The orbital angular momentum of light: Genesis and evolution of the concept and of the associated photonic technology

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Summary. — We present an overview of the main issues concerning the orbital angular momentum of light, starting from the framework of its definition, passing through the developments of the “toolbox” of recent devices and methods introduced for its generation, manipulation and detection, touching on some of the linear and nonlinear radiation-matter interaction phenomena that depend on this quantity, and finally mentioning its most relevant applications to quantum optics and quantum information. This review is meant as a fairly comprehensive survey, although unavoidably incomplete, of a long-standing subject whose relevance has been recognized and established only relatively recently. Of course, the debate on the mechanical degrees of freedom of light, especially those related to the deep nature of the angular momentum, has followed the entire history of the theory of electromagnetic radiation, from its initial formulation in terms of classical waves till the establishment of the quantum field theory. We shall attempt to discuss the reasons of such a tardy acknowledgment, especially compared with the case of the spin angular momentum. Although touching on many different aspects of the subject, our treatment will be somewhat biased towards covering in more detail those aspects of the field which intersected our own research, including the angular momentum exchange with soft materials such as liquid crystals, the spin-orbit optical interaction effects occurring in anisotropic inhomogeneous materials, and the quantum applications of these processes. More generally, we shall try to highlight some of the many problems faced within this research field during the last two decades, as well as the future challenges and potential applications.

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1. Physical foundations of the orbital angular momentum (OAM)

During the last two decades of the nineteenth century, the idea that the electromagnetic radiation field was an autonomous entity with dynamical properties of its own, independently of the sources, gradually took shape. Poynting’s theorems on the conservation of energy and of the linear and angular momenta of the electromagnetic field [1-3] contributed in a prominent way to plant that idea: far from the sources, the time variations of the energy and momenta of the electromagnetic field stored in a specified spatial region are controlled by the corresponding fluxes crossing the boundary of such region, as occurs in ordinary fluids. In the early twentieth century, this picture was further supported and generalized to quantum theory through the experimental evidence for the existence of photons [4], outright elementary particles and not mere excitation quanta associated to a set of normal modes of the electromagnetic field. Photons turned out to be the actual elementary carriers of the energy and momenta of the field, and therefore the intermediaries of their possible transfer to the medium the field interacts with. However, the discoveries unveiling the very essence of the three properties did not always follow neighbouring paths and we need have no qualms about stating that, with the angular momentum, the history has been slightly more “winding”, and perhaps the final chapter is yet to be written.

The most relevant problem in the physics of the angular momentum of light consists in establishing its concept in terms of the wave properties of the field or the quantum states of the photons. Besides, in order to address the mechanical effects induced by the angular momentum transfer between light and matter, it is crucial to single out the overall change of angular momentum suffered by the radiation in the interaction with a transparent medium. The continuity equation expressing the conservation of the total
angular momentum of the electromagnetic field, as given by the electric and magnetic fields $E$ and $B$, respectively, is usually written as follows (in Gaussian units) [5]:

$$\frac{\partial}{\partial t} J_{\text{field}} + \text{div} \, \hat{M} = -r \times \left( \rho E + \frac{1}{c} j \times B \right),$$

where $\rho$ and $j$ are the charge and current densities of the sources, $J_{\text{field}}$ is the angular momentum volume density and $\hat{M}$ the angular momentum flux density given by

$$J_{\text{field}} = \frac{1}{4\pi c} r \times (E \times B),$$

$$\hat{M} = \hat{T} \times r,$$

$\hat{T}$ being

$$\hat{T}_{\alpha\beta} = \frac{1}{4\pi} \left[ E_\alpha E_\beta + B_\alpha B_\beta - \frac{1}{2} (E^2 + B^2) \delta_{\alpha\beta} \right].$$

Far from the sources, in the radiation zone, the densities $\rho$ and $j$ vanish. In free space, the total angular momentum of the radiation field, given by the overall space integral

$$J = \frac{1}{4\pi c} \int r \times (E \times B) \, dV,$$

is conserved in time.

A first important question arises: is it possible to single out some general prescriptions for the evaluation of the angular momentum of a field, without having to identify point by point the entire phase and amplitude structure of the field? In other words, is there any distinct wave property from which the value of the angular momentum can be directly established? To answer this question a deeper insight into the definition of the angular momentum is necessary. According to eq. (2), the total angular momentum of an ideal plane wave of infinite extent is predicted to be exactly zero in any direction and, what’s more, no component of the angular momentum could ever arise along the propagation direction. The linear momentum density $(E \times B)/(4\pi c)$ of a plane wave lies, in fact, entirely along the propagation direction, and therefore the angular momentum density is exactly perpendicular to the propagation direction. On the other hand, it is well known that a left- or right-handed circularly polarized plane wave carries an angular momentum, as demonstrated by Beth in 1936 [6]. The resolution of this seeming paradox lies in the fact that an ideal plane wave is purely a schematization which cannot be applied strictly in the real world. It is a consolidated result [5] the fact that, for a circularly polarized quasi-plane wave of frequency $\omega$ having a finite extent in the transverse plane and a slowly varying amplitude modulation, the ratio between the modulus of the time-averaged component of the angular momentum $J$ parallel to the direction of propagation $z$ and the total energy $U$ is

$$\frac{J_z}{U} = \pm \omega^{-1},$$
the sign depending on the left or right handedness of the circular polarization. Furthermore, for a cylindrically symmetric cross section with the origin on the cylinder axis, the transverse components of the total angular momentum turn out to be zero. The existence of a nonzero longitudinal component of the total angular momentum is related to a nonvanishing longitudinal component of the electric field, arising from the gradient of the transverse amplitude modulation. This analysis allows bridging the circular polarization of a real plane wave to the angular momentum it carries along the propagation direction and ultimately to the angular momentum of photons. Each photon in a circularly polarized wave therefore possesses an angular momentum of $\pm \hbar$, where $\hbar = h/(2\pi)$ is the reduced Planck constant and the sign is fixed by the polarization handedness, as theoretically predicted for its helicity or spin angular momentum (SAM) [7] and beautifully demonstrated experimentally by Beth [6]. In his celebrated experiment, Beth directly measured the mechanical torque exerted on a doubly refracting plate by a circularly polarized real (that is, having finite extent) optical plane wave. Assigning a SAM of $\pm \hbar$ to each photon of left/right circularly polarized light, Beth showed the equivalence between the quantum theory and the wave theory for the torque on a section of a crystal plate, under the assumption that the conservation of the angular momentum holds at the face of the plate. The mechanical torque exerted on a half-waveplate by a circularly polarized light beam was therefore shown to balance the flip of the SAM of light.

Though highly relevant, the case of the circularly polarized plane wave is not the whole story, since it is not the only known case in which the angular momentum in eq. (5) is different from zero for any possible choice of the origin of coordinates, so as to acquire an “intrinsic” value. In 1992 Allen et al. [8,9], by a straightforward calculation, theoretically demonstrated that Laguerre-Gaussian (LG) optical modes, solutions of the paraxial wave equation, indeed, carry an angular momentum, perfectly discernible and profoundly different in nature from SAM, being related to helical wavefronts rather than to polarization. In terms of particles, the helical wavefronts correspond to generally curved Poynting trajectories, representing the paths of photons in the Madelung-Bohm-de Broglie interpretation [10,11]. In this case, the angular momentum of photons is related to the orbital motion rather than to the helicity. The angular momentum possessed by LG modes was accordingly baptized as orbital angular momentum and its actual existence as a mechanical property was first demonstrated by He et al. in 1995 [12]. In this experiment, microscopic isotropic absorptive particles trapped by optical tweezers were transferred the angular momentum carried by an LG beam having an OAM of $\pm \hbar$ per photon (in addition to the SAM). Though, strictly speaking, it cannot be considered as the counterpart of Beth’s experiment, since the final OAM of scattered photons could not be measured so that the full angular momentum balance could not be verified, the experiment by He et al. provided the first experimental evidence of mechanical effects due to the orbital angular momentum. It can be finally concluded that there are at least two distinct properties of light, i.e. circular polarization and helical wavefronts, that represent effective indicators of a nonzero angular momentum, as given in eq. (5). In addition, the angular momenta corresponding to these distinct properties seem to exhibit different mechanical features, particularly in the interaction with transparent media [13,14]. Specifically, circular polarization or SAM plays a crucial role in birefringent bodies, driving the rotation of the local optical axis as for example defined by the average molecular orientation [6,15,16], while helical wavefronts or OAM can set in motion, along trajectories circulating around the beam axis, the molecule centers of mass [17,18] or elemental volumes in inhomogeneous macroscopic bodies [19,12,20,16,21].
orbital parts is not general and is well defined only for paraxial light beams, such as the above-mentioned circularly polarized real plane waves or Laguerre-Gaussian modes. In ref. [22] it has been reported on the partial conversion of the spin carried by a paraxial beam into orbital angular momentum of a nonparaxial beam through a high numerical aperture lens tightly focusing the input beam. The complete one-to-one correspondence between SAM and polarization, on the one side, and OAM and helical wavefronts, on the other side, is hence not valid beyond the paraxial approximation [23-25,14]. On the other hand, a process of spin-to-orbital conversion (STOC) may take place also in the framework of the paraxial approximation [26]. The paraxial STOC, however, as will be discussed later in the present paper, takes place thanks to the interposition of an inhomogenous anisotropic medium endowed with suitable symmetry properties [27], rather than to the intrinsic transformation properties of the electromagnetic field.

Separating the total angular momentum of the electromagnetic field into spin and orbital parts is useful for expressing it in terms of clearly distinct wave or photon properties, thus making it more easily available for optical micromanipulation and trapping, as well as for managing the quantum mechanical properties of light. As it will be clarified in developing this review article, the factual possibility to use the SAM and OAM of photons as distinct degrees of freedom for complex high-dimensional qubits encoding has drawn a large amount of interest in quantum information and cryptography [28-32] and is still doing so.

The formal separation of the total angular momentum into spin and orbital parts has been performed on heuristic grounds by several authors [33-35]. From eq. (5), after replacing $B$ with rot $A_\perp$, with div $A_\perp = 0$ to avoid gauge-dependence problems, a straightforward calculation gives [35]

$$J = L + S + m,$$

where

$$L = \frac{1}{4\pi c} \sum_{i=x,y,z} \int_V E_i (r \times \text{grad}) A_{\perp i} \ dV,$$

$$S = \frac{1}{4\pi c} \int_V E \times A_{\perp} \ dV,$$

$$m = -\frac{1}{4\pi c} \int_{\partial V} (r \times A_{\perp}) (E \cdot n) \ d\sigma,$$

where $\partial V$ is the surface bounding the region $V$ over which the integration is performed and $n$ is the unit outward normal. Assuming that the fields tend to zero sufficiently quickly, the surface term $m$ is equal to zero and $J = L + S$. The formal analogy between $L$ and the expectation value of the orbital angular momentum in quantum mechanics has suggested its association with the orbital contribution to the angular momentum of light. Similarly, the dependence of $S$ on the polarization of the field has suggested its association with the spin. Unfortunately, although tempting, this interpretation is not without fundamental problems. The main problem with the separation of the total angular momentum into $L$ and $S$ does not lie in the intrinsic coherence of their expressions, as given by eqs. (8) and (9), but rather in their physical interpretation as angular momenta and, therefore, as independent generators of the rotations of the radiation fields. Specifically,
the orbital angular momentum would be expected to be the generator of the rotation of the coordinate dependence of the fields, leaving their direction unchanged. Analogously, the spin would be expected to be the generator of the rotations of the vectorial dependence of the fields, leaving their spatial distribution unchanged. This, in turn, would reflect onto the formulation of two independent conservation laws, one for OAM and one for SAM, that relate the time changes of the volume densities to the respective flux densities, and therefore to the interactions with matter and ultimately to the mechanical effects therein induced. In 1994, Van Enk and Nienhuis addressed the problem of the separation of $J$ in the sum of an orbital part $L$ and a spin part $S$ [36,37]. In order to relate the orbital part to helical wavefronts and the spin part to circular polarizations, they performed a noncanonical angular momentum separation [38,39], showing that both parts are observable and consistent with the transversality of the radiation field, but their quantized counterparts do not satisfy the expected commutation relations. This implies that the operators $\hat{L}$ and $\hat{S}$, as defined in refs. [36,37], cannot be regarded as true angular momenta and, therefore, they are not the generators of rotations. In ref. [40], the effect on the electric and magnetic fields of unitary transformations generated by the operators $\hat{L}$ and $\hat{S}$ is considered. The transformations generated by $\hat{L}$ and $\hat{S}$ turn out to be the closest approximations to independent rotations respectively of the spatial distribution and of the directions of the electromagnetic radiation fields that are consistent with the requirements of transversality. Furthermore, at least for paraxial beams, the components of the spin and orbital angular momenta along the propagation direction, say $\hat{L}_z$ and $\hat{S}_z$, turn out to be well defined. In the paraxial limit, in fact, the components of the fields in the direction of propagation are very small and therefore the infinitesimal unitary transformations generated by $\hat{L}_z$ and $\hat{S}_z$ are very close to actual independent rotations of the spatial distribution and of the direction of the fields.

