Pregeometry of $\kappa$-Minkowski/$\kappa$-Poincaré spacetime and relative locality for fuzzy worldlines

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Giovanni Amelino-Camelia, Valerio Astuti, G. R.,
Several arguments suggest that, when both gravitational and quantum effects are
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One way of characterizing such a “quantum spacetime” is through a sort of “Heisenberg uncertainty principle for spacetime”, where, similarly to position and momentum in quantum mechanics (\([\hat{p}, \hat{x}] = i\hbar \Rightarrow \delta \hat{p}\delta \hat{x} \gtrsim \hbar\)), spacetime coordinates do not commute among themselves:

\[ [\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}(\hat{x}) \propto \ell_P \]

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Consider \([\hat{x}^\mu, \hat{x}^\nu] = i\lambda^{\mu\nu}_\rho \hat{x}^\rho\), with \(\lambda^{\mu\nu}_\rho = O(\ell_P)\).

Is this a preferred-frame picture or still a fully relativistic picture?

If \(\lambda^{\mu\nu}\) is the same for all inertial observers, one needs to modify (“deform”) the relativistic laws of transformations between their coordinates.
Several arguments suggest that, when both gravitational and quantum effects are considered, localization measurements in spacetime can be performed down to a minimum length scale, usually thought to be close to the Planck scale
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Is this a preferred-frame picture or still a fully relativistic picture?

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\[ \kappa\text{-Minkowski}\] (\(c = \hbar = 1\))

\[ [x_j, x_0] = \frac{i}{\kappa} x_j \quad [x_j, x_k] = 0 \]

\[ \kappa \sim M_p = \frac{\hbar}{\ell_P c} = \sqrt{\frac{\hbar c}{G}} \sim 10^{19} \text{ GeV}/c^2 \]

\[ \kappa\text{-Poincaré}\] is the deformed (Hopf-)algebra of symmetries
(Lukierski,Ruegg,Majid,...'90)

(See the talks of Gubitosi, Amelino-Camelia,...)
It turns out that in theories where relativistic symmetries are deformed in terms of a (inverse-) momentum invariant scale $1/\kappa$ (DSR theories (Amelino-Camelia, Kowalski-Glikman, Smolin, Magueijo ~ 2000)) the notion of absolute spacetime locality is relaxed in favor of a relativity of locality (Amelino-Camelia+Matassa+Mercati+G.R.(2010)Phys.Rev.Lett.106), similarly to how, in the transition from Galilean to special relativity, the deformation of relativistic symmetries in terms of an invariant speed $c$, enforces relativity of simultaneity.

- What is the description of spacetime points in $\kappa$-Minkowski?
- One expect some kind of fuzziness. How can it be characterized?
- Is it possible to provide a quantum mechanical description of $\kappa$-Minkowski/$\kappa$-Poincaré?

In a “pregeometrical” interpretation, $\kappa$-Minkowski emerges from quantum gravity at some level of effective description, and non-commutativity is represented in terms of standard Heisenberg quantum mechanics introduced at some deeper level.

Some previous attempts

(Ghosh+Pal,Phys.Rev.D75(2007)105021)
(Dabrowski+Piacitelli,ArXiv:1004.5091)
(Mignemi,Phys.Rev.D84(2011)025021)
**\(\kappa\)**-Poincaré algebra and **\(\kappa\)**-Minkowski spacetime (1+1D)

\[\kappa\text{-Minkowski } \mathcal{X}_\kappa\]

\[
[x_1, x_0] = \frac{i}{\kappa} x_1
\]

\[
\Delta (x_\mu) = x_\mu \otimes 1 + 1 \otimes x_\mu
\]

\[
S (x_\mu) = -x_\mu \quad \epsilon (x_\mu) = 0
\]

is the homogeneous space of the **\(\kappa\)**-Poincaré Hopf algebra \(\mathcal{P}_\kappa\)

\[
[P_1, P_0] = 0 \quad [N, P_0] = iP_1
\]

\[
[N, P_1] = i\frac{\kappa}{2} \left(1 - e^{-\frac{2}{\kappa} P_0}\right) - \frac{i}{2\kappa} P_1^2
\]

\[
\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0
\]

\[
\Delta P_1 = P_1 \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes P_1
\]

\[
\Delta N = N \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes N
\]

\[
S (P_0) = -P_0 \quad S (P_1) = -e^{\frac{P_0}{\kappa}} P_1 \quad S(N) = -e^{\frac{P_0}{\kappa}} N_1
\]

\[
\Box_\kappa = (2\kappa)^2 \sinh^2 \left(\frac{P_0}{2\kappa}\right) - e^{\frac{P_0}{\kappa}} \vec{P}^2
\]
Translations and differential calculus

