Exploring the Boundary of Half Positionality*

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Abstract. Half positionality is the property of a language of infinite words to admit positional winning strategies, when interpreted as the goal of a two-player game on a graph. Such problem applies to the automatic synthesis of controllers, where positional strategies represent efficient controllers. As our main result, we describe a novel sufficient condition for half positionality, more general than what was previously known. Moreover, we compare our proposed condition with several others, proposed in the recent literature, outlining an intricate network of relationships, where only few combinations are sufficient for half positionality.

1 Introduction

Games are widely used in computer science as models to describe multi-agent systems, or the interaction between a system and its environment [8, 9, 11, 12]. Usually, the system is a component that is under the control of its designer and the environment represents all the components the designer has no direct control of. In this context, a game allows the designer to easily check whether the system can force some desired behavior (or avoid an undesired one), independently of the choices of the other components. Further, game algorithms may automatically synthesize a design that obtains the desired behavior.

We consider games played by two players on a finite graph, called *arena*. The arena models the interaction between the entities involved: a node represents a state of the interaction, and an edge represents progress in the interaction. We consider turn-based games, i.e. games where each node is associated with only one player, who is responsible for choosing the next node. A sequence of edges in the graph represents a run of the system. Player 0 wants to force the system to follow an infinite run with a desired property, expressed as a language of infinite words called *goal*. The objective of player 1 is the opposite. In this context, a *strategy* for a player is a predetermined decision that the player makes on all possible finite paths ending with a node associated to that player. A strategy is *winning* for a player if it allows him to force a desired path no matter what strategy his opponent uses. A key property of strategies is the amount of memory that they require, in order to choose their next move. The simplest strategies do not need to remember the past history of the game, i.e., their choices only depend on the current state in the game. Such strategies are called *positional*.

We are interested in determining the existence of a winning strategy for one of the players, and possibly compute an effective representation of such a strategy. To this

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aim, suitable techniques have been developed when the desired behavior of a player is specified in particular forms (see [8] for temporal logic specifications, and [3, 9, 10] for parity conditions). In synthesis problems, only positional strategies may be suitable to concrete implementation, due to space constraints. In fact, in principle even a positional strategy, which is a function from states to moves, needs an amount of storage that is proportional to the size of the state-space of the system. Symbolic representations can mitigate such issues [1]. For this reason, it is useful to know when a given goal guarantees that if player 0 (respectively, player 1) has a winning strategy then he has a positional one. This property is called *half positionality* (in the following, HP) for player 0 (resp., player 1). If a goal is HP for both players, the goal is called *full positional* (FP). Notice that HP is more important than FP in the synthesis applications we are referring to. In these applications, player 0 represents the controller to be synthesized and player 1 the environment. Hence, we are only interested in obtaining simple winning strategies for one of the two players, namely for player 0.

Full positionality has been studied and characterized: in [3, 9, 10], it was proved that the parity winning conditions are full positional and in [4] Zielonka and Gimbert defined a complete characterization for full positional determined goals on finite arenas. In that paper, it is proven that a goal is FP if and only if both the goal and its complement satisfy two properties called *monotonicity* and *selectivity*. On the other hand, a goal (but not its complement) being monotone and selective is not sufficient for HP. Moreover, HP has been specifically investigated by Kopczyński in [6, 7]. There, the author defines sufficient conditions for a goal to be HP on all finite arenas. However, no characterization of half positional goals has been found so far. Positionality of games with infinitely many moves has been studied in [2, 5].

In his work, Kopczyński proves that if a goal is *concave* and *prefix-independent* then it is HP. In this paper, we investigate half positionality on finite arenas and we provide a novel sufficient condition for a goal to be HP on all finite arenas. We prove that if a goal is *strongly monotone* and *strongly concave*, then it is HP. As the names suggest, strong monotonicity is derived by the notion of monotonicity in [4] and strong concavity refines the notion of concavity defined in [7]. We prove that our condition constitutes an improvement over that defined in [7], because it allows to classify as HP a broader set of goals. Several examples show that our condition is somewhat robust, in the sense that it is not trivial to further strengthen the result.

Overview. The rest of the paper is organized as follows. In Section 2, we introduce some preliminary notation. In Section 3, we introduce and define the new properties of goals sufficient to ensure half positionality. We prove that such properties describe a wider set of goals than the properties in [7] and we show that some weaker conditions are not sufficient. In Section 4, we prove that our conditions are not necessary to half-positionality. In Section 5, we analyze the conditions of [4], relating them to half positionality. We show that natural stronger forms of such conditions are not sufficient, and we conclude by defining a characterization for half-positionality on game graph whose nodes belong all to one player only. Finally, we provide some conclusions is Section 6.

2 Preliminaries

Let X be a set and i be a positive integer. By X^i we denote the Cartesian product of X with itself i times and by X^* (resp., X^ω) the set of finite (resp., infinite) sequences of elements of X. The set X^* also contains the *empty word* ε. Moreover, by $\mathbb{N}$ we denote the set of non-negative integers.

For a positive integer k, let $[k] = \{0,1,\ldots,k\}$. A *word* on the alphabet $[k]$ is a finite or infinite sequence of elements of $[k]$, a *language* over the alphabet $[k]$ is a set of words over $[k]$. For each element $i \in [k]$, we often use i to denote the language $\{i\}$, when meaning is clear from the context.

Arenas. A *k-colored arena* is a tuple $A = (V_0, V_1, v_{\text{ini}}, E)$, where V_0 and V_1 are a partition of a finite set V of *nodes*, $v_{\text{ini}} \in V$ is the *initial node*, and $E \subseteq V \times [k] \times V$ is a set of *colored edges* such that for each node $v \in V$ there is at least one edge exiting from v. An edge (u,a,v) is said to be *colored* with a. In the following, we also simply call a k-colored arena an *arena*, when k is clear from the context. For a node $v \in V$, we call ${}_vE = \{(v,a,w) \in E\}$ the set of edges exiting from v, and $E_v = \{(w,a,v) \in E\}$ the set of edges entering v.

