\mathsf{dir}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon), \langle g \rangle)$ such that $M_1 = \mathrm{X}'$ and $M_2 = \ldots = M_{g+1} = \emptyset$. At this point, it is evident that there exists a set $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon))$ such that $E = \{(d, \langle \varphi'' \rangle, 1) \mid d \in \mathrm{X}'\}$. Moreover $\delta(\langle \varphi'' \rangle, 1, \sigma) = (\varepsilon, \varphi'')$, so we can construct the following accepting partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on $\mathcal{U}^{\mathcal{K}}_w$: $\mathsf{T}_{\mathsf{pr}} = \{\varepsilon\} \cup \mathbb{N}_{(g-1)} \cup \{i \cdot 0 \mid i \in \mathbb{N}_{(g-1)}\}$, $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$, $\mathsf{prun}(i) = (x_{i+1}, \langle \varphi'' \rangle, 1, 1)$, and $\mathsf{prun}(i \cdot 0) = (x_{i+1}, \varphi'', 0, 0)$, for $i \in \mathbb{N}_{(g-1)}$.
   Now, extending $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ with $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^g$ on $\{(i-1) \cdot 0\}_i^g$, by Lemma 5 we obtain an accepting 1-labeled partial run of $\mathcal{A}'_{\varphi'}$ on the unwinding.

2. Let us suppose that there exists an accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on the unwinding tree $\mathcal{U}_w^{\mathcal{K}}$ in input, with $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$.
   Since $\delta(\mathsf{E}^{\geq g}\mathsf{X}\,\varphi'', 0, \sigma) = (\langle g \rangle, \langle \varphi'' \rangle)$, the transition function is satisfied by the set $S = \{(\langle g \rangle, \langle \varphi'' \rangle)\}$, so there exists a sequence $\{M_i\}_i^{g+1} \in \mathsf{spart}(\mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon), \langle g \rangle)$ such that for all indexes $i \in \mathbb{N}_{(g)+}$ and directions $d \in M_i \setminus M_{i+1}$ there is a node $y \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(\varepsilon)$ such that $\mathsf{prun}(y) = (d, \langle \varphi'' \rangle, i, 1)$. Now, for all $i > 1$, $\delta(\langle \varphi'' \rangle, i, \sigma) = \mathfrak{f}$, so we have that $|M_1| = g$ and $M_2 = \ldots = M_{g+1} = \emptyset$. Moreover, $\delta(\langle \varphi'' \rangle, 1, \sigma) = (\varepsilon, \varphi'')$, so each node $y_i \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(\varepsilon)$, with $\mathsf{prun}(y_i) = (x_i, \langle \varphi'' \rangle, 1, 1)$ has a successor $y_i \cdot j$, with $j \in \mathbb{N}$, labeled by $\mathsf{prun}(y \cdot j) = (x_i, \varphi'', 0, 1)$.
   Now, consider the trees $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ extracted from $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ on the nodes $y_i \cdot j$. By Lemma 6, we obtain that these trees are accepting 1-labeled partial runs of the automata $\mathcal{A}'_{\varphi''}$ on the trees $x_i \triangleleft \mathcal{U}_w^{\mathcal{K}}$, so by inductive hypothesis it holds that $\mathcal{U}_w^{\mathcal{K}}, x_i \models \varphi''$. Let $\mathrm{X} = \{x \in \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \mid \mathcal{U}_w^{\mathcal{K}}, x \models \varphi''\}$. By Lemma 8, $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{U}_w^{\mathcal{K}}, \varepsilon, \mathsf{X}\,\varphi''))| = |\mathrm{X}| \geq |M_1| = g$, so $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \mathsf{E}^{\geq g}\mathsf{X}\,\varphi''$ and then $\mathcal{K}, w \models \mathsf{E}^{\geq g}\mathsf{X}\,\varphi''$.

Inductive case: For all Hypothetical Next. $\varphi' = \mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi''$.