Coproduct $\Delta \implies \text{action } \triangleright : \mathcal{P}_\kappa \otimes \mathcal{X}_\kappa \to \mathcal{X}_\kappa$

$$P_1 \triangleright f(x)g(x) = (P_1 \triangleright f(x)) g(x) + \left( e^{-\frac{P_0}{\kappa}} \triangleright f(x) \right) (P_1 \triangleright g(x))$$

Leibniz

$$P_1 \triangleright [x_1, x_0] = \frac{i}{\kappa} P_1 \triangleright x_1$$

Translations: $x'_\mu = x_\mu - a_\mu \iff [x'_1, x'_0] = \frac{i}{\kappa} x'_1$

$$[a_1^\kappa, x_0] = \frac{i}{\kappa} a_1^\kappa \quad [a_\mu^\kappa, x_1] = 0 \quad [a_0^\kappa, x_0] = 0$$

$$T = 1 + d \quad d = -ia_\mu^\kappa P^\mu$$

differential calculus $df(x)g(x) = (df(x)) g(x) + f(x)dg(x)$

Woronowicz, Majid, Oeckl, ... '90
Pregeometry of $\kappa$-Minkowski/$\kappa$-Poincaré

Covariant quantum mechanics

Pregeometry spacetime noncommutativity arises from a more fundamental theory

Ordinary Heisenberg algebra $\iff$ $\kappa$-Minkowski/$\kappa$-Poincaré

pregeometric representation

But what framework?

$\kappa$-Minkowski

time coordinate = noncommutative observable

$\kappa$-Minkowski

time coordinate = evolution parameter

$\Downarrow$

Covariant formulation of ordinary quantum mechanics

The spatial coordinates and the time coordinate play the same type of role.
Both operators on a "kinematical Hilbert space"
(ordinary Hilbert space of normalizable wave functions)


(See also the talks of Hoehn, Giacomini)
Covariant quantum mechanics


Kinematical Hilbert space

- Space of (1+1D) “spacetime regions” $L^2(\mathbb{R}^2, dq_0, dq_1)$

- Standard multiplicative operators $\hat{q}_0, \hat{q}_1$
  and derivative operators $\hat{\pi}_0 \equiv i \frac{\partial}{\partial q_0}, \hat{\pi}_1 \equiv i \frac{\partial}{\partial q_1}$

- Standard commutator algebra

  $[\hat{\pi}_0, \hat{q}_0] = i$ \hspace{1cm} $[\hat{\pi}_0, \hat{q}_1] = 0$  
  $[\hat{\pi}_1, \hat{q}_0] = 0$ \hspace{1cm} $[\hat{\pi}_1, \hat{q}_1] = -i$

- “Regions of spacetime” $\langle \hat{q}_0 \rangle, \langle \hat{q}_1 \rangle, \delta q_0, \delta q_1$
There is no "evolution"

Kinematical Hilbert space $\leftrightarrow$ Physical Hilbert space

Hamiltonian constraint $\mathcal{H} = 0$

$\Rightarrow$ relationships between the properties of the (partial) observables for spatial coordinates and the properties of the time (partial) observable

$\rightarrow$ "fuzzy worldlines"

"Geometrical interpretation"

(Classical) Special Relativity:
arena = Minkowski spacetime
$\mathcal{H} = 0$
$\downarrow$
dynamics of relativistic classical particles unfolds

Covariant Quantum Mechanics
arena = kinematical Hilbert space
$\mathcal{H} = 0$
$\downarrow$
dynamics of relativistic quantum particles unfolds
Geometrical interpretation

Minkowski spacetime property

classical particles experiment

“spacetime observables”

kinematical Hilbert space property

quantum particles experiment

“spacetime observables” (physical Hilbert space)
Pregeometry of $\kappa$-Minkowski/$\kappa$-Poincaré