For a color $a \in [k]$, we call $E(a) = \{(v,a,w) \in E\}$ the set of edges colored with a. A *finite path* ρ is a finite sequence of edges $\{(v_i,a_i,v_{i+1})\}_{i\in\{0,\ldots,n-1\}}$, and its *length* $|\rho|$ is the number of edges it contains. We denote by $\rho(i)$ the i-th edge of ρ. Sometimes, we write the path ρ as $v_0v_1\ldots v_n$, when the colors are unimportant. An *infinite path* is defined analogously, i.e., it is an infinite sequence of edges $\{(v_i,a_i,v_{i+1})\}_{i\in\mathbb{N}}$. For a finite or infinite path ρ and an integer i, we denote by $\rho^{\leq i}$ the *prefix* of ρ containing i edges. The *color sequence* of a finite (resp. infinite) path $\rho = \{(v_i,c_i,v_{i+1})\}_{i\in\{0,\ldots,n-1\}}$ (resp. $\rho = \{(v_i,c_i,v_{i+1})\}_{i\in\mathbb{N}}$) is the sequence $Col(\rho) = \{c_i\}_{i\in\{0,\ldots,n-1\}}$ (resp. $Col(\rho) = \{c_i\}_{i\in\mathbb{N}}$) of the colors of the edges of ρ. For two color sequences $x,y \in [k]^\omega$, the *shuffle* of x and y, denoted by $x \otimes y$ is the language of all the words $z_1z_2z_3\ldots \in [k]^\omega$, such that $z_1z_3\ldots z_{2h+1}\ldots = x$ and $z_2z_4\ldots z_{2h}\ldots = y$, where $z_i \in [k]^*$ for all $i \in \mathbb{N}$. For two languages $M,N \subseteq [k]^\omega$, the *shuffle* of M and N is the set $M \otimes N = \cup_{n\in N, m\in M} m \otimes n$.

Games. A *k-colored game* is a pair $G = (A,W)$, where $A = (V_0,V_1,v_{\text{ini}},E)$ is a k-colored arena and $W \subseteq [k]^\omega$ is a set of color sequences called *goal*. By $\overline{W}$ we denote the set $[k]^\omega \setminus W$. We assume that the game is played by two players, referred to as player 0 and player 1. The players construct a path starting at v_{ini} on the arena A; such a path is called *play*. Once the partial play reaches a node $v \in V_0$, player 0 chooses an edge exiting from v and extends the play with this edge; once the partial play reaches a node $v \in V_1$, player 1 makes a similar choice. Player 0's aim is to make the play have color sequence in W, while player 1's aim is the opposite. For $h \in \{0,1\}$, let $E_h = \{(v,c,w) \in E \mid w \in V_h\}$ be the set of edges ending into nodes of player h. Let ε be the empty word, a *strategy* for player h is a function $\sigma_h : \varepsilon \cup (E^*E_h) \to E$ such that, if $\sigma_h(e_0\ldots e_n) = e_{n+1}$, then the destination of e_n is the source of e_{n+1}, and if $\sigma_h(\varepsilon) = e$, then the source of e is v_{ini}. Intuitively, σ_h fixes the choices of player h for the entire game, based on the previous choices of both players. The value $\sigma_h(\varepsilon)$ is used to choose the first edge in the game. A strategy σ_h is *positional* iff its choices depend only on the last node of the partial play, i.e., for all partial plays ρ and ρ' with the same last node, it holds that $\sigma_h(\rho) = \sigma_h(\rho')$.

A play $\{e_i\}_{i\in\mathbb{N}} \in E^\omega$ is *consistent* with a strategy σ_h iff *(i)* if $v_{\text{ini}} \in V_h$ then $e_0 = \sigma_h(\varepsilon)$, and *(ii)* for all $i \in \mathbb{N}$, if $e_i \in E_h$ then $e_{i+1} = \sigma_h(e_0 \dots e_i)$. An infinite play ρ is *winning* for player 0 (resp. player 1) iff $Col(\rho) \in W$ (resp. $Col(\rho) \notin W$). Note that, given two strategies, σ for player 0 and τ for player 1, there exists only one play consistent with both of them. We call such a play $P_G(\sigma,\tau)$. A strategy for player h is *winning* iff all plays consistent with that strategy are winning for player h. A game is *determined* iff one of the two players has a winning strategy. A goal is *determined* iff all games $G = (A,W)$ are determined.

Concavity and Prefix Independence. A goal $W \subseteq [k]^\omega$ is *prefix independent* iff for all color sequences $x \in [k]^\omega$, and all finite words $z \in [k]^*$, $x \in W$ iff $zx \in W$. Following [7], a goal W is *concave* iff, for all words $x,y \in [k]^\omega$ and $z \in x \otimes y$, it holds that if $z \in W$ then $x \in W$ or $y \in W$. A goal W is *half positional* on an arena A iff, for all games $G = (A,W)$ if player 0 has a winning strategy he has a positional winning strategy. A goal W is *half positiona* iff it is half positional for all arenas. As proved by Kopczyński, concave and prefix independence properties are sufficient conditions for half positionality.

Theorem 1 ([7]). *All concave and prefix-independent goals are determined and half-positional.*

In the following, for a goal W and for a pair of sets $M,N \in [k]^\omega$ we use the notation $M \leq_W N$ to mean that if M contains a winning word then N contains a winning word too, and the notation $M <_W N$ to mean that M contains only losing words and N contains at least a winning word. For ease of reading, when the goal W is clear from the contest we write $M < N$ and $M \leq N$ replsectively for $M <_W N$ and $M \leq_W N$. With the following two lemmas, we reformulate the definition of concavity and prefix independence in terms of languages, rather than single words.

Lemma 1. *A goal $W \subseteq [k]^\omega$ is prefix-independent iff for all $x \in [k]^*$ and $M \subseteq [k]^\omega$ we have that $xM \leq M$ and $M \leq xM$.*

Proof. Suppose that W is prefix independent. If M contains a winning word m, then xM contains the winning word xm, and we have both $xM \leq M$ and $M \leq xM$. If M contains only losing words m, then xM contains only losing words xm and we have both $xM \leq M$ and $M \leq xM$.