1. If $\mathcal{K}, w \models \mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi''$ it holds that $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi''$. Let $\mathrm{X} = \{x \in \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \mid \mathcal{U}_w^{\mathcal{K}}, x \models \neg\varphi''\} = \{x'_1, \ldots, x'_{|\mathrm{X}|}\}$. By Lemma 8, $|\mathrm{X}| = |\mathsf{minstructs}(\mathsf{paths}(\mathcal{U}_w^{\mathcal{K}}, \varepsilon) \setminus \mathfrak{P}_E(\mathcal{U}_w^{\mathcal{K}}, \varepsilon, \tilde{\mathsf{X}}\,\varphi''))| < g$, so it is possible to choose the set $\mathrm{X}' = \{x_1, x_2, \ldots\} = \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \setminus \mathrm{X} \subseteq \mathbb{N}$. By inductive hypothesis, we have that $\mathcal{A}'_{\varphi''}$ has a 1-labeled accepting partial run $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ on $x_i \triangleleft \mathcal{U}_w^{\mathcal{K}}$, for each index $i \in \mathbb{N}_{(|\mathrm{X}'|)+}$.
   Since $\delta(\mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi'', 0, \sigma) = ([g], [\varphi''])$, the transition function is satisfied by the set $S = \{([g], [\varphi''])\}$. Now, there is a sequence of numbers $\{h_i\}_i^g \in \mathsf{CP}(g)$ with $h_1 = g$ and $h_2 = \ldots = h_g = 0$, so there is a sequence of sets $\{M_i\}_i^{g+1} \in \mathsf{spart}(\mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon), [g])$ such that $M_1 = \mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon)$, $M_2 = \mathrm{X}$, and $M_3 = \ldots = M_{g+1} = \emptyset$. At this point, it is evident that there exists a set $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon))$ such that $E = \{(d, [\varphi''], 1) \mid d \in \mathrm{X}'\} \cup \{(d, [\varphi''], 2) \mid d \in \mathrm{X}\}$. Moreover $\delta([\varphi''], 1, \sigma) = (\varepsilon, \varphi'')$ and $\delta([\varphi''], 2, \sigma) = \mathfrak{t}$, so we can construct the following accepting partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on $\mathcal{U}_w^{\mathcal{K}}$: $\mathsf{T}_{\mathsf{pr}} = \{\varepsilon\} \cup \mathbb{N}_{(|\mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon)|-1)} \cup \{(i + |\mathrm{X}|) \cdot 0 \mid i \in \mathbb{N}_{(|\mathrm{X}'|-1)}\}$, $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$, $\mathsf{prun}(i) = (x'_{i+1}, [\varphi''], 2, 1)$, $\mathsf{prun}(j + |\mathrm{X}|) = (x_{j+1}, [\varphi''], 1, 1)$, and $\mathsf{prun}((j + |\mathrm{X}|) \cdot 0) = (x_{j+1}, \varphi'', 0, 0)$, for $i \in \mathbb{N}_{(|\mathrm{X}|-1)}$ and $j \in \mathbb{N}_{(|\mathrm{X}'|-1)}$.
   Now, extending $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ with $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^{|\mathrm{X}'|}$ on $\{(i + |\mathrm{X}| - 1) \cdot 0\}_i^{|\mathrm{X}'|}$, by Lemma 5 we obtain an accepting 1-labeled partial run of $\mathcal{A}'_{\varphi'}$ on the unwinding.
2. Let us suppose that there exists an accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on the unwinding tree $\mathcal{U}_w^{\mathcal{K}}$ in input, with $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$.
   Since $\delta(\mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi'', 0, \sigma) = ([g], [\varphi''])$, the transition function is satisfied by the set $S = \{([g], [\varphi''])\}$, so there exists a sequence $\{M_i\}_i^{g+1} \in \mathsf{spart}(\mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon), [g])$ such that for all indexes $i \in \mathbb{N}_{(g)+}$ and directions $d \in M_i \setminus M_{i+1}$ there is a node $y \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(\varepsilon)$ such that $\mathsf{prun}(y) = (d, [\varphi''], i, 1)$. Note that $|M_2| < g$. Moreover, $\delta([\varphi''], 1, \sigma) = (\varepsilon, \varphi'')$, so each node $y_i \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(\varepsilon)$, with $\mathsf{prun}(y_i) = (x_i, [\varphi''], 1, 1)$ has a successor $y_i \cdot j$, with $j \in \mathbb{N}$, labeled by $\mathsf{prun}(y \cdot j) = (x_i, \varphi'', 0, 1)$.