Several attempts have been made to perform the separation of the angular momentum of light into orbital and spin parts beyond the paraxial approximation [38, 41, 42], in order to explain the spin-to-orbital angular momentum conversion occurring in tight focusing [22,43-45], scattering by small particles or apertures [46,45], and imaging optical systems [44, 45]. The main outcome of the investigations aimed at separating OAM and SAM in nonparaxial fields lies in the fact that helical wavefronts and polarization cannot be anymore independently ascribed to OAM and SAM respectively [23] and that the nonparaxial correction to the OAM is proportional to polarization rather than to topological charge of the helical wavefronts [23,38]. Most of the approaches reported in the literature address the problem of separating the spin and orbital parts of Poynting energy flow in vacuum [47-49]. In a recent paper [50], the authors attempted to clarify, on both theoretical and experimental grounds, the link between the amplitude, phase, and polarization profiles of vector beams and the forces and torques these beams exerted on trapped microparticles. In particular the role played in optical traps by a radially nonuniform spin angular momentum distribution in an elliptically polarized light beam was singled out, concluding that this term contributes to the mechanical properties of light with terms that are similar to those arising from OAM (for example driving an orbital motion of the particles around the beam axis).

In ref. [14], the possibility was explored of discerning the OAM and SAM of radiation on the grounds of the distinct mechanical effects they induce into a medium, such as liquid crystals, endowed with both extrinsic and intrinsic degrees of freedom, which independently couple to spin and orbital angular momenta. This approach will be discussed in detail later in this paper and, in principle, it is suitable to be applied also beyond the paraxial approximation.
2. – Generation and manipulation of light beams carrying orbital angular momentum

In the present section, the most important methods to create and process OAM in light beams will be surveyed. The efficient generation of photons in a preselected OAM state, including superpositions of different OAM components, is a key problem for both fundamental investigations and practical applications. Until the early 90s, due to the dearth of tools suitable for its generation and manipulation, the OAM of photons has played an auxiliary role in the study of both classical and quantum optics, being adopted essentially only to label the properties of wave functions under rotations [7]. The SAM degree of freedom has been used (and still is) much more widely than OAM, mainly thanks to the availability of many efficient and convenient polarization optical elements. Nowadays, several tools are available to handle OAM, including spiral phase plates, computer-generated holograms (CGH) printed on photographic plates or displayed on spatial light modulators (SLMs), cylindrical-lenses mode converters, Dove prisms and q-plates or q-plate-like devices. A q-plate is a recently introduced liquid-crystal-based birefringent plate with uniform retardation \( \delta \) and space-variant optical axis oriented according to a transverse pattern having topological charge \( q \) [26]. A q-plate essentially is a bidirectional SAM-to-OAM converter, capable of transposing both classical and quantum information from the spin to the OAM degree of freedom of photons and vice versa. In explaining the operation of a q-plate, we will take the opportunity to discuss the physics of this linear SAM-to-OAM conversion process, arising from birefringence inhomogeneities in a way reminiscent of the nonlinear interactions described in subsect. 3’2, where both SAM and OAM of light were transferred to liquid crystals.

Before inspecting the several tools just mentioned, it is worth briefly mentioning some general properties of light modes endowed with OAM. Light beams carrying a well-defined value of OAM are usually designated as helically phased beams or simply helical beams, in order to emphasise their helical wavefront structure. This structure originates from the azimuthal phase dependence \( \exp(i\ell\phi) \) of these beams, where \( \phi \) is the azimuthal coordinate and \( \ell \) is an integer. Such modes carry a well-defined OAM of \( \hbar \ell \) per photon, that is they are OAM eigenstates. Among such class of modes, a special role is played by the Laguerre-Gaussian (LG) ones, due to their profile stability under free-space propagation. Their amplitude distribution, \( LG_{p\ell} \), is the following [24, 51, 52]:

\[
LG_{p\ell} = \sqrt{\frac{2p!}{\pi (p + |\ell|)!}} \frac{1}{w(z)} \left[ \frac{r}{w(z)} \right]^{|\ell|} \exp \left( -\frac{r^2}{w^2(z)} \right) L_p^{|\ell|} \left( \frac{2r^2}{w^2(z)} \right) \exp i\ell\phi \\
\times \exp \frac{ik_0r^2z}{2(z^2 + z_R^2)} \exp \left[ -i(2p + |\ell| + 1) \arctan \frac{z}{z_R} \right],
\]

where \( r, \phi, z \) are the cylindrical coordinates, with the \( z \)-axis coincident with the propagation direction, the \( 1/e \) radius of the Gaussian term is given by \( w(z) = w(0)\left( (z^2 + z_R^2) / z_R^2 \right)^{1/2} \), with \( w(0) \) being the beam waist, \( z_R \) the Rayleigh range, \( (2p + |\ell| + 1) \arctan(z/z_R) \) the Gouy phase and \( L_p^{|\ell|}(x) \) is an associated Laguerre polynomial. \( p \) is the radial number and corresponds to the number of radial nodes (excluding the origin) in the intensity distribution (fig. 1). Though very popular in theoretical analyses, pure LG
modes do not represent the first choice in experimental situations, since both amplitude and phase must be controlled to generate them starting from a standard TEM\(_{00}\) laser mode. Indeed, in order to control the complex amplitude, i.e., the phase and the intensity structure, of an LG mode, several phase-only modulation methods are available [53-56].

LG beams are not the only important family of spatial helical modes. High-order Bessel beams, for example, form another outstanding family of spatial modes having a well-defined orbital angular momentum. They are propagation-invariant beams [57,58]—in fact, they are often designated as nondiffracting beams—that are exact solutions of both the Helmholtz and the paraxial wave equations. This remarkable feature makes them ideal for investigating the transition from the paraxial to the nonparaxial regime [23]. The amplitude distribution of Bessel beams is the solution to the scalar Helmholtz equation in cylindrical coordinates \(r, \phi, z\), given by

\[
\psi_\ell(r, \phi, z) = J_\ell(k_t r) \exp(i \ell \phi + ik_z z),
\]

where \(J_\ell\) is the \(\ell\)th-order Bessel function, and \(k_t\) and \(k_z\) are the transversal and longitudinal components of the wave vector (fig. 2). The azimuthal phase factor \(\exp(i \ell \phi)\) in eq. (12) clearly indicates that Bessel beams carry orbital angular momentum. However, the radial dependence \(J_\ell(k_{t \rho})\) of the scalar wave makes the total energy and momenta of Bessel beams diverge. For this reason, exact Bessel beams cannot represent realistic fields but only approximate them in limited space regions, analogously to plane waves. A rigorous vector analysis for Bessel beams is also available [59]. This analysis has shown that Bessel beams with pure azimuthal or pure radial polarization carry a vanishing total angular momentum. This can be ascribed to the fact that both azimuthally and radially polarized Bessel beams result from equal-weight superpositions of two beams: one having \(\ell = 1\) and right-circular polarization (\(\sigma = -1\)), and the other \(\ell = -1\) and left-circular polarization (\(\sigma = 1\)). For linearly and circularly polarized Bessel beams, on the contrary, the total angular momentum turns out to be nonzero and directed along the propagation direction. For linearly polarized nonparaxial beams (\(k_t/k_z\) not negligible), the radial symmetry characterizing the total angular momentum density along the beam axis \(J_z\) spontaneously breaks in the plane of polarization. Indeed, this symmetry breaking is recognized to be a distinctive feature of vector beams [60-62], i.e., optical beams formed by the nonseparable combinations of spatial and polarization modes.

In several applications, the details of the radial profile of a helical mode are not particularly relevant, and the OAM state preparation essentially involves imposing a helical phase to the input fundamental Gaussian mode at a given optical plane. The modes
obtained in this way can be described as a subfamily of the so-called Hypergeometric-Gaussian modes [64].

It should be noticed that all the beams mentioned above are scalar solutions of the wave equation and hence independent of the polarization of the light. In the paraxial limit, the full vectorial modes can be obtained by multiplying these scalar modes by given states of polarization, thus forming separable polarization-orbital states. By taking superpositions of such modes, one can then also form nonseparable combinations of spatial and polarization modes, thus giving rise to the so-called “vector beams”.

2.1. Spiral phase plates. – Spiral phase plates operate by directly imposing a helical phase shift on the incident light, through a suitable space-variant optical path. These elements typically consist of a transparent material slab with a gradually increasing and spiraling thickness. This corresponds to having a transversly space-variant optical path of the slab as a linear function of the azimuthal angle $\phi$, i.e. $h = \ell \lambda \phi / 2\pi (n - 1)$, where $n$ is the medium refractive index (see fig. 3). The spiraling thickness of a phase plate then creates the spiraling phase distribution of the helical mode. In order to be
effective, however, the phase plate must be smooth and shaped within an accuracy of a small fraction of a wavelength. This is why, in earlier implementations, spiral phase plates were immersed in an index-matched fluid bath, the temperature of which could be controlled to give precisely the index mismatch required to tune the optical-path step height at the operating wavelength [65]. A major limitation of these devices is that they are only applicable to a specific wavelength of light and generate a fixed topological charge of the generated wave, \( i.e. \) a fixed \( \ell \), including its handedness. For this reason, more versatile adjustable spiral phase plates have later been demonstrated [66]. These phase plates can be used with multiple wavelengths and produce a range of topological charges. They were created by twisting a piece of cracked Plexiglas and orienting the device so that one tab of the phase plate was directly perpendicular to the incident light, and the other tab was bent at some angle \( \phi \) away from the other. A laser directed at the end of the crack will then produce an optical vortex because of the azimuthally varying tilt around the center of the phase plate (fig. 4).