Pregeometric representations


\[
\hat{x}_0 = \hat{q}_0 \\
\hat{x}_1 = \hat{q}_1 e^{\hat{\pi}_0 / \kappa}
\]

\[
\begin{align*}
P_0 \triangleright f(\hat{x}_0, \hat{x}_1) &\leftrightarrow [\hat{\pi}_0, f(\hat{q}_0, \hat{q}_1 e^{\hat{\pi}_0 / \kappa})] \\
P_1 \triangleright f(\hat{x}_0, \hat{x}_1) &\leftrightarrow e^{-\hat{\pi}_0 / \kappa} [\hat{\pi}_1, f(\hat{q}_0, \hat{q}_1 e^{\hat{\pi}_0 / \kappa})]
\end{align*}
\]

\[
\hat{a}_0^\kappa = a_0 \\
\hat{a}_1^\kappa = a_1 e^{\hat{\pi}_0 / \kappa}
\]

\[
\begin{align*}
[\hat{\pi}_0, \hat{q}_0] &= i \\
[\hat{\pi}_0, \hat{q}_1] &= 0 \\
[\hat{\pi}_1, \hat{q}_0] &= 0 \\
[\hat{\pi}_1, \hat{q}_1] &= -i
\end{align*}
\]

\[
\Rightarrow [\hat{x}_1, \hat{x}_0] = \frac{i}{\kappa} \hat{x}_1
\]

\[
\Rightarrow P_1 \triangleright f g = (P_1 \triangleright f) g + \left(e^{-\frac{P_0}{\kappa} \triangleright f}\right) (P_1 \triangleright g)
\]

\[
\Rightarrow [\hat{a}_0^\kappa, \hat{x}_0] = 0 \\
[\hat{a}_1^\kappa, \hat{x}_0] = 0 \\
[\hat{a}_0^\kappa, \hat{x}_1] = 0 \\
[\hat{a}_1^\kappa, \hat{x}_1] = \frac{i}{\kappa} \hat{a}_1^\kappa
\]

Translutions

\[
T = 1 + \mathbf{d}_P \\
\mathbf{d}_P = -i a_\mu^\kappa P^\mu = -i a_\mu^\mu [\hat{\pi}_\mu, \cdot] \\
\mathbf{d}_P f g = (\mathbf{d}_P f) g + f \mathbf{d}_P g
\]
Pregeometry of $\kappa$-Minkowski/$\kappa$-Poincaré

Boosts

Infinitesimal boost transformation

\[ B = 1 + d_N \quad d_N = i\hat{\xi}^\kappa N \]
\[ d_N f g = (d_N f) g + f d_N g \]

\[ [\hat{\xi}^\kappa, \hat{x}_0] = i\frac{\hat{\xi}^\kappa}{\kappa} \quad [\hat{\xi}^\kappa, \hat{x}_1] = 0 \]

\[ \Delta P_1 = P_1 \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes P_1 \]
\[ \Delta N = N \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes N \]
\[ [N, P_1] = i\frac{\kappa}{2} \left( 1 - e^{-\frac{2}{\kappa}P_0} \right) - \frac{i}{2\kappa} P_1^2 \]

\[ \hat{\xi}^\kappa = \xi e^{\frac{\pi_0}{\kappa}} \]

\[ N \triangleright f(\hat{x}) \equiv e^{-\frac{\pi_0}{\kappa}} [\hat{\eta}, f(\hat{x})] \]
\[ \hat{\eta} \equiv \left( \frac{e^{2\frac{\pi_0}{\kappa}} - 1}{2\kappa} + \frac{\pi_1^2}{2\kappa} \right) \hat{q}_1 - \hat{\pi}_1 \hat{q}_0 \]

\[ B = 1 + i\xi [\hat{\eta}, \cdot] \]

Finite boost

\[ B_{\text{finite}} \triangleright \hat{O} \rightarrow \hat{B}^\dagger \hat{O} \hat{B} = e^{i\xi \hat{\eta}^\dagger} \hat{O} e^{-i\xi \hat{\eta}} \]

On-shell condition

\[ \hat{H} = \Box_{\kappa} = (2\kappa)^2 \sinh^2 \left( \frac{\pi_0}{2\kappa} \right) - e^{-\frac{\pi_0}{\kappa}} \pi_1^2 \]
Pregeometry of $\kappa$-Minkowski/$\kappa$-Poincaré

Deformed measure and hermiticity

“Pregeometric Momentum-space representation” $\psi(\pi_\mu)$ of states: square-integrable functions of $\hat{\pi}_0$ and $\hat{\pi}_1$