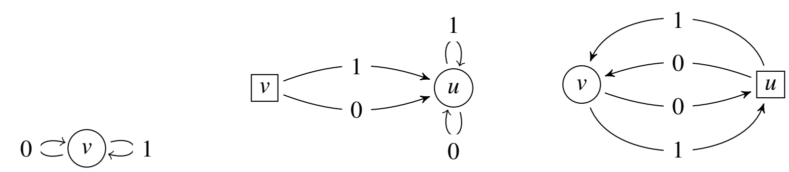
Suppose now that, for all languages $M \subseteq [k]^\omega$, we have $xM \leq M$ and $M \leq xM$. Moreover, suppose by contradiction that $W \neq xW$. Then, there exists a word m such that $xm \notin W$. Hence, for the language $M = \{m\}$ we do not have $M \leq xM$. □

Lemma 2. *A goal $W \subseteq [k]^\omega$ is concave iff for all languages $M,N \subseteq [k]^\omega$ we have that $M \otimes N \leq M \cup N$.*

Proof. Suppose that W is concave. For all $M,N \subseteq \overline{W}$, we have that $M \otimes N \subseteq \overline{W}$. So, for all languages $M,N \in [k]^\omega$, if M or N contains a winning word in W, we have in both cases $M \otimes N \leq M \cup N$; conversely, if M and N contain only losing words, by hypothesis, so does $M \otimes N$. Hence, we have that $M \otimes N \leq M \cup N$.

Suppose now that for all languages $M,N \subseteq [k]^\omega$ we have $M \otimes N \leq M \cup N$. Then, if M and N contain only losing words, $M \otimes N$ must contain only losing words too. Thus, for all $M,N \in \overline{W}$ we have that $M \otimes N \subseteq \overline{W}$. □

3 Novel Properties of Goals



(a) (b) (c)

Fig. 1. Three game arenas.

In this section, we present two properties of goals: *strong monotonicity* and *strong concavity*. Their aim is to refine the properties of prefix-independence and concavity in such a way they still imply half positionality for player 0. The property of monotonicity was first defined in [4]. It states that two color sequences with a common prefix cannot exchange their winning value by switching to another prefix. Here, we define a stronger version of monotonicity.

Definition 1. *A goal* $W \subseteq [k]^\omega$ *is* strongly monotone *if, for all words* $x \in [k]^*$*,* $m, n \in [k]^\omega$*, such that* $xm \notin W$ *and* $xn \in W$*, for all* $y \in [k]^*$ *it holds that either* $ym \notin W$ *or* $yn \in W$*.*

In the following we make use of an equivalent definition of strong monotonicity that operates on languages.

Lemma 3. *A goal* $W \subseteq [k]^\omega$ *is strongly monotone iff, for all words* $x \in [k]^*$ *and languages* $M, N \subseteq [k]^\omega$*, it holds that* $xM < xN$ *implies that for all* $y \in [k]^*$ *it is* $yM \leq yN$*.*

Strong monotonicity represents a weakening of the property of prefix independence that requires, instead, that the winning nature of a word does not change by changing a finite prefix. Indeed, the following lemma holds.

Lemma 4. *All prefix-independent goals are strongly monotone. Moreover, there is a goal which is strongly monotone, but not prefix-independent.*

Proof. For the first part, we have by hypothesis that, for all $x \in [k]^*$, and $M \subseteq [k]^\omega$, it holds that $M \leq xM \leq M$. Now, take two languages $M, N \subseteq [k]^\omega$, and suppose that there exists an $x \in [k]^*$ such that $xM < xN$, then for all $y \in [k]^*$ we have $yM \leq M \leq xM \leq xN \leq N \leq yN$.

For the second part, let $k = 1$, a strongly monotone and prefix-dependent goal is given by the language of all words containing at least one 0, i.e., $W = [k]^* 0 [k]^\omega$. It is easy to see that the goal is not prefix-independent, because the word 1^ω is losing while the word 01^ω is winning. We show that the goal is strongly monotone. Consider two languages $M,N \subseteq [k]^*$, and suppose that there exists an $x \in [k]^*$ such that $xM < xN$, then xN contains a winning word and xM contains only losing words. Observe first that x cannot contain 0, or else all words in xM would be winning. So $x \in 1^*$, there exists a word in N that contains 0, and all words in M contain only 1's. So, for each $y \in [k]^*$, there is always a word in yN containing 0. Since yN contains a winning word, we have $yM \leq yN$. □

We investigate the usefulness of strong monotonicity. First, we show that strong monotonicity cannot replace prefix-independence in the hypotheses of Theorem 1.

Lemma 5. *There is a strongly monotone and concave goal which is not half-positional.*

Proof. For $k = 1$, the strongly monotone and concave goal is $W = [k]^*01^\omega$. We prove first that the goal is strongly monotone and concave. A word is losing if and only if it is either 1^ω or it does not have 1^ω as a suffix. Let $x \in [k]^*$, $n,m \in [k]^\omega$ with $xn, xm \notin W$. There are two situations to discuss. First, assume that x does not contain 0. Then, n and m may be both 1^ω in which case $x(m \otimes n) = 1^\omega$ or at least one between n and m contains 0 infinitely often, thus the shuffle of n and m contains only words that pick colors from both the sequences infinitely often and thus only words that contain 0 infinitely often. So, $x(m \otimes n)$ contains losing word even in this case. Instead, assume that x contains 0. Then, n and m contain 0 infinitely often and the same reasoning above applies. So the goal is concave. Let $x \in [k]^*$, $n,m \in [k]^\omega$ such that $xm \notin W$ and $xn \in W$. We prove strong monotonicity by showing that for all $y \in [k]^*$ it holds that $ym \notin W$ or $yn \in W$. We again distinguish two cases. First, assume that x does not contain a 0. Then, n contains 0 and a suffix 1^ω thus for every $y \in [k]^*$, we have $yn \in W$ since it contains 0 and a suffix 1^ω. Instead, assume that x contains a 0. Then, m contains 0 infinitely often, thus for every $y \in [k]^*$ the word $ym \notin W$ since it contains 0 infinitely often. The above goal is not half-positional in the following arena $(\{v\}, \emptyset, v, \{(v,0,v),(v,1,v)\})$ (Fig. 1(a)), in such a game graph player 0 wins by choosing at least once the edge with color 0 and then always the edge with color 1. □

Observe that, in the previous counterexample, the key element that does not allow half positionality is the fact that player 0 prefers switching between two different behaviors finitely often and then progressing indefinitely along one of them. However, concavity just requires that player 0 prefers following a fixed behavior rather than switching between two different ones *infinitely often*. Thus, we introduce a modification to the property of concavity, requiring not only that alternating infinitely often between two losing words yields a losing word, but also that alternating *finitely* often between two losing words and then progressing along one of them yields a losing word.