2.2. Computer-generated holograms for OAM generation. – The currently most widely used method to reshape the spatial transverse profile of optical beams, and in particular to generate pure helical modes or their superpositions, is that based on computer-generated holography. In practice, an amplitude or phase hologram behaves as an optical filter, whose transmittance/reflectance modulates the complex amplitude of the input beam. The phase distribution of such filter is typically added to a linear phase ramp and the sum expressed as modulo \( 2\pi \), as shown in fig. 5, yielding a diffraction grating that produces the desired beam in the first diffraction order. The phase profile of the diffracted beam is controlled by the hologram fringe pattern, while the amplitude can be addressed by modulating the fringe contrast. The transmittance of the hologram is designed and controlled through a computer, according to the experimental requirements. To produce helical modes, as an alternative to pitchfork diffraction gratings [67-69], also spiral Fresnel lenses are sometimes used [70]. More complex holograms have been realized to produce high-purity higher-order LG modes or their superpositions [53-55].
Fig. 5. – Example of pitch-fork holographic pattern for generating a horizontally deflected helical mode with $\ell = 5$, as determined by the number of branches in the central fork pattern. The phase shift is represented as a varying colour brightness, with the interval $0 - 2\pi$ corresponding to dark to the maximum brightness range. It can be noticed that the phase gradient is asymmetric, so as to obtain a blazed hologram having higher efficiency on a prescribed diffraction order.

Holograms may be printed on photographic plate, to obtain a fixed passive holographic optical element, or displayed on spatial light modulators. The latter are electro-optical devices whose main merit consists in their full computer-controlled reconfigurability, since a single hardware component enables one to dynamically change the displayed hologram at will (at about kHz maximum rate). The most widespread type of SLMs essentially are black and white liquid crystals (LC) displays, in all respects similar to those adopted in modern high-resolution LCD TV screens, except that they are realized on miniaturized scale (typically 1 cm × 1 cm). LC-based SLMs are connected to a computer and can be programmed through a video interface to act as arbitrary holograms. The adoption of LCs restricts the wavelength range application to the visible region. Another important class of spatial light modulators is based on deformable mirror devices (DMD), whose main merits are their applicability to wavelengths lying beyond the visible range and a relatively fast time response. As already mentioned above, computer-generated holograms can also be used to produce arbitrary superpositions of helical modes, by manipulating both the amplitude and the phase of the generated beams.

Holograms were used in 1995 to confirm the prediction that light was actually able to carry OAM in an experiment where micro-particles trapped by optical tweezers were set in rotation by the OAM of light [12]. Photographic CGHs were also used in a breakthrough experiment, in 2001, in which the existence of quantum OAM correlations between pairs of photons generated by spontaneous parametric down-conversion (SPDC) was first demonstrated [28]. In sect. 6 the relevance of photon OAM in quantum optics will be discussed at some length. Let us just mention here that the relevance of OAM for quantum optics is today mainly linked to the fact that it is defined in an infinite-dimensional Hilbert space, yet with discrete quantum numbers, so that it lends itself to the implementation of single-photon qudits (i.e., higher-dimensional quantum state units for quantum information applications). The use of high-dimensional qudits instead of qubits is desirable since it may, for example, lead to simplifying quantum computations [71, 72] and improve quantum cryptography [73]. Though holograms, especially those displayed on an SLM, are very versatile, they turn out to be scarcely practical in applications where high efficiencies are required independently of the input/output state, such as for example in detection stages of quantum optics experiments. Pure phase holograms, suitably designed and printed on high-quality photographic plates, may have efficiencies as high as 90%. However, the same holograms displayed on SLM, due to a
Fig. 6. – A \( \pi/2 \) mode converter comprising two cylindrical lenses of focal length \( f \) separated by a distance \( \sqrt{2}f \). An input Hermite-Gaussian mode of Rayleigh range \((1+1/\sqrt{2})f\) is converted to a Laguerre-Gaussian mode. The angular momentum transfer takes place exclusively at the lens on the left-hand side. (Reprinted with permission from ref. [75]. Copyright 2002, IOP publishing.)

lower resolution, will usually exhibit lower efficiencies (typically 40\%) [74]. Moreover, the hologram blazing can only optimize a single transformation at the time (\emph{i.e.}, the output of a single diffraction order), while often quantum measurements require simultaneous multiple-channel detection.

2'3. \emph{OAM generation via cylindrical lenses.} – Another class of methods for preparing light modes carrying nonzero OAM is that based on astigmatic optical systems. The prototypical example is that based on cylindrical-lens mode converters [9]. By combining a pair of identical cylindrical lenses of focal length \( f \), separated by \( \sqrt{2}f \) (\( \pi/2 \)-converter), an
Fig. 7. Example of helical mode (a) and petal mode (b) having $\ell = 3$. The petal mode is obtained as a balanced superposition of two opposite-$\ell$ helical modes. The relative phases of the superposition coefficients will fix the petal azimuthal orientation.

$\text{LG}_{p\ell}$ mode can be obtained starting from a Hermite-Gaussian (HG) mode with Cartesian indices $m, n$ (HG$_{mn}$), sharing the same beam waist, provided that the input HG$_{mn}$ mode axes are aligned at 45° with respect to the axes of the lenses. The relation between the input Cartesian indices and the output polar ones is the following: $\ell = m - n$ and $p = \min(m, n)$ (see fig. 6). The operation of this mode converter is based on manipulating the Gouy phase and has been fully described by several authors [9,75]. Unlike the spiral phase plate and the holographic converter, this method can, in principle, produce pure LG modes, but requires higher-order HG modes at input, whereas the former methods introduce an azimuthal phase dependence into a standard Gaussian mode beam at input. An advantage of cylindrical-lens mode converters over holograms is that the optical efficiency of conversion is much higher, limited only by the quality of the antireflection coatings of the lenses.

Another remarkable cylindrical lens converter is again formed by a pair of identical cylindrical lenses of focal length $f$, but this time separated by $2f$. Such converter transforms any mode into its own mirror image and is optically equivalent to a Dove prism [76]. This converter is also named $\pi$-converter [9], since it implements for an OAM state an action that is analogous to that implemented by a half-waveplate for polarization. It is worth underlining that an ideal $\pi$-converter exists only in the geometrical optical limit. In a real diffraction-limited system the phase conversion can be only $\pi - \epsilon$, with $\epsilon$ arbitrary small parameter depending on the scale size of both the system and the beam. Similarly, the reason why the HG-LG converter is named $\pi/2$-converter is related to the idea that this device is the OAM counterpart of the SAM quarter-waveplate. However, this is not generally the case if we construct the OAM two-dimensional space of modes as the set of all linear combinations of two opposite OAM eigenstates (that is, the so-called OAM-Poincaré sphere [77]). Indeed, in order to transfer to the OAM space the map representing, in the SAM space, the action of a quarter-waveplate, one should translate in terms of OAM the capability of a quarter-waveplate to transform a linear polarization aligned at 45° with respect to the axis of the waveplate into a circular polarization and vice versa. Therefore the most appropriate OAM counterpart of a $\pi/2$-converter would be a device that transforms, irrespective of any radial dependence, a helical mode $\ell$ into a...
2ℓ-petal mode (fig. 7), i.e. a balanced superposition of two opposite helical modes, ℓ and −ℓ, both sharing the same radial dependence and having equal-modulus superposition coefficients. The π/2-cylindrical mode converter has this effect only for ℓ = 1, but not for higher ℓ values. A possible setup to implement such quarter-wave-like OAM operation in a more general way will be presented later (subsect. 2.7).

We finally notice that cylindrical lenses are useful not only to realize mode converters able to generate helical modes starting from higher-order HG modes, but also to change the OAM distribution of arbitrary optical fields. A couple of cylindrical lenses having their axes rotated by an angle different from 0 or π can be used to convert the fundamental Gaussian mode into a nonzero OAM average superpositions of LG modes [78]. The change in the mean OAM per photon of an astigmatic Gaussian beam, with waists $w_x$ and $w_y$, on passing through a cylindrical lens aligned at an angle $\alpha$ with respect to the $x$-axis, is given by [79]

$$\delta\langle L_z \rangle = \frac{\hbar k}{f} \frac{w_x^2 - w_y^2}{4} \sin 2\alpha.$$  

2.4. SAM-to-OAM conversion devices. – In 2006, Marrucci et al. realized that photons crossing anisotropic inhomogeneous media, such as liquid crystals, can suffer a variation of SAM, due to the medium’s birefringence, that causes the simultaneous appearance of OAM, as a consequence of the medium’s inhomogeneity [26]. In a liquid crystal cell, prescribed inhomogeneities can be easily obtained by suitable treatment of the internal cell surfaces, so as to impose inhomogeneous boundary conditions for the liquid crystalline molecular alignment. Some features of the generated OAM state, for example the handedness, can be controlled by the input SAM, that is the input polarization state. In particular, in a specific geometry characterized by a full cylindrical symmetry, the SAM variation per photon imposed by the birefringence is entirely transferred into the OAM variation (that is in the generated OAM, when starting with a zero OAM mode), so that the process can be described as an all-optical SAM-to-OAM conversion (STOC), with the medium only playing an intermediate role. In the other geometries the medium picks up (or gives up) some angular momentum as well. However, we shall use the STOC acronym to denote this method of generating OAM also in this more general situation. The STOC process can be interpreted as resulting from the introduction of a space-variant Pancharatnam-Berry phase acting through a local change of the beam polarization over the transverse wavefront of the radiation field [80-82].

In order to practically implement the STOC process, topologically charged birefringent waveplates, named $q$-plates, have been used [26,83]. A $q$-plate is realized as a slab of a birefringent material, such as a liquid crystal, having a uniform birefringent phase retardation $\delta$ across the slab thickness and a space-variant transverse optical axis distribution exhibiting a topological charge $q$. The charge $q$ represents the number of rotations of the local optical axis in a path circling once around the center of the plate, where a topological defect must be present. The sign of $q$ may be positive or negative depending on whether the spinning direction of the axis is the same as the path orientation or not. The modulus of $q$ can in general be any real number, which can be set within an accuracy limited only by the precision in manufacturing methods. However, as we shall see in more detail below, integer and semi-integer values of $q$ are special in the fact that there will be no discontinuity line in the plate, but only the central point defect, and as a consequence the resulting $q$-plates are useful to access individual OAM eigenmodes or their finite superpositions.
In the simplified limit in which the q-plate is ideally thin, transverse diffraction effects arising from propagation inside the device can be neglected (such propagation effects have been discussed in ref. [84], although only within an approximate treatment), so that the q-plate acts as an ideal phase optical element. In this approximation, the birefringence-induced Pancharatnam-Berry phase can be derived by using a simple Jones matrix approach \[26,83,85,86\]. The Jones matrix representing the transformation of an optical field propagating along the z-axis, relative to the circular basis \(\{|L\rangle_\pi, |R\rangle_\pi\}\) of polarization space \(\pi\), is

\[
Q_P = \cos \frac{\delta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin \frac{\delta}{2} \begin{pmatrix} 0 & e^{-i2q\phi-i2\alpha_0} \\ e^{i2q\phi+i2\alpha_0} & 0 \end{pmatrix},
\]

where \(\alpha_0\) is the angle formed by the q-plate optical axis at \(\phi = 0\), that is on the x-axis, with respect to the orientation of the same x-axis. \(\alpha_0\) together with the topological charge \(q\) fully define the q-plate pattern (fig. 8). As evident from eq. (14), in order to convert the fundamental Gaussian mode into a helical mode with a well-defined value \(\hbar \ell\) of the OAM per photon, \(q\) must be integer or semi-integer. It can be semi-integer because the optical axis has no polarity, so it is physically equivalent to its opposite. In principle, however, in manufacturing a q-plate, the modulus of \(q\) can be fixed to any real number, except that a discontinuity defect line will be generated along some radial line in the q-plate plane. In this case, the optical field emerging from a q-plate having a real \(q\) is also not continuous over the transverse plane and it corresponds to an infinite superposition of helical modes.