Measure of integration

Scalar product

$$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle = \int D(\pi_\mu) \psi^*(\pi_\mu) O(\pi_\mu) \psi(\pi_\mu)$$

B:

$$\begin{align*}
\pi_0' &= \pi_0 - \xi \pi_1 \\
\pi_1' &= \pi_1 - \xi \left( \kappa e^{\frac{\pi_0}{\kappa}} - 1 + \frac{\pi_1^2}{2\kappa} \right)
\end{align*}$$

$\Rightarrow$ Invariant measure

$$D(\pi_\mu) = d\pi_0 d\pi_1 e^{-\frac{\pi_0}{\kappa}}$$

$D(\pi_\mu)$: $\hat{\eta}^\dagger = \hat{\eta}$

$\hat{B}$:

$$\langle \psi' | \psi' \rangle = \langle \psi | e^{i\xi \hat{\eta}} e^{-i\xi \hat{\eta}} | \psi \rangle = \langle \psi | \psi \rangle$$

Scalar product is invariant
\[ \hat{q}_1 \equiv i \partial_{\pi_1} \Rightarrow \hat{q}_1 e^{-\frac{\pi_0}{\kappa}} = e^{-\frac{\pi_0}{\kappa}} \hat{q}_1 \Rightarrow \hat{x}_1^\dagger = \hat{x}_1 \]

\[ \hat{q}_0 \equiv -i \partial_{\pi_0} \Rightarrow \hat{q}_0 e^{-\frac{\pi_0}{\kappa}} = e^{-\frac{\pi_0}{\kappa}} \left( \hat{q}_0 + \frac{i}{\kappa} \right) \Rightarrow \hat{x}_0^\dagger = \hat{x}_0 + \frac{i}{\kappa} \quad (\hat{x}_0 = \hat{q}_0) \]

- \( \hat{x}_0^* \equiv \hat{q}_0 - \frac{i}{2\kappa} \quad \hat{x}_0^\dagger = \hat{x}_0 \)

- Anyway: \( \hat{x}_0 \) not observable: \[ [\hat{H}, \hat{x}_0] \neq 0 \]

- As a “partial” observable we are interested in “differences in time”
No state in the pregeometric Hilbert space gives absolutely sharp values to $\hat{x}_0$ and $\hat{x}_1$:

$$\hat{x}_0 = \hat{q}_0, \quad \hat{x}_1 = \hat{q}_1 e^{\ell\hat{\pi}_0}$$

a sharply specified $\hat{x}_0$ requires an eigenstate of $\hat{q}_0$ but on such eigenstates of $\hat{q}_0$ one finds that $\hat{\pi}_0$ is infinitely fuzzy $\delta \hat{\pi}_0 = \infty$ which in turn implies that $\hat{x}_1 = \hat{q}_1 e^{\ell\hat{\pi}_0}$ cannot be sharp

So all points in $\kappa$-Minkowski must be fuzzy
Fuzzy Points

Gaussian states

\[ \Psi_{\pi, \sigma, \bar{q}}(\pi) = Ne^{-\frac{(\pi_0 - \pi)^2}{4\sigma^2_0} - \frac{(\pi_1 - \pi_1)^2}{4\sigma^2_1}}e^{i\pi_0 \bar{q}_0 - i\pi_1 \bar{q}_1} \]

\[ N^2 = \frac{e^{\frac{\bar{q}_0}{\kappa} - \sigma^2_0}{2\kappa^2}}{2\pi\sigma_0\sigma_1} \]

\[ \bar{x}_0 = \langle \hat{q}_0 \rangle \quad \bar{x}_1 = \langle \hat{q}_1 e^{\frac{\pi_0}{\kappa}} \rangle \]

\[ \delta \hat{x}_0 = \sqrt{\langle \hat{q}_0^2 \rangle - \bar{x}_0^2} \]

\[ \delta \hat{x}_1 = \sqrt{\langle \left( \hat{q}_1 e^{\frac{\pi_0}{\kappa}} \right)^2 \rangle - \bar{x}_1^2} \]

\[ \langle \hat{x}_0 \rangle = \bar{q}_0 - \frac{i}{2\kappa} \quad \delta \hat{x}_0 = \frac{1}{2\sigma_0} \]