Definition 2. *For two color sequences $x, y \in [k]^\omega$, the* strong shuffle *of x and y, denoted by $x \otimes_s y$ is the language containing*

1. *the set $x \otimes y$;*
2. *the words $z_1 z_2 \dots z_l z' \in [k]^\omega$, for odd l, $z_i \in [k]^*$ and $z' \in [k]^\omega$, such that it holds $x = z_1 z_3 \dots z_l z'$ and $y = z_2 z_4 \dots z_{l-1} y'$, for some $y' \in [k]^\omega$;*
3. *the words $z_1 z_2 \dots z_l z' \in [k]^\omega$, for even l, $z_i \in [k]^*$ and $z' \in [k]^\omega$, such that it holds $x = z_1 z_3 \dots z_{l-1} x'$ and $y = z_2 z_4 \dots z_l z'$, for some $x' \in [k]^\omega$.*

For two languages $M, N \subseteq [k]^\omega$, the strong shuffle *of M and N is the set $M \otimes_s N = \cup_{n \in N, m \in M} (m \otimes_s n)$.*

Definition 3. *A goal $W \subseteq [k]^\omega$ is* strongly concave *iff, for all words $x \in [k]^*$, $n, m \in [k]^\omega$, and $z \in x(m \otimes_s n)$, it holds that if $z \in W$ then either $xn \in W$ or $xm \in W$.*

It is immediate to see that a strongly concave goal is concave too. In the following, we make use of an equivalent definition of strong concavity that operates on languages.

Lemma 6. *A goal $W \subseteq [k]^\omega$ is strongly concave iff, for all words $x \in [k]^*$ and languages $M, N \subseteq [k]^\omega$, it holds that $x(M \otimes_s N) \leq xM \cup xN$.*

Even the property of strong concavity is not sufficient to ensure half positionality.

Lemma 7. *There is a strongly concave goal which is not half-positional.*

Proof. For $k = 1$ the strongly concave goal is $W = 0^\omega \cup 1^\omega$. Two losing words n and m contain at least an occurrence of the color 1 and an occurrence of the color 0, thus every word in their strong shuffle will contain at least an occurrence of color 1 and an occurrence of color 0 and it will be losing. So the strong concavity of the goal is proved. The above goal is not half-positional in the following 2-colored arena $(\{u\}, \{v\}, v, \{(v, 0, u), (v, 1, u), (u, 0, u), (u, 1, u)\})$, showed in Figure 1(b). In this arena player 0 wins the game by choosing forever the edge $(u, 0, u)$ or the edge $(u, 1, u)$ depending on what color was chosen by player 1 to reach u from v. □

In the previous counterexample, by choosing a different prefix, player 1 can exchange the winning nature of the following choices of player 0. That is why strong monotonicity is essential since it somehow allows player 0 to operate while forgetting the past decisions taken by player 1.

We argue now that the two introduced properties of strong monotonicity and strong concavity are strictly less restrictive than the properties of prefix independence and concavity.

Lemma 8. *Concave and prefix-independent goals are strongly monotone and strongly concave.*

Proof. By Lemma 4 we already have that a prefix-independent goal is strongly monotone. It remains to show that a concave and prefix-independent goal is strongly concave.

For a language $M \subseteq [k]^\omega$, let $\mathit{suff}(M)$ and $\mathit{pref}(M)$ be the sets of suffixes and prefixes of words in M, respectively. By concavity, for all $M, N \subseteq [k]^\omega$ we have $M \otimes N \leq M \cup N$ and by prefix independence we have for all $M \in [k]^\omega$ and for all $x \in [k]^*$ $M \leq xM \leq M$. Take any word $x \in [k]^*$, and any two languages $M, N \subseteq [k]^\omega$. Then we have $x(M \otimes_s N) = x(M \otimes N) \cup x \cdot \mathit{pref}(M \otimes N) \cdot \mathit{suff}(N) \cup x \cdot \mathit{pref}(M \otimes N) \cdot \mathit{suff}(M)$. First, by prefix independence and then by concavity we have $x(M \otimes N) \leq M \otimes N \leq M \cup N \leq x(M \cup N) = xM \cup xN$. Then, $x \cdot \mathit{pref}(M \otimes N) \cdot \mathit{suff}(T) \leq \mathit{suff}(T) \leq xT \leq xM \cup xN$, where $T \in \{M, N\}$. So, we have $x(M \otimes_s N) \leq xM \cup xN$. □

Lemma 9. *There exists a strongly monotone and strongly concave goal which is not prefix independent.*

Proof. Let $k = 1$, the goal is given by the set of words that either start with 1, or start with 0 and contain infinitely many 0's, i.e., $W = 0(1^*0)^\omega \cup 1[k]^*$. It is easy to see that the goal is not prefix-independent: indeed, for $M = 1^\omega$ we have that $0M \leq M$, but not $M \leq 0M$ since M contains only winning words and $0M$ only losing ones.

Next, we prove that the goal is strongly monotone. Consider $M,N \subseteq [k]^*$ and $x \in [k]^*$ and suppose that $xM < xN$, so xN contains a winning word and xM contains only losing ones. Observe that x does not start with 1, otherwise all words in xM would be winning. So, there are two situations to discuss: $x = \varepsilon$ or x starts with 0. If $x = \varepsilon$ then all words in M starts with 0 and have a suffix equal to 1^ω. Now for all $y \in 1[k]^*$ we have $yM \leq yN$ since all the words in all languages are winning; for all $y \in 0^*[k]^*$ we have $yM \leq yN$ because all the words in yM are losing since they start with 0 and have a suffix 1^ω. If instead x starts with 0 then there exists a word $n \in N$ that contains infinitely many 0, for every $y \in [k]^*$ the word yn will contain infinitely many 0 and it will be winning, thus for all $y \in [k]^*$ we will have $yM \leq yN$.

Now we prove that the goal is strongly concave. Consider $x \in [k]^*$, $M,N \subseteq [k]^\omega$ and $K \subseteq [k]^*$. We want to prove that $x(M \otimes_s N) \leq xM \cup xN$. If the r.h.s. of the inequality contains a winning word, the inequality trivially holds. So, suppose that the r.h.s. does not contain a winning word, so it cannot be $x \in 1[k]^*$ but it must be $x \in 0[k]^* \cup \{\varepsilon\}$. If x starts with 0, every word in M,N contains a suffix 1^ω and all words in $M \otimes_s N$ contain a suffix 1^ω. So, $M \otimes_s N$ contains only losing words. If $x = \varepsilon$, every word in M,N contains a suffix 1^ω and starts with 0, so all words in $M \otimes_s N$ contain a suffix 1^ω and start with 0, and therefore they are losing. □

4 A Sufficient Condition for Half Positionality

In this section, we prove that determinacy, strong monotonicity and strong concavity are sufficient but not necessary conditions to half positionality for player 0.