For \(\delta = \pi\), \(i.e., \) for optimal conversion tuning, a q-plate changes the OAM state \(\ell\) of a circularly polarized input light beam by the amount \(\pm 2q\), the actual sign depending on the input polarization, positive for left-circular and negative for right-circular. The handedness of the output circular polarization is also inverted, \(i.e., \) the SAM is flipped. In other words the medium is always transferred the maximum SAM available, \(i.e., \) \(\pm 2\hbar\). The medium, in turn, transfers part of this change and its own reaction to the OAM degree of freedom of the output beam. In particular, in the rotationally symmetric case \(q = 1\), the OAM variation balances the SAM one, as already mentioned, so that the total angular momentum of the light beam is conserved. This behaviour is pictorially illustrated in figs. 9 and 10. An untuned q-plate with \(\delta \neq \pi\) will give rise to a superposition of a wave transformed just as for the tuned case and an unmodified wave, with amplitudes respectively given by \(\sin \delta/2\) and \(\cos \delta/2\). A tuned q-plate allows one to generate a pure helical
beam carrying nonzero OAM (with $\ell = \pm 2q$) starting with a circularly polarized Gaussian input mode (having $\ell = 0$), with very high efficiency (ideally close to 100%), no deflection of the propagation axis and with a polarization-controlled handedness [84,87,88].

The STOC process has also been demonstrated for single photons and correlated photon pairs [89], demonstrating that also coherent superpositions of opposite SAM/OAM states can be transferred, a result that turns out to be of great relevance for quantum applications of the photon OAM, as it will be discussed in detail in sect. 6.

Summarizing, in general the action of a $q$-plate consists in converting photon SAM into photon OAM and vice versa, and in transposing the quantum information from the spin to the OAM degree of freedom of photons and vice versa. Specifically, a $q$-
plate modifies the angular momentum state of an incident photon by introducing a finite probability, depending on the retardation $\delta$, of finding each output photon in a polarization state with inverted helicity and OAM quantum number $\ell$ increased or decreased by the amount $\Delta \ell = 2q$, for a positive or negative initial helicity, respectively. Besides the topological charge $q$, another key parameter of a $q$-plate is the retardation $\delta$ between the ordinary and extraordinary waves crossing it, allowing to regulate the probability of switching between $\ell$ and $\ell \pm 2q$, i.e. the STOC efficiency for any given wavelength of the input photons. STOC efficiencies exceeding 95% (not considering reflection losses) have been demonstrated on the grounds of an accurate tuning of the retardation $\delta$, as achieved by temperature control [88] or, more conveniently, by an electrical control of the effective birefringence [87]. $q$-plates can therefore provide a very convenient approach to generating OAM beams, which in several applications can present advantages over computer-generated holograms and spatial light modulators. The polarization control of the OAM sign allows high-speed switching with rates that in principle can reach GHz values [83].

In 2009, Brasselet et al. [90] have experimentally demonstrated the possibility of achieving the STOC process through an optically trapped radial nematic liquid crystal droplet of a few micron in diameter, acting as a miniaturized $q$-plate, able to generate optical vortex for both mono- and polychromatic light (fig. 11). However, as the droplet birefringence is not uniform (and is not easily tunable), this STOC was not complete, so that the input circular polarization must be filtered out in order to isolate the emerging vortex beam.

Inhomogeneous birefringent media such as $q$-plates are not the only systems in which STOC can take place. An inhomogeneous dichroic medium, such as a space-variant polarizer with a space-variant dichroic axis geometry, can give rise to similar phenomena (with the advantage of an achromatic response and the disadvantage of significant optical
losses) [91]. An electro-optical device allowing a polarization-controlled OAM manipulation quite similar to the $q$-plate one, based on a pair of opposite spiral phase plates having electrically controlled refractive index, has been theoretically proposed few years ago [92].

Another quite different optical system that gives rise to a pure STOC phenomenon is that based on back-reflection in a conical mirror [93,94]. In this system, a circularly polarized input wave undergoes a double 45° reflection and ends up being retroreflected. For ideal metallic reflections, the final circular polarization after the double reflection is inverted twice and hence remains the same (in terms of helicity), which means that SAM per photon has finally flipped its orientation. The SAM variation cannot be transferred to the mirror, owing to its cylindrical symmetry, just as for the $q$-plate with $q = 1$, and hence it must remain in the light, being turned into OAM. Therefore, the outgoing wave acquires a helical phase as shown in fig. 12.

A STOC phenomenon bearing many similarities to that taking place in a $q$-plate with $q = 1$ but not relying on inhomogeneity of the material system may also occur in a (homogeneous) uniaxial birefringent crystal, when an optical beam propagates along the optical axis of the crystal. This was first proved theoretically by Ciattoni et al. [95,96] and then experimentally by Fadeyeva et al. [97]. Recently Brasselet et al. have demonstrated the possibility of nonlinearly inducing a topological charge within the texture of nematic liquid crystals [98, 99]. A similar phenomenon has been shown to occur in a biaxial crystal by internal conical diffraction [100, 101]. Unlike the case of $q$-plates, however, this approach is limited to generating OAM $\ell = \pm 2$, due to the rotational symmetry of the medium (and $\ell = \pm 1$ in the case of biaxial media, where the rotational symmetry

---

Fig. 12. – (a) Michelson interferometer setup for observing the spiral interference pattern of the generated helical mode. (b) Intensity distribution of the reflected wave from the conical reflector. (c), (d) Interference patterns generated by the reflected beam from the conical reflector and a quasi-spherical reference wave. The area enclosed by the green dashed line corresponds to the beam diffracted from the recess at the bottom of the conical reflector. Such tiny recess originates from the grinding process of the fabrication of the conical reflector shown in the inset (e). (Reprinted with permission from ref. [94]. Copyright 2012, Optical Society of America.)
is broken in a specific way). Moreover, the conversion efficiency in the paraxial limit cannot be higher than 50%.

Another interesting situation in which a form of STOC takes place is when an initially paraxial circularly polarized beam passes through a short-focal-length lens. The resulting strongly focused nonparaxial beam exhibits an OAM content, as demonstrated experimentally by particle manipulation experiments [22,102]. In this case, however, the OAM per photon remains small and its effects are clearly visible only close to the beam focus. The possibility of an electro-optical modulation of this effect has also been reported [103]. Another recent work showed that optical beams having a radially varying SAM also acquire a mechanical effect that acts like an OAM-like angular momentum [104], although it has been later shown that this angular momentum is still purely SAM in nature [50]. Moreover, the interaction of SAM and external OAM, that is at the basis of the so-called optical spin Hall effect, has also been recently conceived and experimentally demonstrated [105,106]. Related spin-orbit optical phenomena are the polarization “geometrodynamics” [107,108] and the polarization-based optical sensing of nano-particle displacements [44]. It should be furthermore mentioned that several works in the field of singular optics [109], that is strictly related with that of OAM, have recently tackled issues concerning the interaction between polarization and wavefront structures in the optical field (see, e.g., [110-112]).

In the following three subsections, we will consider in some details some specific aspects and applications of the $q$-plate technology.

2.5. Liquid crystal $q$-plate manufacture and control of the SAM-to-OAM conversion efficiency. – As above mentioned, liquid-crystal-based $q$-plates are characterized by a topologically charged distribution of the local optical axis. Liquid crystals (LC) are soft birefringent materials allowing for a flexible spatial patterning of the average molecular orientation defining the optical axis. The molecular alignment across the bulk of an LC film can be set essentially in two ways: by means of external static electric or magnetic fields, directly acting on the bulk molecular alignment, or by means of so-called “surface anchoring”, i.e. a treatment of the bounding substrates that fixes the alignment of the LC molecules directly in contact with the surface. In surface anchoring, the boundary alignment is then transmitted to the bulk via elastic orientational forces internal to LCs. The latter method is the most widely adopted to set the equilibrium configuration of LC devices, while the field control is used for dynamical control and switching. Typical $q$-plates have been manufactured as thin (order of 5 μm) LC films, sandwiched between two glass substrates that have been previously coated with a suitable alignment layer, typically made of polymer, such as polyimide, or polymer-dispersed azo-dye molecules. These materials, properly cured, are suitable for planar anchoring, i.e. for aligning the LC optical axis parallel (or slightly tilted) to the bounding surfaces. In order to create a preferential alignment direction within the anchoring plane, essentially two methods are available, specifically the mechanical rubbing (using velvet or other fabrics) of the polymer-coated substrate and the photoinduced alignment of the molecules forming the coating layer. Though largely exploited in manufacturing LC-based displays, mechanical rubbing is clearly not the most suitable method to introduce an arbitrary pattern in $q$-plate. This approach has turned to be convenient only in the case of the simplest geometry, corresponding to $q = 1$, which is rotationally symmetric. To overcome this restriction and produce $q$-plates with arbitrary values of the topological charge $q$, the photoinduced alignment has proven to be very powerful and effective [113] and lends itself to realizing optical axis distributions that, in principle, are arbitrary functions of
both radial and azimuthal coordinates [114]. In this approach, the anisotropy of the coating polymers or azo-dye-polymers is controlled by the linear polarization of the “writing light”, which defines the material optical axis (either parallel or perpendicular to the writing field polarization). The orienting effects of light on polyimide coatings is generally photochemical, i.e. based on selectively destroying or creating chemical links by preferential absorption. Conversely, the orienting effects of light on diluted solutions of azo-dye-polymers is mainly photophysical, i.e. based on the photoinduced selective

Fig. 13. – (a) A $q = 1$ $q$-plate prepared by a photoalignment technique, as seen between crossed polarizers. (b), (c) Interference patterns of the outgoing beam from the $q$-plate with (b) planar and (c) spherical reference waves, for a left-circular input polarization and Gaussian input. (Reprinted with permission from ref. [27]. Copyright 2011, IOP publishing.)

Fig. 14. – Time decay of the power $P_c$ of the STOC converted components of the output beam for a 20 $\mu$m thick $q$-plate (a) and a 2 $\mu$m thick $q$-plate (b) after the voltage switch-off, relative to the total power $P_0$. The time scale in panel (a) is slower by a factor of 100 with respect to panel (b). (Reprinted with permission from ref. [87]. Copyright 2010, AIP Publishing LLC.)
Fig. 15. – Measured contrast ratio $\rho(v) = \cos(\delta(v))$ of the STOC process as a function of applied AC voltage amplitude $v$, for a 20 $\mu$m thick $\varphi$-plate (a) and for a 2 $\mu$m thick $\varphi$-plate (b). The continuous lines give the theoretical behavior. (Reprinted with permission from ref. [87]. Copyright 2010, AIP Publishing LLC.)

reorientation of dye molecules dispersed in the polymer. One can use this approach to directly write an anisotropic pattern in a thin polymer film that becomes itself a $\varphi$-plate, as for example recently reported in [115]. However, polymer $\varphi$-plates are not dynamically tunable, as their birefringent retardation $\delta$ is fixed by the film thickness and by the polymer degree of alignment and corresponding birefringence. In fig. 13, an LC $\varphi$-plate manufactured by the photoalignment method is shown as seen between crossed polarizers, together with the interference patterns demonstrating the helical structure of the outgoing wavefront. The tuning of an LC $\varphi$-plate, that is controlling the birefringence phase retardation $\delta$, useful for optimizing the STOC process or to adjust it for different wavelengths, can be achieved by different methods, including mechanical pressure, thermal methods, and external-field-induced LC reorientation. No doubt the most versatile and effective method to control the $\varphi$-plate retardation has proved to be the application of an external low-frequency electric field, which enables to exploit the same electro-optics effects the common LCD are based on [87]. This method is practical since the applications of an electric field to the cell is easily implemented by assembling the cell with indium tin oxide (ITO) coated glass substrates electrically connected to a commercial AC voltage generator. Electro-optical control, besides, allows for a relatively fast dynamical control of the retardation (fig. 14). Since the STOC process is accompanied by polarization helicity inversion, in the case of a pure circularly polarized input beam the STOC and non-STOC components of the output light can be simply separated by a
quarter-waveplate followed by a polarizing beam-splitter (PBS), because the converted and nonconverted light will have orthogonal polarization states. This allows for a very simple measurement of the STOC efficiency and of the phase retardation $\delta$ that controls it, as shown for example in fig. 15. In the experiment reported in ref. [87], the optimal STOC efficiency exceeded 99%, neglecting reflection and diffraction losses (85% taking into account losses, which, however, were not minimized with anti-reflection coatings).