\[ \langle \hat{x}_1 \rangle = \langle \hat{q}_1 \rangle \langle e^{\frac{\pi_0}{\kappa}} \rangle = \bar{q}_1 e^{\frac{\pi_0}{\kappa}} e^{-\frac{\sigma^2_0}{2\kappa^2}} \]

\[ \delta \hat{x}_1 = e^{\frac{\pi_0}{\kappa}} \left[ \frac{1}{4\sigma^2_1} + \bar{q}_1^2 \left( 1 - e^{-\frac{\sigma^2_0}{\kappa^2}} \right) \right]^{1/2} \]

\[ \langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle = \int D(\pi) \psi^*(\pi) O(\pi) \psi(\pi) \quad \hat{q}_\mu \equiv -i \partial_\mu \]
Translations and relative locality for fuzzy points

\[ T = 1 - i a^\mu [\hat{\pi}_\mu, \cdot] \]

\[ T \triangleright \hat{x}_0 = \hat{x}_0 - \hat{a}_0^\kappa = \hat{q}_0 - a_0 \]

\[ T \triangleright \hat{x}_1 = \hat{x}_1 - \hat{a}_1^\kappa = e^{\frac{\pi_0}{\kappa}} (\hat{q}_1 - a_1) \]

\[ \langle T \triangleright \hat{x}_0 \rangle = \bar{q}_0 - a_0 - \frac{i}{2\kappa} \]

\[ \delta(T \triangleright \hat{x}_0) = \frac{1}{2\sigma_0} \]

\[ \langle T \triangleright \hat{x}_1 \rangle = (\bar{q}_1 - a_1) e^{\frac{\pi_0}{\kappa}} e^{-\frac{\sigma_0^2}{2\kappa^2}} \]

\[ \delta(T \triangleright \hat{x}_1) = e^{\frac{\pi_0}{\kappa}} \left[ \frac{1}{4\sigma_1^2} + (\bar{q}_1 - a_1)^2 \left( 1 - e^{-\frac{\sigma_0^2}{\kappa^2}} \right) \right]^{1/2} \]
Hamiltonian constraint

\[ \mathcal{H} = (2\kappa)^2 \sinh^2 \left( \frac{\pi_0}{2\kappa} \right) - e^{-\frac{\pi_0}{\kappa}} \pi_1^2 \]

Physical observables invariant under \( \mathcal{H} \rightarrow \) new scalar product \( \langle \cdot | \cdot \rangle_{\mathcal{H}} \)
projecting all the orbit of the gauge transformation generated by \( \mathcal{H} \) on the same state

\[
\langle \psi | \phi \rangle_{\mathcal{H}} = \langle \psi | \delta (\mathcal{H}) \Theta(\pi_0) | \phi \rangle = \int e^{-\frac{\pi_0}{\kappa}} d\pi_1 d\pi_0 \delta (\mathcal{H}) \Theta(\pi_0) \psi^*(\pi) \phi(\pi)
\]

expectation value

\[
\langle \Psi_{\bar{q}_0, \bar{q}_1} | \hat{O} | \Psi_{\bar{q}_0, \bar{q}_1} \rangle_{\mathcal{H}}
\]

\[
\Psi_{\bar{q}_0, \bar{q}_1} (\pi_\mu, \bar{\pi}_\mu, \sigma_\mu) = Ne^{-\frac{(\pi_0 - \bar{\pi}_0)^2}{4\sigma_0^2} - \frac{(\pi_1 - \bar{\pi}_1)^2}{4\sigma_1^2}} e^{i\pi_0 \bar{q}_0 - i\pi_1 \bar{q}_1}
\]

\[
N^{-2} = \int e^{-\frac{\pi_0}{\kappa}} d\pi_1 d\pi_0 \delta (\mathcal{H}) \Theta(\pi_0) | \Psi_{\bar{q}_0, \bar{q}_1} (\pi_\mu, \bar{\pi}_\mu, \sigma_\mu) |^2
\]
\( \hat{x}_1 \) and \( \hat{x}_0 \) are not self-adjoint operators on our physical Hilbert space \( (\hat{\mathcal{H}}, \hat{x}_0, \hat{x}_1) \neq 0 \)

\[
\hat{A} = e^{\frac{\pi_0}{\kappa}} \left( \hat{q}_1 - \hat{V} \hat{q}_0 - \frac{1}{2} [\hat{q}_0, \hat{V}] \right) 
\]

\[
\mathcal{V} \equiv \frac{\partial \mathcal{H}/\partial \pi^1}{\partial \mathcal{H}/\partial \pi^0} \quad [\hat{\mathcal{H}}, \hat{A}] = 0 \quad \hat{A}^\dagger = \hat{A}
\]