Theorem 2. *All determined, strongly monotone and strongly concave goals are half-positional.*

Proof. The proof proceeds by induction on the number of edges exiting from the nodes controlled by player 0 in the game arena. As a base case in the graph G for each node controlled by player 0 there exists only one exiting edge. In such a graph player 0 has only one possible strategy which is positional. So, the result is trivially true. Suppose that in the arena there are n edges exiting from nodes of player 0 and that, for all graphs with at most $n-1$ edges exiting from nodes of player 0, if player 0 has a winning strategy he has a positional one. Let t be a node of player 0 in G such that there is more than one edge exiting from t. We can partition the set of edges exiting from t in two disjoint non-empty sets E_α and E_β. Let G_α and G_β be the two subgraphs obtained from G by removing the edges of E_β and E_α, respectively. There are two cases to discuss.

First, suppose that in G_α or G_β player 0 has a winning strategy. Then, by inductive hypothesis he has a positional winning strategy. It is easy to see that such a strategy is winning in G too. Indeed, since player 0 controls the node t, he is able to force the play to stay always in G_α or G_β. Suppose now that player 0 has no winning strategy in G_α and in G_β. We prove the thesis by showing that player 0 has no winning strategy in G.

By determinacy, there exist two strategies τ_α and τ_β winning for player 1 in G_α and G_β, respectively.

Let σ be a strategy of player 0 in G, we show that there exists a strategy of player 1 in G winning in G against σ. If one of the plays $P(\sigma,\tau_\alpha)$ or $P(\sigma,\tau_\beta)$ does not pass through t then that play is in G_α and G_β and so it is winning for player 1 who is using his winning strategy on one of the graphs.

Suppose now that both of the above plays pass through t. Let x_α and x_β be respectively the color sequences of the prefixes of $P(\sigma,\tau_\alpha)$ and $P(\sigma,\tau_\beta)$, up to the first occurrence of t. Let M_α and M_β be the sets of color sequences of suffixes after respectively a prefix x_α and x_β of plays consistent respectively with τ_α and τ_β. Observe that $x_\alpha M_\alpha$ and $x_\beta M_\beta$ contain plays consistent respectively with τ_α in G_α and τ_β in G_β, and such plays are losing for player 0. We prove now that either $x_\alpha M_\beta$ or $x_\beta M_\alpha$ contains only losing words for player 0. Indeed, if $x_\alpha M_\beta$ contains a winning word, we have that $x_\alpha M_\alpha < x_\alpha M_\beta$, since $x_\alpha M_\alpha$ contains only plays losing for player 0. Then, by strong monotonicity we have that, for all $y \in C^*$, it holds $yM_\alpha \leq yM_\beta$ and hence $x_\beta M_\alpha \leq x_\beta M_\beta$. Since $x_\beta M_\beta$ contains only losing words, so does $x_\beta M_\alpha$.

Suppose without loss of generality that $x_\beta M_\alpha$ contains only losing words. Then, we construct the strategy τ'_α, which behaves like τ_α on all partial plays which do not have a prefix x_β. When the partial play has a prefix x_β, it behaves like τ_α when it sees x_α in place of x_β. More formally $\tau'_\alpha(x_\beta\pi) = \tau_\alpha(x_\alpha\pi)$, and in the other cases $\tau'_\alpha(\pi) = \tau_\alpha(\pi)$. Let $\tau'_\beta = \tau_\beta$. We construct a strategy τ in G: at the beginning the strategy behaves like τ_β; when the play passes through t, depending on what subgraph the last edge from t chosen by player 0 belongs to, the strategy τ behaves like τ'_α or τ'_β when they are applied only to the initial prefix up to t and all the loops from t to t, where the first edge belongs to G_α or G_β, respectively.

Formally, for all prefixes π that do not pass through t, we have $\tau(\pi) = \tau_\beta(\pi)$; if π_{i,γ_i} is a loop from t to t with first edge in G_{γ_i}, for all prefixes $\pi = x\pi_{1,\gamma_1},\ldots,\pi_{n,\gamma_n}\pi_\gamma$, we have $\tau'(\pi) = \tau'_\gamma(x(\prod_{\gamma_i=\gamma}\pi_{i,\gamma_i}\pi_\gamma))$. The play $P(\sigma,\tau)$ coincides with $P(\sigma,\tau_\beta)$ up to t, so it has a prefix with color sequence x_β. After that prefix, the play develops in parallel and alternates pieces of two plays: one in G_β consistent with τ_β, and the other in G_α consistent with τ'_α. So, the color sequence of the two suffixes are respectively in M_β and in M_α.[1] Hence, the color sequence of the suffix after x_β of the play $P(\sigma,\tau)$ lies in the shuffle of M_α and M_β. By strong concavity we have that $Col(P(\sigma,\tau)) \in x_\beta(M_\alpha \otimes_s M_\beta) \leq x_\beta M_\alpha \cup x_\beta M_\beta$. Since both $x_\beta M_\alpha$ and $x_\beta M_\beta$ contain only losing words, we have that $Col(P(\sigma,\tau))$ is a losing word for player 0. Hence, for all strategies σ of player 0 there exists a strategy τ of player 1 winning over 0. We conclude that player 0 has no winning strategy. □

Since strongly concavity implies concavity, the following result states that the conditions appearing as the hypothesis of the previous theorem and of Theorem 1 are not a complete characterizations for half positional goals.

Lemma 10. *There exists a goal that is half positional but not concave.*

Proof. The half positional goal is $W = [k]^*1[k]^*1[k]^\omega$. The goal states that player 0 tries to make color 1 occur at least twice. It is half positional because in every point in a

[1] Note that it is possible that one of the two suffixes does not progress indefinitely.

play player 0 does not need to look at the past, but just tries to form a path that passes through as many edges colored with 1 as possible. For a more formal proof, see Lemma 14.