2.6. Azimuthal mode control based on SAM-to-OAM conversion. – The $q$-plate and the resulting SAM-OAM coupling and STOC process have provided the basis for several novel optical devices and setups for manipulating the OAM of light. Many of these devices, although they are essentially classical, find a natural application in the quantum information setting, as reported in sect. 6. The basic idea underlying these applications consists in the possibility of exploiting STOC to establish a one-to-one mapping between the SAM space $\pi$ and the OAM space $\alpha_\ell$ spanned by the helical modes $|\ell = 2q\rangle$ and $|-\ell = -2q\rangle$, defined independently of the radial mode (this is for example appropriate if the radial mode is separable and can be factorized). The SAM (or polarization) space $\pi$ is intrinsically bidimensional, since any polarization state can be obtained as a linear combination of the two opposite circularly polarized states $|L\rangle = |s = 1\rangle$ and $|R\rangle = |s = -1\rangle$, whereas $\alpha_\ell$ is merely a bidimensional subfolder of the infinite-dimensional OAM space. Both $\pi$ and $\alpha_\ell$ spaces are two-dimensional complex vector spaces having the same structure as the Hilbert space of a quantum spin, or as the representations of the $SU(2)$ group. Neglecting a global phase, any point in such a space can be represented as a point on a three-dimensional sphere, the Poincarè sphere in the case of polarization, or an analogous sphere in the OAM case [77] (both being analogous to Bloch’s sphere used for quantum spins). An arbitrary state in an OAM $\alpha_\ell$ subspace is generally not a pure helical beam, but it can always be written as the superposition of the two opposite OAM basis states. This is just the same as for the polarizations, which in general are not associated with well-defined values of the photon SAM, but can always be decomposed into a superposition of the two circularly polarized waves with opposite handedness.

To see how such a map can be physically implemented by exploiting a $q$-plate, consider its action (eq. (14)) on a helical mode with helicity $\ell$ in an arbitrary polarization state, $|\psi\rangle = \alpha|L, \ell\rangle + \beta|R, \ell\rangle$. In general, the state $|\psi\rangle$ is mapped into the superposition state $|\psi'\rangle$ given by

\[
|\psi'(\delta)\rangle = \sin \frac{\delta}{2} (\alpha|R, \ell + 2q\rangle + \beta|L, \ell - 2q\rangle) + \cos \frac{\delta}{2} |\psi\rangle,
\]

where $\delta$ is the $q$-plate retardation. Even for $\ell = 0$, such linear mapping does not work as expected, since it sends the space $\pi$ to a subspace of the OAM larger than $\alpha_{2q}$, due to the presence of the input $|\psi\rangle$ in the output $|\psi'\rangle$. This is due to the fact that, for $\delta \neq \pi$, the STOC process is not complete and both converted and unconverted states occur. Therefore a necessary condition to perform the above-mentioned mapping is the optimal tuning of the $q$-plate, i.e. $\delta = \pi$. From eq. (15), it is also evident that, even in the case of optimal tuning, in the output state $|\psi'\rangle$, SAM and OAM degrees of freedom turn out to be entangled. In order to make them disentangled and obtain, as overall effect, the transfer of the coefficients $\alpha$ and $\beta$ from the polarization states $|L\rangle$ and $|R\rangle$ to the helical modes $|2q\rangle$ and $|-2q\rangle$, the simplest method consists in linearly polarizing the output state $|\psi'(\delta = \pi)\rangle$. Notice that, for $\ell = 0$, the radial part of the $q$-plate output mode depends only on the modulus of $2q$ [84], and then is the same for both $2q$ and $-2q$. 


as required. This implementation choice of the mapping is simple and practical and it has been exploited in several works [116]. However, the polarizer introduced into the scheme to disentangle SAM and OAM makes the transformation nonunitary, with a 50% efficiency (or success probability for the photon). The opposite transfer, from OAM to SAM, can be obtained by combining a q-plate with a single-mode fiber, aimed at filtering \( \ell = 0 \) states [116]. This is again a scheme with 50% efficiency.

A 100% efficient mapping, that is a unitary scheme, can be obtained by combining a q-plate and one or two Dove prisms inserted into an interferometer, such as a Mach-Zehnder or (more conveniently) a Sagnac [116,117]. This scheme, illustrated in fig. 16, is fully reciprocal and can therefore work in both directions (see also [118] for another proposed optical scheme, not based on the q-plate, in which the OAM state is controlled by polarization via a Sagnac interferometer). This scheme was demonstrated experimentally in both the classical regime [117] and the quantum regime [119]. An additional feature of this setup is that also the geometrical Pancharatnam-Berry phase arising from the polarization manipulations is transferred to the OAM output beam. Examples of the resulting modes on the OAM-Poincaré sphere for \( \ell = 2 \) are given in fig. 17. The interference fringes with a reference beam, also shown, were used to analyze the output mode phase structure and to measure the geometric phase. It must be emphasized that this setup can generate, with a theoretical efficiency of 100%, a class of azimuthal transverse modes that include all the modes having the same azimuthal structure as the Hermite-Gaussian modes. The choice of the generated mode is entirely controlled by the input polarization, which can be manipulated at very fast rates. This should be contrasted with the more limited efficiency of the spatial light modulators (typically below 70%) and slow response (less than 1 kHz). The possibility of controlling even a four-dimensional OAM subspace, including both \( \ell = \pm 2 \) and \( \pm 4 \) states, by a single q-plate inserted in an optical loop scheme has also been reported recently [120].

A different optical scheme, still based on the q-plate, can be used as a spin-orbit SAM-OAM four-dimensional mode sorter and detector [88]. The four-dimensional space is the tensor product of the polarization space \( \sigma \) and an OAM subspace \( \alpha \). In other words, arbitrary optical states are defined as linear combinations of four basis states

![Fig. 16. – Scheme of an experimental setup based on a q-plate and a Dove prism in a polarizing Sagnac interferometer, that allows for 100% efficient transfer of an arbitrary SAM-encoded input state into an OAM bi-dimensional state, or vice versa. Legend: QHQH: set of waveplates to generate an arbitrary polarization state; PR: polarization rotator; PBS: polarizing beamsplitter; DP: Dove prism; M: mirror. (Reprinted with permission from ref. [117]. Copyright 2010, America Physical Society.)](image-url)
Fig. 17. – A possible closed path over the OAM-Poincaré sphere. The path starts and ends at the pole. (a) Intensity profiles of the generated beam at different points of the path. (b) Corresponding interference patterns with a TEM$_{00}$ linearly polarized reference beam. (Reprinted with permission from ref. [117]. Copyright 2010, America Physical Society.)

$|L, \ell\rangle$, $|L, -\ell\rangle$, $|R, \ell\rangle$ and $|R, -\ell\rangle$. The sorting is based on two steps. First, there is a $q$-plate-induced shifting of the OAM value of the beam. Assuming that the input beam has an OAM given by the number $\ell$ (in units of $\hbar$), the $q$-plate will convert it into either $\ell' = 0$ or $\ell' = 2\ell$, depending on the input polarization handedness. This requires using a $q$-plate with $q = \ell/2$ (in the reported experiment, $q = 1$ and $\ell = 2$ were used). Next, the beam is split according to the outgoing circular polarization and further separated by radial sectioning, e.g. by using a mirror with a hole to reflect only the external doughnut component and let the central spot pass. This radial sectioning exploits the coupling between the OAM and the radial profile that emerges during free propagation. The small residual overlap of the two radial modes, however, gives rise to a nonperfect contrast ratio. The contrast ratio can be improved without limitations at the expense of detection efficiency by blocking the annular region where mode overlap occurs. Alternatively, the contrast ratio can be improved without losing efficiency by working with a higher value of $\ell$, that leads to smaller radial overlaps.

Another related device has been proposed theoretically to perform arbitrary unitary transformations in the spin-orbit four-dimensional space $\pi \otimes o_\ell$, while remaining within a single-beam geometry (i.e., avoiding any beam splitting within an interferometer scheme) [121]. This device is based on a complex combination of $q$-plates, birefringent waveplates and lenses, and it again exploits the coupling between the radial mode and the OAM arising in the free propagation. This setup was proposed mainly with quantum applications in mind, as it provides a universal quantum gate for the spin-orbit Hilbert space of a single photon. Being a single-photon device, however, it can be also discussed as a classical-optics device. The working principle of this device is similar to that of the spin-orbit mode sorter just described, as it exploits the spatial separation of the OAM 0 and $2\ell$ modes occurring in the radial coordinate in the free propagation.

An important element is the so-called “$q$-box” (QB), which is made of two $q$-plates and a unitary polarization gate sandwiched between them (which is essentially a combination of suitable waveplates and isotropic phase retardation plates). The radius of the waveplates of the spin gate is selected so to act only on the OAM 0 mode, leaving the $2\ell$ one unchanged. The propagation-induced coupling between the OAM and the radial coordinate is controlled by suitable lenses to switch between near-field and far-field and
back. A schematic illustration of the $q$-box device is given in fig. 18. A sequence of four $q$-boxes separated by quarter- and half-waveplates (QWP and HWP) in the following order: QB $\rightarrow$ QWP $\rightarrow$ QB $\rightarrow$ HWP $\rightarrow$ QB $\rightarrow$ QWP $\rightarrow$ QB will make a 16-parameter unitary gate that will correspond to a $4 \times 4$ unitary matrix which is universal, meaning that by adjusting the parameters one can realize any unitary operation on the spin-orbit optical state. Such a highly complex setup, however, is not always necessary. Many important gates can be realized with many fewer elements. For example, the CNOT gate can be realized with a single $q$-box having a single half-waveplate inside. Because of the presence of residual overlaps of 0 and $2\ell$ modes in the radial coordinate, the fidelity of the $q$-box is not 100%. An optimal selection of the radius of the waveplates in the $q$-box can provide a minimum fidelity of about 83% for $\ell = 2$ (but the fidelity increases if higher-order OAM modes are considered and it can become close to 100% for specific input-output states). As for the mode sorter, the fidelity can be increased at the expense of efficiency (success probability in the quantum applications) by stopping the annular regions where the overlap takes place. The efficiency can be however recovered by working with higher $\ell$ values.