(\( \kappa \)-deformed Newton-Wigner operator)

\[
\hat{X}(T) = \hat{A} + e^{\frac{\pi_0}{\kappa}} \hat{V} T \quad \left( \frac{\partial x_1/\partial x_0}{\sim e^{\frac{\pi_0}{\kappa}} \mathcal{V}} \right)
\]

\[
\hat{A} = \hat{x}_1 - e^{\frac{\pi_0}{\kappa}} \hat{V} \hat{x}_0 \quad \left( \frac{\hbar \to 0}{\kappa \to \infty} \to x - vt \right)
\]

\[
\mathcal{V} \equiv \frac{e^{-\frac{\pi_0}{\kappa}} \pi_1}{\kappa \sinh \left( \frac{\pi_0}{\kappa} \right) + e^{-\frac{\pi_0}{2\kappa}} \pi_1^2} \quad \frac{\mathcal{H}=0}{m=0} \quad \frac{1}{1 + \frac{\pi_1}{\kappa}} \sim 1 - \frac{\pi_0}{\kappa}
\]

Now we can ask what is the observed fuzziness for a particle traveling an arbitrary distance (even cosmological)
Translations: Alice and Bob

Alice: \[ q_0 = q_1 = 0 \quad (x_0 = x_1 = 0) \quad \Rightarrow \quad \langle \Psi_{0,0} | A | \Psi_{0,0} \rangle_\mathcal{H} = 0 \]

\[ \delta A^2_{[\kappa]} = \left( \langle \Psi_{0,0} | A^2 | \Psi_{0,0} \rangle_\mathcal{H} \right)_{[\kappa]} \approx \frac{\langle \pi_0 \rangle}{2 \kappa \sigma^2} \]

\[ \sigma_1 \ll \sigma_0, \quad \bar{\pi}_1 \pi_1 \quad \rightarrow \text{saddle point approximation} \]

in the \[ \pi_1 \] integration: \[ \frac{1}{\sigma^2} = \frac{1}{\sigma_1^2} + \frac{<V>^2}{\sigma_0^2} \]

Bob (distant observer reached by the particle):

\[ \Psi_{a_0,a_1} \quad <A> = \langle \Psi_{a_0,a_1} | A | \Psi_{a_0,a_1} \rangle_\mathcal{H} = 0 \]

\[ a^0, a^1 : \quad \langle \Psi_{0,0} | T^{-1} A T | \Psi_{0,0} \rangle_\mathcal{H} = 0 \quad \Rightarrow \quad a^1 = \langle V \rangle a^0 \approx a^0 \left( 1 - \frac{\langle \pi_0 \rangle}{\kappa} \right) \]

\[ \delta A^2_{[\kappa]} = \left( \langle \Psi_{a_0,a_0} | V \rangle a^0 | A^2 | \Psi_{a_0,a_0} \rangle_\mathcal{H} \right)_{[\kappa]} \approx \frac{\langle \pi_0 \rangle}{2 \kappa \sigma^2} + \frac{a_0^2 \sigma^2}{\kappa^2} \]

(one parameter) family of observers “on the worldline”
\[ \langle \Psi_{q_0,q_1} | \hat{X} (T) | \Psi_{q_0,q_1} \rangle_{\mathcal{H}} = \bar{x}_1 - \bar{x}_0 + T + O \left( \frac{1}{\kappa^2} \right) \]

\[ \delta A^2_{[\kappa]} \approx \frac{\langle \pi_0 \rangle}{2\kappa \sigma^2} + \frac{a_0^2}{\kappa^2} \sigma^2 \approx \frac{a_0 \sigma}{\kappa} \]

\[ a^1 \approx a^0 \left( 1 - \frac{\langle \pi_0 \rangle}{\kappa} \right) \]

\[ \delta A \big|_{m=0; \sigma_1 \ll \sigma_0} \sim \delta x_1 \]
\[ \langle \Psi_{q_0, q_1} | \hat{X}(T) | \Psi_{q_0, q_1} \rangle_H = \bar{x}_1 - \bar{x}_0 + T + O \left( \frac{1}{\kappa^2} \right) \]

\[ x_1 = q_1 e^{\frac{\pi_0}{\kappa}} \]

\[ \delta A^2_{[\kappa]} \approx \frac{\langle \pi_0 \rangle}{2 \kappa \sigma^2} + \frac{a_0^2}{\kappa^2} \sigma^2 \approx \frac{a_0 \sigma}{\kappa} \]

\[ a^1 \approx a^0 \left( 1 - \frac{\langle \pi_0 \rangle}{\kappa} \right) \]

\[ \delta A|_{m=0; \sigma_1 \ll \sigma_0} \approx \delta x_1 \]
Thanks!
Galilean relative rest