We show that the goal is not concave: let $x = \varepsilon$, $n, m = 10^\omega$, then we have $xn, xm \notin W$, but $t = 110^\omega \in m \otimes n$ with $xt \in W$, hence the goal is not concave. □

5 Selectivity

Here, we discuss how the properties presented in [4] as a characterization of full positionality relate to half positionality. The property of monotonicity is almost the same as the one of strong monotonicity presented in this paper with the exception that the original property needs to hold only for automata-recognizable (i.e., regular) languages, i.e. with M, N and K as regular languages.

Definition 4. *Let $M \subseteq [k]^*$. Then, with the notation $\langle M \rangle$ we define the set of all words $m \in [k]^\omega$ such that every prefix of m is a prefix of a word in M.*

Definition 5. *A goal $W \subseteq [k]^\omega$ is* monotone *iff for all words $x \in [k]^*$ and all regular languages $M, N \subseteq [k]^\omega$ it holds that $xM < xN$ implies that for all $y \in [k]^*$ it is $yM \le yN$.*

The second property introduced in [4] is similar to the property of strong concavity with the exception that in the shuffle the interleaving of words is allowed only at certain points.

Definition 6. *A goal W is* selective *iff for all $x \in [k]^*$ and for all regular languages $M, N, K \subseteq [k]^*$ we have that $x\langle (M \cup N)^* K \rangle \le x\langle M^* \rangle \cup x\langle N^* \rangle \cup x\langle K \rangle$.*

The two conditions of selectivity and monotonicity provide a complete characterization of full positionality. Precisely, a goal W is full positional iff both W and $\overline{W}$ are selective and monotone [4]. The proof makes use of the fact that, assuming that player 1 uses a positional strategy, player 0 can play on the graph induced by that strategy, and hence construct paths whose prefixes are recognizable by the automaton described by the game graph. We investigated the hypothesis that monotonicity and selectivity of W were sufficient to half positionality. However, the two conditions are not directly applicable, since they operate on regular languages. Indeed, when player 1 can use a non-positional strategy, the path constructed by player 0 is taken from a simple graph no more and it does not belong to a language recognized by an automaton. Hence, we strengthened the conditions of monotonicity and selectivity in order to take into account all possible paths that could be formed by player 0 together with a non-positional strategy of player 1.

Definition 7. *A goal W is* strongly selective *iff for all $x \in [k]^*$ and for all languages $M, N, K \subseteq [k]^*$ we have that $x\langle (M \cup N)^* K \rangle \le x\langle M^* \rangle \cup x\langle N^* \rangle \cup x\langle K \rangle$.*

Selectivity and strong selectivity represent two weaker properties than strong concavity.

Lemma 11. *Every strongly concave goal is strongly selective.*

Proof. For all words $x \in [k]^*$, for all languages $M,N,K \subseteq [k]^*$, we have that $x\langle (M \cup N)^* K\rangle \subseteq x((\langle M^*\rangle \otimes_s \langle N^*\rangle) \otimes_s \langle K\rangle) \leq x\langle M^*\rangle \cup x\langle N^*\rangle \cup x\langle K\rangle$.

Unfortunately, the strong versions of selectivity and concavity proved not to be sufficient conditions to half positionality[2].

Lemma 12. *There is a strongly monotone and strongly selective goal which is not half-positional.*

Proof. Let $k \in \mathbb{N}$, for all colors $i \in [k]$ and finite paths π, let $|\pi|_i$ be the number of edges colored by i on π, and let $|\pi|$ be the number of edges in π. Moreover for all $n \in \mathbb{N}$ let $\pi^{\leq n}$ be the prefix of length n of π. The strongly monotone and strongly selective goal is the set W of all the infinite words m such that, for all colors $i \in [k]$, the limit $\lim_{n\to+\infty} \frac{|m^{\leq n}|_i}{|m^{\leq n}|}$ exists and is finite. The goal is prefix independent. Indeed, let $\pi = x\pi'$ then for all $i \in [k]^*$ we have $\lim_{n\to+\infty} \frac{|\pi'^{\leq n}|_i}{|\pi'^{\leq n}|} = \lim_{n\to+\infty} \frac{|\pi^{\leq n+|x|}|_i - |x|_i}{|\pi^{\leq n+|x|}| - |x|} = \lim_{m\to+\infty} \frac{|\pi^{\leq m}|_i}{|\pi^{\leq m}|}$. The goal is also strongly selective. Indeed, suppose by contradiction that there exist a sequence $x \in [k]^*$, and three languages $M,N,K \subseteq [k]^*$ such that $x\langle (M\cup N)^* K\rangle$ contains one winning word and $x\langle M^*\rangle \cup x\langle N^*\rangle \cup x\langle K\rangle$ contains only losing words. In this case, M and N must be empty else any periodic word $\pi = m^\omega \in M^* \cup N^*$ with $m \in M \cup N$ has a finite limit $\lim_{n\to+\infty} \frac{|\pi^{\leq n}|_i}{|\pi^{\leq n}|} = \frac{|m|_i}{|m|}$, for all colors i. So, the set $\langle x(M\cup N)^* K\rangle = x\langle K\rangle$ and contains only losing words which is a contradiction. The above goal is not half-positional in the following arena $(\{u\},\{v\},u,\{(v,0,u),(v,1,u),(u,0,v),(u,1,v)\})$ with $k = 1$ (Fig 1(c)). Player 0 can win with a strategy with memory by choosing from V' to V the opposite of the color that player 1 chose from V to V' right before, thus yielding a path in $[k]^*(10)^\omega$ which has limit $\frac{1}{2}$ for both colors. However if player 0 uses a positional strategy, it will only choose one color from V' to V, let suppose without loss of generality that he chooses color 0. The player 1 can force a path $\pi = \prod_{i=0}^{+\infty} (00)^{2^i} (10)^{2^i}$. Then we have $|\prod_{i=0}^{l} (00)^{2^i} (10)^{2^i}| = \sum_{i=0}^{l} 4\cdot 2^i = 4(2^{l+1}-1)$, and $|(\prod_{i=0}^{l-1} (00)^{2^i} (10)^{2^i}) \cdot (00)^{2^l}| = 4(2^l + 2^{l-1} - 1)$. Moreover, $|\prod_{i=0}^{l} (00)^{2^i} (10)^{2^i}|_1 = \sum_{i=0}^{l} \cdot 2^i = (2^{l+1}-1)$, and $|(\prod_{i=0}^{l-1} (00)^{2^i} (10)^{2^i}) \cdot (00)^{2^l}|_1 = \sum_{i=0}^{l-1} \cdot 2^i = 2^l - 1$. So we have $\frac{|\prod_{i=0}^{l} (00)^{2^i} (10)^{2^i}|_1}{|\prod_{i=0}^{l} (00)^{2^i} (10)^{2^i}|} = \frac{1}{4}$, moreover $\frac{|(\prod_{i=0}^{l-1} (00)^{2^i} (10)^{2^i}) \cdot (00)^{2^l}|_1}{|(\prod_{i=0}^{l-1} (00)^{2^i} (10)^{2^i}) \cdot (00)^{2^l}|} = \frac{1}{4} \frac{2^l - 1}{2^l - 2^{l-1} - 1} > \frac{1}{4} \frac{2^l - 2^{l-2}}{2^l - 2^{l-1}} = \frac{1}{4} \frac{3 \cdot 2^{l-2}}{2 \cdot 2^{l-2}} = \frac{3}{8}$. This shows that in the limit $\frac{|\pi^{\leq n}|_1}{|\pi^{\leq n}|}$ oscillates between $\frac{1}{4}$ and something more than $\frac{3}{8}$. □

