Balance Games on Colored Graphs

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Abstract. We consider games played on finite colored graphs for an infinite number of rounds, whose goal is to visit all colors with the same asymptotic frequency. Such games may represent scheduling problems with special fairness constraints. We show that the main corresponding decision problems are Co-NP-complete.

Recently, the following two problems on *colored graphs* have been addressed and solved [BFMM09]. Consider a directed graph G whose edges are labeled with tags, called colors, belonging to a fixed finite alphabet. The first problem asks whether there exists in G an infinite path ρ where all colors occur with the same asymptotic frequency, i.e., the long-run average number of occurrences for each of them is the same. The second problem addresses a stronger criterion, namely whether there exists an infinite path ρ for which there is a numerical constant c bounding the difference between the number of occurrences of any two colors, for all prefixes of ρ. The first question is called the *balance problem*, while the second one is called the *bounded difference problem*. Both can be solved in polynomial time, by reducing them to the feasibility of a linear program.

These two problems, besides being natural and canonical questions on colored graphs, can be also regarded as refinements of the well established notion of fairness [Fra86], whose simplest form checks whether all colors occur infinitely often. Classical definitions treat the frequencies of the relevant events in isolation and in a strictly qualitative manner. For example, some of these only distinguish between zero frequency (not fair), limit-zero frequency (fair, but not finitarily so [AH98]), and positive frequency (finitarily fair). The considered proposal, instead, introduces a quantitative comparison between competing events. In particular, these notions provide stronger criteria suitable for scheduling applications based on a coarse-grained model of the jobs involved.

A natural extension along this line of research consists in introducing a second decision agent in the system, thus switching from graphs to games. Games provide the appropriate framework for modeling control problems, where a controller tries to ensure a property of interest while interacting with an uncontrollable environment. As a specific example for balance games, consider the problem of synthesizing a fair scheduler for a given set of concurrent jobs with shared resources [dAFMR05]. Assume that the jobs are known as data-abstract control-flow graphs. Then, the resulting problem can be modeled as a two-player game between the scheduler and the internal non-determinism of each job. The scheduler (which in this context can be called the *ordering* player O), tries to choose a correct sequence of jobs satisfying one of the two criteria discussed above, regardless of the non-deterministic choices made by the jobs (i.e., the moves of the *disordering* player D).

More formally, we consider a game $\mathcal{G} = (A, g)$, where the *arena* A is a colored directed graph whose set of vertices is partitioned into two subsets corresponding to the positions from which each player can move, and the *goal* g of player O is to find a balanced or a bounded difference path, no matter how player D behaves. We refer to the first case as a *balance game*, and to the second one as a *bounded difference game*. We study the problem of deciding whether O has a winning strategy starting from a given node of the arena. We show that this problem is Co-NP-complete for both types of games. For the lower bound, we use a reduction from the validity problem for boolean formulas. For the upper bound, by the determinacy theorem for Borel goals [Mar75], from each given node of the arena one of the two players has a winning strategy. Thus, to show that our problem is solvable in Co-NP, we show that the problem of deciding whether player D has a winning strategy from the same node is in NP. First, we prove that both in the balance and in the bounded difference games, there exists an optimal *memoryless* strategy for player D. We obtain this result by extending the known characterizations of goals admitting memoryless winning strategies ([GZ05]) to asymmetric situations, like ours, in which only one player has a memoryless winning strategy. Then, for all possible memoryless strategies f of player D, we check if there exists a balanced or bounded difference path in the colored graph obtained by pruning the arena A in accordance with f. This check can be performed in polynomial time using the algorithms of [BFMM09].

We further show that a closely related problem is PSPACE-hard: the problem of checking whether there exists a strategy for player O such that for all strategies of D the outcome is a perfectly balanced finite path connecting two given nodes in the arena.

Finally, we consider a generalization of the balance game in which the asymptotic frequencies of all colors do not have to be equal to one another, but have to be equal to some *a priori* assigned fractions, whose sum is equal to 1. Such a game is proved to be Co-NP-complete too, using techniques similar to the ones discussed above.

References

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