2.7. $q$-plate-like devices for direct conversion of helical modes into petal modes. – Despite the recent progress in the manipulation of OAM light states and in their coupling with the SAM degree of freedom, the possibility of achieving full control of a two-dimensional quantum state defined in a subspace of OAM remains challenging. In fact, to be able to perform an arbitrary path on the Poincaré sphere, both a $\pi$ and a $\pi/2$ phase-retarders are required [122]. In the SAM space, such tools are provided by half- and quarter-wave birefringent retardation plates, respectively. In the OAM space, as mentioned in subsect. 2.3, $\pi$ phase retarders changing $\ell$ into $-\ell$ are already available and handy to be used, e.g. Dove prisms and cylindrical-lens $\pi$-converters [9]. In contrast, convenient OAM $\pi/2$ phase retarders are still missing, except for the case $\ell = 1$, for which a $\pi/2$-cylindrical lens mode converter is appropriate. According to the definition
provided in subsect. 2.3, in fact, an ideal OAM $\pi/2$-converter is one that transforms, irrespective of any radial dependence, a helical mode $\ell$ into a superposition of two helical modes, $\ell$ and $-\ell$, both sharing the same radial dependence and having equal-modulus superposition coefficients. In the light of the methods described in the previous section, a currently feasible—though rather complex—scheme to this purpose may exploit the $q$-plate-based SAM-to-OAM (STO) state transferrers [117,119], which can convert, with theoretical 100% efficiency, any polarization state $\alpha |L\rangle + \beta |R\rangle$ into a corresponding OAM state $\alpha |\ell\rangle + \beta |-\ell\rangle$ and vice versa. Such devices perform the transformation in a deterministic way and are realized through a combination of a $q$-plate, which interfaces OAM and SAM spaces of the photon, and a polarizing Sagnac interferometer with a Dove prism (PSID) [123]. Therefore, in order to implement the desired OAM state transformation, a transferrer should be first used to transfer the OAM state into the SAM space. Then, a quarter-waveplate induces the desired $\pi/2$ phase transformation and, finally, a second transferrer is used to return to the OAM space. This approach is shown schematically in fig. 19. The same scheme can be used to give rise to any other unitary transformation in the OAM space, as the quarter-waveplate can be replaced by a suitable set of wave plates (e.g., a half-waveplate, sandwiched between two quarter-waveplates) that together, can perform any unitary transformation in the SAM Hilbert space [122]. Recently, it was introduced a liquid-crystal Pancharatnam-Berry optical element (PBOE)—a $q$-plate like device—that is able to change in a single pass the OAM state of the beam from a helical mode $|\ell\rangle$ into a superposition state $|\theta\ell\rangle = 1/\sqrt{2} (|\ell\rangle + e^{i\theta} |-\ell\rangle)$ (petal mode rotated by $\theta$) and vice versa, so as to partly mimic the behavior of a $\pi/2$ phase transformation in OAM space. It belongs to a larger class of PBOEs able to convert a specific $\ell$ helical mode (e.g., a fundamental Gaussian input) directly into a petal mode $|\theta\ell\rangle$, $\ell$ and $\ell'$ being not necessary equal. A device of this kind is realized as a birefringent waveplate of uniform phase retardation $\delta$, whose angle $\alpha$ between the slow optical axis and the $x$-axis of the
fixed laboratory reference frame is not constant, but is described by a prescribed function
\( \alpha = f(r, \phi) \), where \( r \) and \( \phi \) are the polar coordinates in the waveplate transverse plane. It can be easily demonstrated with the Jones matrix approach, that the corresponding transformation matrix in the circular polarization basis \( \{|L\rangle_\pi, |R\rangle_\pi\} \) is given by

\[
T(r, \phi) = \cos(\delta/2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(\delta/2) \begin{pmatrix} 0 & e^{2if(r,\phi)} \\ e^{2if(r,\phi)} & 0 \end{pmatrix}.
\]

In other words, a fraction of the incident circularly polarized light undergoes a helicity inversion and gains an additional phase factor of \( \pm 2f(r, \phi) \), where the sign depends on the input polarization helicity. As for the \( q \)-plate, this additional phase factor is not due to refractive index change or nonuniform thickness of the optical element, but to the geometrical phase arising from the space-variant polarization manipulation. The efficiency of the beam conversion depends on the phase retardation \( \delta \) and is maximum for the half-wave \( \delta = \pi \) condition. To calculate the pattern \( f(r, \phi) \) needed to perform a desired state transformation, one should use the phase difference between the output and input beam states. To transform the helical mode \( |\ell_1\rangle \) into a petal mode \( |\theta \ell\rangle \) (with \( \theta = 0 \)) and vice versa, the corresponding pattern is given by

\[
f_\ell(\phi, \ell, \ell_1) = \frac{1}{2}(-\ell_1 \phi + \arg(e^{i\ell\phi} + e^{-i\ell\phi})).
\]

Setting \( \ell_1 = \pm \ell \) yields a \( \pi/2 \) mode converter (MC) device, while setting \( \ell_1 = 0 \) and arbitrary \( \ell \) yields a mode generator (MG) device that can generate a \( |\theta \ell\rangle \) state from the Gaussian \( \ell = 0 \) input directly. Setting \( \ell \neq \ell_1 \) or tuning the phase retardation of the device (and thus its conversion efficiency) allows more complex state manipulations even beyond the original bidimensional OAM subspace. Some examples of these optical axis patterns are given in fig. 20, together with images of corresponding fabricated samples.
3. Orbital angular momentum exchange between light and matter

3.1. OAM transfer to trapped particles and to soft matter. When the coupling between the light OAM and matter is very strong, matter can be set in motion by the OAM exchange with photons. The light propagation in turn is modified by the matter dynamics, thus generally leading to a complex highly nonlinear behavior of the system. The light-matter interaction must be described in detail by a suitable set of coupled nonlinear equations for light and matter. As already mentioned, the first demonstration of angular momentum transfer from light to matter was made by Beth [6], who measured with a torsion balance the torque exerted by circularly polarized light on a half-waveplate. In this case $2\hbar$ of SAM per photon are transferred, and this gives a constant mechanical torque acting on the waveplate. In general, photon SAM may be transferred either to isotropic materials, together with the photon energy, via absorption, or to birefringent transparent materials via an intrinsically nonlinear process named Self-Induced Stimulate Light Scattering (SISLS) [124, 125]. The SISLS was first introduced, in relation to SAM transfer, to explain the collective rotation of liquid crystal molecules in the field of a normally incident circularly polarized laser beam [15]. Beth’s experiment can be put into such a nonlinear framework and corresponds to the special case of half-wave retardation. SISLS was also recognized to be at the root of complex light-induced dynamical regimes [126, 127] and suitable for operating light-driven molecular motors [128]. Exploiting the same principle, manipulation of small transparent and birefringent particles trapped by optical tweezers was also achieved [16]. The first experiment realizing the orbital counterpart of the SISLS is much more recent [21] and puts into practice an idea coming from a theoretical work by Allen et al. [8], according to which the measurement of the mechanical torque arising from the orbital angular momentum was to be performed from Beth’s experiment mould. The orbital SISLS was also exploited to control the transverse orientation of small transparent isotropic particles in optical tweezers (fig. 21) [129, 130].

The photon OAM can be transferred together with SAM to absorbing media. In this case, photon SAM and OAM transfers are indistinguishable as can be inferred from the operation of the optical spanner reported in ref. [19]. The optical spanner or wrench [131] has been realized by adding a rotational degree of freedom to optical tweezers, which made use of tightly focused beams of light to trap microscopic particles in three dimensions within a surrounding fluid. The extra degree of freedom was provided by the angular momentum of photons that, under appropriate conditions, could be transferred to the trapped particles. In ref. [19], weakly absorbing dielectric particles were trapped by a tightly focused circularly polarized LG beam, having a transverse cross-section smaller than the micron-sized particles. The helical beam was $\ell = 1$ so that its total angular momentum could be set at $h + h = 2h$ or $h - h = 0$ switching by means of a half-waveplate the handedness of the spin from left to right and vice versa, so that the trapped particles could be set in rotation or stopped at will.

In most cases, the paraxial approximation applies, so that OAM and SAM are independent quantities when light propagates in or through a homogeneous and isotropic transparent medium. There is some experimental evidence, however, that SAM-to-OAM conversion occurs in such media when a Gaussian beam [132] or circularly polarized vortex beam [22] is tightly focused through a high-numerical-aperture (NA) lens. These experiments, however, do not actually enable to infer a violation of the interaction scheme just sketched, since they have been performed in nonparaxial regime. In this regime, in fact, SAM and OAM are not well defined (see sect. 1) and they cannot be independently
related to polarization or helical wavefronts. For arbitrary optical fields it is the total angular momentum the only observable actually involved in conservation laws. Since such observable is not related to directly measurable properties of the optical field (see sect. 1), the physical effects are still represented in terms of the two paraxial quantities SAM and OAM, but also SAM-OAM coupling must be considered. As already mentioned, a useful side-effect of the SAM-OAM interaction under very strong focusing is the possibility to confine the light in a sub-wavelength region [62,133,134] and to obtain in the focal region an optical electric field mainly polarized along the beam propagation axis [135,92,136].

Finally, we would like to mention here several outstanding processes of interaction between OAM-carrying light and matter, that could be used for direct laser micro- and nano-writing of both chiral and nonchiral structures, as controlled by the OAM beam structure. It has been recently reported on the optical vortex-induced mass transport in an azobenzene-containing polymer film [137]. When an azobenzene-containing polymer film is exposed to nonuniform illumination, a light-induced mass migration process may be induced, leading to the formation of relief patterns on the polymer-free surface. When the polymer film is illuminated by focused Laguerre-Gauss beams, spiral-shaped relief patterns appear on the polymer film (fig. 22). Such spiral reliefs turn out to be sensitive to the vortex topological charge and to the wavefront handedness, due to the surface-mediated interference of the longitudinal and transverse components of the optical field. Besides polymers, also metals can be used as light OAM-assisted writing materials. Optical vortices have been used to create chiral needle-shaped metal nanostructures via laser ablation [138]. The conical surface of the needle turns out to be twisted azimuthally
with the sign of the optical vortex helicity. Moreover, the tip curvature of these chiral nanoneedles was measured to be less than 40 nm, i.e. two orders of magnitude smaller than the laser wavelength (1064 nm). The process has been interpreted assuming that the focused vortex pulse used to ablate the metal transfers orbital angular momentum to the melted material and puts it into rotation about the annular intensity profile of the optical vortex, clockwise or counterclockwise depending on the sign of the optical helicity. The laser energy used in the experiments in ref. [138] was \( \lesssim 0.3 \) mJ. For higher energy the fabricated needles turn out to be nonchiral [139], but the height and tip diameter of the needles have shown to be still related to the angular momentum transferred from the light pulse to the melted material and/or to the laser-induced plasma. Specifically, when the signs of the OAM and SAM of the light pulses are concordant, the height and the tip diameter of the needles turn to be respectively larger and smaller than in the case the signs are discordant. Finally, doughnut beams have been recently exploited to experimentally implement the two-beam optical lithography based on the polymerization and photoinhibition strategy, irradiating a newly developed photoresin with large two-photon absorption cross-section, high mechanical strength and sufficient photoinhibition function [140]. The smallest fabricated feature size and the highest resolution presently available respectively are 9 nm (\( \lambda/42 \) for the wavelength of the inhibition beam) 52 nm (\( \lambda/7 \)) two-line resolution.
The peculiar features of OAM and SAM transfer mechanisms, however, go beyond the different nature of the isotropic particles they interact with. When a micron-sized birefringent particle, for instance, is placed off-axis in the field of a wide, circularly polarized, helically phased beam, a particle spinning around its own axis will be observed, as expected, owing to the SAM transfer, but also an orbital motion along a circular trajectory centered in the phase singularity will take place, due to the azimuthal component of the scattering force arising when the center of mass of the particle is displaced from the singularity center [141]. The orbital effect originates from the fact that the off-axis particle samples a portion of the wavefront that appears as approximately plane or not completely helically shaped. On an atomic scale, the differences in the dynamical effects produced by SAM and OAM are even more striking. It is well known that the selection rules for transition between Zeeman levels depend on the polarization state of the radiation field impinging on the atom, and consequently on the SAM transferred from the photon to the atom. OAM, by contrast, plays no role, since over the angstrom-scale diameter of an atom, any helical wavefront in the visible range is effectively perceived as an inclined plane wave and cannot affect the dynamics of electrons (except if the atom is located exactly on the vortex core, in which case the light intensity in the atom region will vanish almost perfectly, leading to no effect). As a consequence, the typical effect of a helical phase front on an atom originates directly from the radiation pressure locally exerted on the center of mass and results in a spiral-type motion of the atom itself [142]. This property can be very useful to implement atom manipulation techniques [143,144], based on the effective potential well located in the central dark spot of the intensity profile of a light beam carrying a well-defined nonzero OAM. In order to effectively transfer OAM to a medium, it is mandatory that the medium samples a relevant portion of the helical phase-front. This may occur, for example, in Bose-Einstein condensates [145-147].