- In Galilean relativity we can say that one has relativity of "spatial locality"

We can use this scheme to describe the paradigmatic situation in which Bob is on a boat moving at velocity $v_x$ respect to Alice, who is standing on the dock. Imagine that Alice is bouncing a ball on the dock, and that the two points mark the position and time of two of the ball’s bounces on the ground. While Alice evidently observes the ball bouncing at the same point in space, Bob, who is moving with velocity $v_x$ relative to Alice, observes, in its reference frame, the ball bouncing in two different positions: if Bob is approaching the dock, for example, Bob sees the second bounce closer then the first.
Galilean relative rest

- In Galilean relativity we can say that one has relativity of "spatial locality"

\[
\begin{align*}
(t_1, \bar{x}) & \quad (t_2, \bar{x}) \\
(t_1, \bar{x} - v_x t_1) & \quad (t_2, \bar{x} - v_x t_2)
\end{align*}
\]

the rest is relative

\[
\begin{align*}
(t, x_1) & \quad (\bar{t}, x_2) \\
(\bar{t}, x_1 - v_x \bar{t}) & \quad (\bar{t}, x_2 - v_x \bar{t})
\end{align*}
\]

while time simultaneity is still absolute
SR: relative simultaneity

- In special relativity:

  invariant scale "c" ⇒ absolute (time) simultaneity → relative (time) simultaneity.
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  invariant scale \( c \) \( \Rightarrow \) absolute (time) simultaneity \( \rightarrow \) relative (time) simultaneity.

Thus one has relative space locality and relative (time) simultaneity, but still absolute spacetime locality.
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Thus one has relative space locality and relative (time) simultaneity, but still absolute spacetime locality.

- There is no observer-independent projection from spacetime to separately space and time. We can say that one "sees" spacetime as a whole.
Alice and Bob, distant observers in relative motion (with constant speed), have stipulated a procedure of clock synchronization and they have agreed to build emitters of blue photons (blue according to observers at rest with respect to the emitter). They also agreed to then emit such blue photons in a regular sequence, with equal time spacing $\Delta t^*$. Bob's worldlines are obtained combining a translation and a boost transformation (Bob $= B \triangleright T \triangleright$ Alice),

$$\begin{align*}
x_1^B \left( x_0^B \right)_p &= \gamma \left( \bar{x}_A^A - a_1 - \beta \left( \bar{x}_0^A - a_0 \right) \right) + \frac{p_1^A - \beta p_0^A}{p_0^A - \beta p_1^A} \left( x_0^B - \gamma \left( \bar{x}_0^A - a_0 - \beta \left( \bar{x}_A^A - a_1 \right) \right) \right),
\end{align*}$$

We arranged the starting time of each sequence of emissions so that there would be two coincidences between a detection and an emission, which are of course manifest in both coordinatizations, so to obtain a specular description. Relative simultaneity is directly or indirectly responsible for several features that would appear to be paradoxical to a Galilean observer (observer assuming absolute simultaneity). In particular, while they stipulated to build blue-photon emitters they detect red photons, and while the emissions are time-spaced by $\Delta t^*$ the detections are separated by a time greater than $\Delta t^*$. 
Relative locality: an insight

We don’t actually “see” spacetime, but we “see” (detect) time sequences of particles, and then abstract spacetime by inference:

We actually “see” (detect) only what we **locally witness**
Relative locality: an insight

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![Diagram showing relative locality and Einstein clock principle](image)

We actually “see” (detect) only what we locally witness.
Relative locality: DSR theories

- **DSR theories**: invariant (inverse-momentum) scale $\ell \Rightarrow$ absolute spacetime locality $\rightarrow$ relative spacetime locality

There is no observer-independent projection from a one-particle phase space to a description of the particle separately in spacetime and in momentum space. We thus can say that one "sees" phase space as a whole.
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