Although the following theorem is obtained easily from the techniques developed in [4], we think that it is worth mentioning that half positionality on arenas controlled only by player 0 is equivalent to the selectivity of the goal. Since the selectivity is similar in a way to strong concavity, we show that strong concavity is a condition useful to assert that, on decisions independent from player 1, player 0 prefers a fixed behavior rather than switching between two different ones. We prove the above statement by making use of the following lemma proved in [4].

[2] We thank W. Zielonka and H. Gimbert for pointing out the counterexample.

Lemma 13 ([4]). *Let A be a finite co-accessible* [3] *automaton recognizing a language $L \subset [k]^*$ and having starting state q. Then, $\langle L \rangle$ is the set of infinite color sequences on the graph of A starting in q.*

Theorem 3. *A goal is selective iff it is half-positional on all arenas controlled by player 0.*

Proof. [only if] Suppose that a goal W is half-positional on all game graph controlled by player 0 but non-selective. Let $x \in [k]^*$ and $M, N, K \subseteq [k]^*$ be three recognizable languages such that $x\langle (M \cup N)^* K \rangle \not\subseteq x\langle M^* \rangle \cup x\langle N^* \rangle \cup x\langle K \rangle$. This means that there is a winning word in $x\langle (M \cup N)^* K \rangle$ and $x\langle M^* \rangle \cup x\langle N^* \rangle \cup x\langle K \rangle$ contains only losing words. Let G_x, G_M, G_N be the minimized finite automata recognizing the languages $\{x\}, M, N$, respectively, and having only one starting state with no transition returning to it and one final state with no transition exiting from it. Let G_K be the minimized finite automaton recognizing the language K, having only one starting state with no transition returning to it. We construct the game graph G by combining together the graphs G_x, G_M, G_N, G_K. Precisely we glue together the final state of G_x, the initial and final states of G_M and G_N and the initial state of G_K in a new node t. Observe that, by gluing together the initial and final states, the automata G_M, G_N recognize M^* and N^*, respectively. The initial state of G is the starting state of G_x. Thus the graph G recognizes the language $x(M \cup N)^* K$. Hence by Lemma 13, every infinite path in G is in $\langle x(M \cup N)^* K \rangle = x\langle (M \cup N)^* K \rangle$. Since this set contains a winning word, there is a winning strategy for player 0. However, if player 0 uses a positional strategy he cannot win. Indeed, player 0 reaches first the node t by constructing the color sequence x on G_x. In the node t player 0 chooses once and for all which of the subgraphs G_M, G_N, G_K he will use, so the infinite play will be of the form xm where m is an infinite path in G_M,G_N or G_K. By Lemma 13, $xm \in x\langle M^* \rangle \cup x\langle N^* \rangle \cup x\langle K \rangle$. But this set contains only losing words. Hence, xm is losing.

[if] Suppose that a goal W is selective, we prove by induction on the number of edges exiting from the nodes of the arena G controlled by player 0 that if there exists a winning strategy for player 0 then there exists a positional one. As base case there exists only one edge exiting from the nodes of G, hence player 0 has only one strategy, which is trivially positional. Suppose that in the arena there are n edges exiting from nodes of player 0 and that for all graphs with at most $n-1$ edges exiting from nodes of player 0, if player 0 has a winning strategy he has a positional one. Let t be a node of player 0 in G such that there is more than one edge exiting from t. We can partition the set of edges exiting from t in two disjoint non-empty sets E_α and E_β. Let G_α and G_β be the two subgraphs obtained from G by removing the edges of E_β and E_α, respectively. There are two cases to discuss. First, suppose that either in G_α or G_β player 0 has a winning strategy. Then, by inductive hypothesis he has a positional winning strategy. It is easy to see that such a strategy is winning in G too, indeed player 0 is able to play always in G_α or G_β since he controls every node.

Suppose now that player 0 has no winning strategy in G_α and in G_β. We prove the thesis by showing that player 0 has no winning strategy in G. Let M_α and M_β be the sets of all finite color sequences from t to t and K_α and K_β be the sets of all finite color

[3] An automaton is co-accessible iff from every state there is a path reaching an accepting state. It's easy to see that a minimized automaton is co-accessible.

sequences starting from t, in G_α and G_β, respectively. Such sets are regular languages: M_α and M_β are recognized by the automata having respectively G_α and G_β as state graphs, with starting node t and accepting set $\{t\}$. The sets K_α and K_β are the languages accepted by the automata with state graphs G_α and G_β, respectively, with starting node t and accepting set given by all the states.