Now, in order to get a deeper insight into the mechanism ruling over the OAM transfer to matter and the difference in its nature with respect to the SAM transfer, we are going to discuss the role played by SAM and OAM transfer in the optical reorientation of liquid crystals [148], which is a particular case of a giant Kerr effect. This case will be discussed in detail, since liquid crystals, to the best of our knowledge, are the only materials able to dynamically interact on a local scale with both OAM and SAM and therefore, in our opinion, might provide the most appropriate arena for investigating the fundamental issues of the radiation angular momentum transfer to matter.
3.2. OAM transfer in the optical reorientation of liquid crystals. – Liquid crystals are fluids composed of organic molecules with a specific, anisometric shape, with an arrangement exhibiting a long-range order that is however not complete, as for a crystalline solid. The most typical materials are rod-like molecules or rod-like molecular aggregates (fig. 23), which give rise to conventional nematic and smectic phases. Nematic liquid crystals (NLCs) are characterized by long-range orientational order only, with no long-range correlations in the molecular center-of-mass positions. The axes of the molecules locally share a common average direction—specified by a unit vector \( \mathbf{n} \) called molecular director—and the centers of mass of the molecules are randomly spread over space. Smectic liquid crystals (SLC), on the other hand, are characterized by both the orientational order and one-dimensional positional order, i.e. involving one Cartesian coordinate only (more exotic columnar phases may exhibit two-dimensional positional order).

The molecular orientational order of LCs results in the anisotropy of their mechanical, electrical, magnetic, and optical macroscopic properties [149,150]. In a sense, liquid crystalline phases, or “mesophases” as they are also called, combine the anisotropic properties of a solid crystal and the mechanical ones of a liquid, thus giving rise to very peculiar optical phenomena, having no counterparts in solids or normal (isotropic) liquids. In what follows we shall consider nematic mesophases only. They are birefringent and locally uniaxial, the director \( \mathbf{n} \) being coincident with the direction of the optical axis. Most of the specific liquid crystal optical effects must be traced back to the reorientation of the director in the macroscopic volume of the material under the influence of an external field or the flow of the liquid. Liquid crystals can be aligned also by suitable coatings on the sample walls. Isotropic coatings lead to molecular anchoring perpendicular to the wall (homeotropic anchoring). Anisotropic coatings with a preferred direction on the surface, such as for example rubbed polymers, typically lead to molecular anchoring parallel to the wall (planar anchoring). Due to elastic orientational interactions, the anchoring determined by the wall then extends its effect inside the bulk of the material. Usually, LC cells are prepared with given (homeotropic or planar) uniform orientation. Subsequently, they can be reoriented by flow or external fields, so as to obtain dynamical changes in their optical properties. The possibility of reorienting liquid crystals by applying static magnetic or electric fields has been known since long time [151]. The director \( \mathbf{n} \) reorients in a static electric (magnetic) field under the action of a dielectric (diamagnetic) torque, which is proportional to the dielectric (diamagnetic) anisotropy \( \Delta \varepsilon = \varepsilon_\parallel - \varepsilon_\perp \) (\( \Delta \chi = \chi_\parallel - \chi_\perp \)). The director \( \mathbf{n} \) tends to be aligned along the field, \( \mathbf{n} \parallel \mathbf{E} \), if \( \Delta \varepsilon > 0 \) and perpendicular to \( \mathbf{n} \), \( \mathbf{n} \perp \mathbf{E} \), if \( \Delta \varepsilon < 0 \). Analogous expressions hold for the case of magnetic field induced reorientation. Nematic liquid crystals may be reoriented also by the optical field of a laser beam. The reorientation with optical fields was discovered only in 1980 by Zel’dovich and Tabiryan [152] and by Zolot’ko et al. [153] and later quantitatively discussed by Durbin, Arakelian, and Shen in 1981 [154]. The laser-induced reorientation may be described in terms of an optical torque, in a similar way as its static counterpart. It represents the main contribution to the third-order nonlinearity of liquid crystals, and the effect appears in the form of a refractive index change \( \delta n \) depending on the laser intensity \( I \). Actually even with moderate laser intensities, the effect can be so large that \( \delta n \) appears as a complex nonlinear function of \( I \), including both nonlocality and time-retarded effects. It has been clarified that the molecular correlations and the ensuing collective reorientation are at the basis of the strong field-induced reorientation effect [155]. The electronic response certainly also contributes to the third-order nonlinearity of liquid crystals [156-158], but it is not expected to be significantly larger than that of other organic molecules with delocalized electrons.
NLCs are particularly interesting from the point of view of angular momentum transfer with light because they have both “orbital” and “spin” material degrees of freedom. In fact, these materials are fluids and birefringent simultaneously. In the NLC mesophases, in fact, each volume element \( dV \), located at a position \( \mathbf{r} \) at a time \( t \), is characterized not only by the mass density \( \rho \), as in conventional isotropic fluid, but also by the orientation of the molecular director \( \mathbf{n} \), representing, as above stated, the common average direction of the molecules contained inside the volume element \( dV \) (see fig. 23). NLCs, therefore, exhibit clearly distinguishable orbital and intrinsic degrees of freedom. The fluid motion is evidently related to the orbital part of the angular momentum, while the molecular reorientation refers to the rotation of the director \( \mathbf{n}(\mathbf{r},t) \) in the volume element \( dV \) at fixed position \( (\mathbf{r},t) \), which is associated with the intrinsic (spin) part of the angular momentum of the material. It will come out quite naturally that the average orientation \( \mathbf{n}(\mathbf{r},t) \) of the molecules contained in \( dV \) is directly affected by the spin part of the angular momentum of the optical field, while the motion of center of mass of the volume element \( dV \) is directly affected by the orbital part. The problem of the dynamics of nematic liquid crystals evidently intersects the problem of the separation of the angular momentum of light in its intrinsic (or spin) and orbital parts, especially beyond the paraxial optics approximation \([34,8,78,159,24,160,23,25]\). In fact, the separation of the rotational motions inside the liquid crystalline material and the separation of the torques they arise from has repercussions on the corresponding fluxes in the radiation incident from the outside. This makes liquid crystals a very good arena to unambiguously define the orbital and spin angular momentum \( \text{fluxes} \) also beyond the paraxial approximation by looking to their physical effect on matter.

The different effects of OAM and SAM when light interacts with a NLC fluid is made clear directly from the dynamical equations governing the fluid motion and the molecular reorientation, viz. \([14]\)

\[
\begin{align*}
\rho \dot{\mathbf{v}} &= \mathbf{f} = \text{div} \, \mathbf{\sigma} - \text{grad} \, p, \\
\rho \mathbf{r} \times \dot{\mathbf{v}} &= \mathbf{r} \times \mathbf{f} = \text{div} \, \mathbf{L} - \mathbf{w}, \\
I \mathbf{n} \times \ddot{\mathbf{n}} &= \mathbf{\tau} = \text{div} \, \mathbf{S} + \mathbf{w},
\end{align*}
\]

where \( \rho \) is the fluid density (assumed to be constant), \( \mathbf{v} \) is the velocity of the fluid (flow of the centers of mass of volume elements \( dV \)), \( \mathbf{f} \) is the force per unit volume acting on the fluid, \( p \) is the hydrostatic pressure, \( I \) is the momentum of inertia per unit volume associated to the rotation of \( \mathbf{n} \), \( \mathbf{\tau} \) is the torque density acting on \( \mathbf{n} \), and, finally \( \mathbf{\sigma} \) is the stress tensor and \( \mathbf{w} \) is the vector dual to its antisymmetric part, i.e. \( w_{\alpha} = \epsilon_{\alpha\beta\gamma} \sigma_{\beta\gamma} \).

The dot in eqs. (18)-(20) stands for the material derivative. The terms on the left in eqs. (18)-(20) may be unambiguously interpreted as the densities per unit time of linear momentum, orbital angular momentum, and intrinsic angular momentum of matter, respectively. The terms on the right of eqs. (18)-(20) can be consistently interpreted as the densities of force, orbital torque and intrinsic torque acting in the bulk of the medium. Such densities are represented by expressions that contain the divergence of tensors involving the external fields and then make evident their relationship with the fluxes of force and momentum from the outside. It is therefore quite natural to refer to the tensors \( \mathbf{L} \) and \( \mathbf{S} \) in eqs. (19) and (20) as the external orbital and the intrinsic (spin) angular momentum flux densities acting on the volume element, respectively. Because tensors \( \mathbf{\sigma} \), \( \mathbf{L} \) and \( \mathbf{S} \) enter the equations of motion eqs. (18)-(20) with divergences, we may add to
them any divergence-free tensor without changing the motion. As shown in ref. [14], this ambiguity is fully resolved by requiring that in the limit of isotropic medium ($\Delta \epsilon \to 0$) the electromagnetic parts $\hat{L}^{em}$ and $\hat{S}^{em}$ of $\hat{L}$ and $\hat{S}$ are both divergence-free, leading to separate conservation laws of the spin and orbital electromagnetic angular momentum. The explicit expressions of these tensors are reported in ref. [14]. Here we just repeat that they are not only both conserved in vacuum, but act on different degrees of freedom of matter (see fig. 24), so it is natural to identify such $\hat{L}^{em}$ and $\hat{S}^{em}$ as the spin and orbital fluxes carried by the radiation incident on the sample, respectively. This distinction holds beyond the paraxial approximation and is based on physical grounds, rather than formal properties (as the relationship with rotation generators), thus providing an attractive possible solution to the long-standing controversial ambiguity about the decomposition of the electromagnetic angular momentum in its spin and orbital parts [34, 159, 24]. In most experiments with liquid crystals, laser beams are used to drive the material and small deformations of the molecular alignment are induced. For small deformations of the molecular director, in the paraxial optics approximations, the explicit expression for the overall spin angular momentum transferred from the electromagnetic field to the liquid crystalline medium simplifies to [161]

$$\Delta S_z = -\frac{1}{\omega} \int \int dx dy I(x,y) \Delta s_3(x,y),$$

where $I(x,y)$ is the intensity profile of the beam, $\omega$ is the optical frequency, and $\Delta s_3$ is the change suffered by the reduced Stokes’ parameter $s_3 = 2 \text{Im}(E_x E_y^*)/(|E_x|^2 + |E_y|^2)$ in traversing the medium ($s_3 = \mp 1$ for left/right handed polarization, respectively, and $s_3 = 0$ for linear polarization). The integral is carried out across the $x, y$-plane orthogonal to the beam axis. In the same approximations, the explicit expression for the overall
orbital angular momentum transferred from the electromagnetic field to the medium, involved in eqs. (18), simplifies to [161]

\[ \Delta L_z = \frac{1}{\omega} \int \mathrm{d}x \mathrm{d}y I_e(x, y) (\mathbf{r} \times \nabla)_z \Delta \Psi_e(x, y) \]

\[ = -\frac{1}{\omega} \int \mathrm{d}x \mathrm{d}y \Delta \Psi_e(x, y) (\mathbf{r} \times \nabla)_z I_e(x, y), \]

where \( I_e(x, y) \) is the beam intensity transverse profile and \( \Delta \Psi_e(x, y) = \Psi_e(x, y, L) - \Psi_e(x, y, 0) \) is the phase change of the extraordinary wave given by \( \Delta \Psi_e(x, y) = \frac{2\pi}{\lambda} \int_0^L [n_e(\theta) - n_o] \mathrm{d}z \approx \tilde{L} (n_x^2 + n_y^2) \), where \( \tilde{L} \) is a characteristic length of the sample and \( n_x \) and \( n_y \) are the components of the molecular director in the plane transverse to the light propagation direction. The ordinary wave suffers a phase change which is uniform in the transverse plane, so that it does not contribute to \( \Delta L_z \). From eq. (22) we see that no angular momentum is deposited in the medium if either \( \Delta \Psi_e(x, y) \) or \( I_e(x, y) \) is cylindrically symmetric around the propagation direction. In particular, Laguerre-Gauss beams (which are eigenstates of \( L_z \)) have an intensity profile which is cylindrically symmetric so that they cannot transfer their own orbital angular momentum to transparent media such as liquid crystals\(^{(1)} \). In general, light SAM and OAM can be simultaneously transferred to LC media, leading to complex dynamics [162], exhibiting OAM-induced chaotic features and intermittence [163,164] as shown in fig. 25.