Now, consider the trees $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ extracted from $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ on the nodes $y_i \cdot j$. By Lemma 6, we obtain that these trees are accepting 1-labeled partial runs of the automata $\mathcal{A}'_{\varphi''}$ on the trees $x_i \triangleleft \mathcal{U}^{\mathcal{K}}_w$, so by inductive hypothesis it holds that $\mathcal{U}^{\mathcal{K}}_w, x_i \models \varphi''$. Let $\mathrm{X} = \{x \in \mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon) \mid \mathcal{U}^{\mathcal{K}}_w, x \models \neg\varphi''\}$. By Lemma 8, $|\mathsf{minstructs}(\mathsf{paths}(\mathcal{K}, w) \setminus \mathfrak{P}_E(\mathcal{U}^{\mathcal{K}}_w, \varepsilon, \tilde{\mathsf{X}}\,\varphi''))| = |\mathrm{X}| \leq |M_2| < g$, so $\mathcal{U}^{\mathcal{K}}_w, \varepsilon \models \mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi''$ and then $\mathcal{K}, w \models \mathsf{A}^{<g}\mathsf{X}\,\varphi''$.

Inductive case: Does not exist a successor. $\varphi' = \mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f}$.

1. If $\mathcal{K}, w \models \mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f}$, by Lemma 7, it holds that $\mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon) = \emptyset$. Then, we can construct the following accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on $\mathcal{U}^{\mathcal{K}}_w$: $\mathsf{T}_{\mathsf{pr}} = \{\varepsilon\}$ and $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$. This partial run is also a valid run. Indeed, $\delta(\mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f}, 0, \sigma) = ([1], \mathfrak{f})$, so we can choose the set $S = \{([1], \mathfrak{f})\}$ and then, accordingly to $\mathsf{exec}(S, \mathsf{dir}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon))$, for all $x \in \mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon)$ it holds that ε has a successor with label $(x, \mathfrak{f}, 1, 1)$, but $\delta(\mathfrak{f}, 1, \sigma) = \mathfrak{f}$, so the construction is correct since $\mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon) = \emptyset$.
2. Let us suppose that there exists an accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on the unwinding tree $\mathcal{U}^{\mathcal{K}}_w$ in input, with $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$.
   Since $\delta(\mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f}, 0, \sigma) = ([1], \mathfrak{f})$, the transition function is satisfied by the set $S = \{([1], \mathfrak{f})\}$, so accordingly to $\mathsf{exec}(S, \mathsf{dir}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon))$, for all $x \in \mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon)$ it holds that ε has a successor with label $(x, \mathfrak{f}, 1, 1)$, but $\delta(\mathfrak{f}, 1, \sigma) = \mathfrak{f}$, so it must hold that $\mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon) = \emptyset$ then, by Lemma 7, $\mathcal{K}, w \models \mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f}$.

Inductive case: There exists a successor. $\varphi' = \mathsf{AX}\,\mathfrak{t}$.

1. If $\mathcal{K}, w \models \mathsf{AX}\,\mathfrak{t}$, by Lemma 7, it holds that $\mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon) \neq \emptyset$. Then, we can construct the following accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on $\mathcal{U}^{\mathcal{K}}_w$: $\mathsf{T}_{\mathsf{pr}} = \{\varepsilon, 0\}$, $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$, and $\mathsf{prun}(0) = (x, \mathfrak{t}, 1, 1)$, with $x \in \mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon)$. This partial run is also a valid run.
   Indeed, $\delta(\mathsf{AX}\,\mathfrak{t}, 0, \sigma) = (\langle 1 \rangle, \mathfrak{t})$, so we can choose the set $S = \{(\langle 1 \rangle, \mathfrak{t})\}$ and then, accordingly to $\mathsf{exec}(S, \mathsf{dir}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon))$, there exists $x \in \mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon)$ such that ε has a successor 0 with label $(x, \mathfrak{t}, 1, 1)$. Moreover, since $\delta(\mathfrak{t}, 1, \sigma) = \mathfrak{t}$, we can choose the set $S = \emptyset$ and thus 0 does not need to have any successor.
2. Let us suppose that there exists an accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on the unwinding tree $\mathcal{U}^{\mathcal{K}}_w$ in input, with $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$.
   Since $\delta(\mathsf{AX}\,\mathfrak{t}, 0, \sigma) = (\langle 1 \rangle, \mathfrak{t})$, the transition function is satisfied by the set $S = \{(\langle 1 \rangle, \mathfrak{t})\}$, so accordingly to $\mathsf{exec}(S, \mathsf{dir}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon))$, there exists $x \in \mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon)$ such that ε has a successor 0 with label $(x, \mathfrak{t}, 1, 1)$, with $x \in \mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon)$, so it must hold that $\mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_w}(\varepsilon) \neq \emptyset$ then, by Lemma 7, $\mathcal{K}, w \models \mathsf{AX}\,\mathfrak{t}$.

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