Suppose now by contradiction that there exists a winning strategy for player 0 in G. Then this strategy will form a winning path π. Such a path cannot be in G_α or G_β, or else player 0 has a winning strategy in one of those subgraphs. So the path is in G and passes through t. Let x be the shortest prefix of π ending in t, then π belongs to the set $x\langle(M_\alpha \cup M_\beta)^*(K_\alpha \cup K_\beta)\rangle$, since it starts with x, then either loops forever from t to t in G_α and G_β, or possibly ends with an infinite path that never comes back to t. However, for $\gamma \in \{\alpha, \beta\}$, the sets $x\langle M_\gamma^*\rangle$ and $x\langle K_\gamma\rangle$ contain only paths in G_γ, so they are losing. Thus, we have $x\langle(M \cup N)^*K\rangle \not\subseteq x\langle M^*\rangle \cup x\langle N^*\rangle \cup x\langle K\rangle$, which contradicts selectivity. □

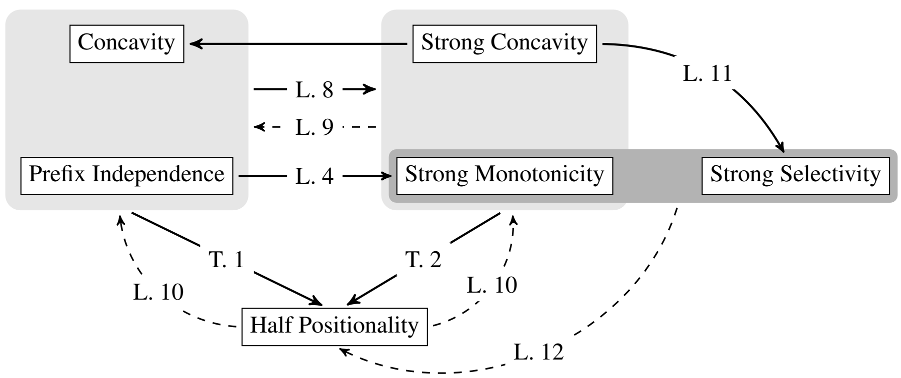


Fig. 2. Summary of results. Continuous arrows represent a holding implication and dashed ones a false one. Arrows are labeled with the corresponding lemma or theorem. Moreover, a gray box represents a conjunction of conditions.

After discussing monotonicity and selectivity we can formally complete the proof of Lemma 10.

Lemma 14. *Let $k = 1$, the goal $W = [k]^*1[k]^*1[k]^\omega$ is full positional.*

Proof. Using the characterization of [4], we prove the statement by showing that the goals W and $\overline{W}$ are selective and monotone. Observe that W is the set of all the words having at least two 1's and $\overline{W}$ is the set of all the words contain at most one 1.

1. W is selective. Suppose by contradiction that W is not selective. Then, there exist $x \in [k]^*$ and $M, N, K \subseteq [k]^*$ such that $x\langle(M \cup N)^*K\rangle = x\langle(M \cup N)^*\rangle \cup x(M \cup N)^*\langle K\rangle$ contains a winning word and $x\langle M^*\rangle \cup x\langle N^*\rangle \cup x\langle K\rangle$ contains only losing words.

Observe that no word in M or N contains 1, or else if $m \in M \cup N$ contains a 1, $xm^{\omega} \in x\langle M^* \rangle \cup x\langle N^* \rangle$ contains infinitely many 1's and it is a winning word. So, the words in the set $x\langle (M \cup N)^* \rangle$ do not contain 1 and they are losing. Moreover, since $x\langle K \rangle$ does not contain more than one 1, the words in $x(M \cup N)^* \langle K \rangle$ do not contain more than one 1 and they are all losing too. So, the set $x\langle (M \cup N)^* K \rangle$ contains only losing words, hence a contradiction.

2. W is monotone. Suppose by contradiction that W is not monotone. Then there exist $x, y \in [k]^*$ and $M, N \subseteq [k]^*$ such that $xM < xN$ and $yN < yM$. So, xM and yN contain only losing words, xN and yM contain a winning word. If x contains more than one 1, all words in the first two sets are losing, hence a contradiction. If x contains one 1, then no word in M contains 1. However, there is a winning word in yM, so y contains two 1's. Hence, yN contain only winning words, which is a contradiction. If x does not contain a 1, there is a word in N with two 1's. Hence, yN contains at least a winning word, which is again a contradiction.
3. $\overline{W}$ is selective. Suppose by contradiction that $\overline{W}$ is not selective. Then, there exist $x \in [k]^*$ and $M, N, K \subseteq [k]^*$ such that $x\langle (M \cup N)^* K \rangle = x\langle (M \cup N)^* \rangle \cup x(M \cup N)^* \langle K \rangle$ contains a winning word and $x\langle M^* \rangle \cup x\langle N^* \rangle \cup x\langle K \rangle$ contains only losing words. Observe that no word in M or N does not contain 1, else if $m \in M \cup N$ does not contain a 1, $xm^{\omega} \in x\langle M^* \rangle \cup x\langle N^* \rangle$ does not contain 1's and it is a winning word. So the words in the set $x\langle (M \cup N)^* \rangle$ contain infinitely many 1's and they are losing. Moreover, since $x\langle K \rangle$ contains more than one 1, the words in $x(M \cup N)^* \langle K \rangle$ contain more than one 1 and they are all losing. So, the set $x\langle (M \cup N)^* K \rangle$ contains only losing words, hence a contradiction.
4. $\overline{W}$ is monotone. Suppose by contradiction that $\overline{W}$ is not monotone. Then there exist $x, y \in [k]^*$ and $M, N \subseteq [k]^*$ such that $xM < xN$ and $yN < yM$. So, xM and yN contain only losing words, xN and yM contain a winning word. If x contains more than one 1, all words in the first two sets are winning, hence a contradiction. If x contains one 1, then there is a word in N that does not contain 1's. Since yN contains only losing words, y contains more than one 1. So, all words in yM are losing, hence a contradiction. If x does not contain 1, then all words in M contain more than one 1, so all words in yM are losing, hence a contradiction. □

6 Conclusions

In this paper, we defined a new sufficient condition for half-positionality on finite arenas, which turns out to be strictly weaker (i.e., broader) than that defined by Kopczyński in [7], as long as determined goals are considered. We discussed the conditions presented in [4] for full-positionality and we proved that a stronger partial form of them does not ensure half positionality.

The main open problem left by this research is the formulation of a complete characterization of half-positionality. Another interesting question for further research is whether or not the properties of strong monotonicity and strong concavity imply determinacy. The answer to this question may simplify the statement of Theorem 2 by removing the hypothesis of determinacy. Finally, another open problem consists in developing algorithms for checking whether a goal, given in input in some effective way such as an automaton or a temporal logic formula, satisfies the conditions outlined in this paper and is therefore HP. Such an algorithm may be used as a preliminary step in controller synthesis tools, in order to estimate the amount of memory that the synthesized controller will need.

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