\(^{(1)} \) This statement is no longer true when defects and/or dislocations are present in the LC texture.
4. Nonlinear optics with OAM

4.1. Second-order nonlinear interactions. When a vortex beam propagates in a nonlinear medium, a variety of interesting effects can be observed. Nonlinear optical media are characterized by an electromagnetic response that depends on the intensity of the propagating light. The dielectric polarization of such a medium can be typically described as

\[ P(E) = \chi^{(1)} E + \chi^{(2)} E^2 + \chi^{(3)} E^3 + \ldots, \]

where \( E \) is the amplitude of the optical electric field and the \( \chi^{(n)} \) coefficients characterize the linear and nonlinear response of the medium (we are omitting here the tensorial features of this relation, for simplicity). The \( \chi^{(1)} \) coefficient determines the linear refractive index of the medium. The second-order nonlinear optical effects are the strongest, but they vanish in the important case of centrosymmetric materials, where \( \chi^{(2)} \) vanishes. In such media, only third-order nonlinear optical effects are observable.

The distinctive feature of second-order nonlinear processes is the frequency mixing between three waves, or, from a quantum point of view, the exchange of photons between three components of the field, having frequencies \( \omega_1, \omega_2 \) and \( \omega_3 \). Assuming that the optical field incident upon the medium consists of two components with distinct frequencies \( \omega_1 \) and \( \omega_2 \), the possible combinations can be represented as

\[ \omega_3 = m\omega_1 + n\omega_2, \]

with \( m \) and \( n \) integers. The process occurring for \( m = n = 1 \) and \( \omega_1 = \omega_2 \) is Second-Harmonic Generation (SHG), for \( m = 0 \) and \( n = 0 \) Optical Rectification (OR), for \( m = 1 \) and \( n = 1 \) Sum-Frequency Generation (SFG) and for \( m = 1 \) and \( n = -1 \) Difference-Frequency Generation (DFG). The efficiency and therefore the occurrence of each process depends on certain “phase matching” conditions, which are related to conservation of the momentum within the components of the field. Typically just one of these frequency combinations occurs with a higher probability. The DFG process is also known as optical parametric amplification, since, in an ideal lossless medium, conservation of energy requires that for each photon created at the frequency difference \( \omega_3 = \omega_1 - \omega_2 \), a photon at the higher frequency \( \omega_1 \) must be destroyed and a photon at the lower frequency \( \omega_2 \) must be accordingly created, thus feeding the input. Two-photon emission from the medium can still occur even if the stimulating field at the lower frequency \( \omega_2 \) does not impinge on the medium. In this case the process is designated as Spontaneous Parametric Down Conversion (SPDC), as opposite to up-conversion processes of SHG and SFG. In SPDC, the conservation of energy allows the splitting of the input wave into two lower frequency waves for a range of different values of the output frequencies. Also the phase matching condition turns to be flexible in this process: a change of the crystal orientation may result in a directional and/or frequency redistribution of the emitted field, rather than in a dramatic change of the emitted power. However, similarly to SHG and SFG, dramatic changes in the power of the SPDC light can be observed provided that the output power is measured within a narrow spectral bandwidth and within a well-defined range of wave vectors. SPDC plays a central role in optics and especially in quantum optics to generate correlated photon pairs.
The role played by OAM in second-order nonlinear optical processes was first explored in SHG [165,166]. It was demonstrated that frequency-doubling LG modes yields eigen-modes of the orbital angular momentum having the azimuthal phase index doubled for all modes, i.e. $\ell_\omega = 2\ell_\omega$ for any $\ell$ and $p$, consistently with conservation of orbital angular momentum within the light beams. This was shown to be a consequence of the phase-matching conditions, which turned to be independent of the mode indices. For a $p = 0$ mode, the frequency-doubled beam is also a Laguerre-Gaussian mode with $p = 0$, doubled $\ell$ and beam waist reduced by $\sqrt{2}$. For modes having $p > 0$, the amplitude distribution of the second-harmonic beam may be described as a Gegenbauer-Gaussian. These modes do not propagate in a structurally stable fashion and exhibit an energy transverse distribution closer to the beam axis than in Laguerre-Gaussian modes [166]. Berzanskis et al. [167] observed conservation of the orbital angular momentum also in SFG of beams having the same frequency but different azimuthal mode structure. In ref. [167], the authors also checked the topological charge conservation in the parametric amplification of optical vortices. Recently, generation of optical vortex beams via three-wave mixing in a non centrosymmetric photonic crystal has been reported [168]. Radially polarized vortex beams under strong focusing can produce subdiffraction transverse beam size ($w \approx 0.43\lambda$) in the focal region [62,133,134,136] and, therefore, are particularly suitable for SHG. SHG far-field intensity pattern from a nonlinear crystal considering excitation with vortex beams with radial and azimuthal polarization under strong focusing conditions were calculated in [169-171]. SHG from nonlinear crystal pumped by vector Gaussian beams [172] and surface SHG induced by polarization vortices [173] has been also reported recently.

In ref. [174], the authors perform the first experimental investigation of the spontaneous parametric down-conversion for light beams possessing orbital angular momentum. They utilized an extended pump beam produced by a commercial cw laser source and holographically added to it an azimuthal helical phase $e^{i\ell\phi}$, with both even and odd values of $\ell$, so to realize an LG-like beam with $p \approx 0$. The conclusion was that in the SPDC process they experimentally observed, unlike in SHG and SFG, the orbital angular momentum was not conserved and this behaviour was ascribed to the fact that the spatial coherence of the pump beam was not transferred to each individual down-converted mode. In 2001, however, Mair et al. [28] experimentally demonstrated transfer of orbital angular momentum to parametric down-converted photons, using Laguerre-Gauss pump beams with $\ell = 0, \pm 1$. The results from that experiment did confirm the conservation of the orbital angular momentum in parametric down-conversion at the single-photon level. It was in 2002 that Barbosa et al. [175] definitively clarified that orbital angular momentum is conserved in SPDC and established it in terms of appropriate “phase matching” conditions. In essence, each down-converted beam has individually only a limited spatial coherence [176], but the two down-converted beams are coherent with respect to each other and the photons belonging to the distinct generated modes are correlated. In practice, when a nonlinear crystal is pumped by TEM$_{00}$ laser mode, photon pairs are generated in the OAM superposition $|\psi\rangle = \sum_{\ell=0}^{\infty} C_{\ell}|+\ell\rangle|-\ell\rangle$ as experimentally observed using a parametric generator with large numerical aperture [28]. With similar techniques, hyperentangled photon pairs can be generated where SAM, OAM and time-energy are all entangled [177].

SPDC process has been also exploited to generate high-dimensional two-photons states entangled in orbital angular momentum [178], promising high-density encoding of quantum information [179,180]. Specifically, the authors, in this paper, have experimentally realized a genuine 11-dimensional entanglement in the OAM of signal and idler photons, as indicated by the violation of Bell’s inequalities generalized to $d$-dimensional...
systems with $d = 12$. It has been also reported on the generation via SPDC of high-dimensionally spatially entangled photon in the full azimuthal-radial Laguerre-Gauss-like basis [181]. The importance of accessing the radial modes in spatial entanglement rests upon the fact that the number of useful entangled modes is approximately the square of number of merely azimuthal modes, therefore high-density encoding of quantum information turns to be considerably favoured.

It has been shown that both stimulated parametric down-conversion and optical parametric oscillators are able to generate well-defined down-converted spatial modes [182, 183]. Moreover, a continuous-wave LG mode has been utilized for pumping a degenerate, type-I optical parametric oscillator in order to control the stabilization of domain walls through the topological charge of the phase [184]. It came out that when an $LG_{0\ell}$ mode is used for pumping, the output signal turned out to be stable with transverse phase profile $\exp(i\ell \phi/2)$. From this result, it is evident that for even $\ell$ the output phase profile is stable, since $\ell/2$ is integer, and the number of domain walls it is able to trap is even (possibly zero). For odd $\ell$, in contrast, the output is in a discontinuous state with fractional angular momentum. The trapped domain walls, in this case, turn out to be odd in number (possibly one) and have radially vanishing intensity profiles.

4.2. Optical vortices and orbital angular momentum in Kerr media. – It is well known that the distinctive feature of nonlinear Kerr media is the dependence of the refractive index on the intensity of the incident beam. Depending on whether the refractive index change is positive or negative, the incident beam is self-focused or self-defocused while propagating within the medium. Being a third-order nonlinear process, self-focusing may occur in isotropic materials as glass. Under self-focusing, the annular intensity of an OAM beam can induce a column of high-refractive-index material along which a secondary beam can be guided [185]. In the late 90’s, Firth and Skryabin [186, 187] predicted the existence of annular-shaped OAM-carrying solitary waves in nonlinear Kerr media. They were shown to be unstable under propagation and to split into fundamental solitons. The input OAM determines the number of sprouted solitons and causes the solitons to fly out tangentially from the initial ring, in the same way as free Newtonian particles. Recently, self-induced SAM-to-OAM conversion has been also reported in nonlinear tellium-gallium garnet [188] and in silver-doped glasses [189]. Another recently demonstrated effect concerns the capability of inducing stable and localized matter vortices in nematic liquid crystal valves irradiated with circularly polarized light beams [190].

5. – Measuring the orbital angular momentum of light

In the last two decades, the interest for OAM raised considerably thanks also to the technological achievements enabling more and more efficient methods for generating and manipulating the OAM. Nevertheless, a method for measuring the average angular momentum carried by an arbitrary light field, with an accuracy and efficiency comparable to those available for SAM, is still missing. In other words, the problem of fast and reliably measuring the angular momentum related to the optical ray-torsion either in pure helical-modes or in arbitrary superpositions of them is still open, even in the framework of the paraxial approximation, where the separation between SAM and OAM satisfactorily applies. Passing over the difficulties of a full quantum statement [37], the similarity between paraxial propagation for light waves and evolution of wave functions in the Schrödinger picture has suggested to read the longitudinal component of the OAM as the quantum average of the operator $\hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ in a given state of radiation. Consistently, the
fraction of the total intensity having $\exp(i\ell \phi)$ form of the transverse modes and carrying, therefore, $\hbar \ell$ per photon has come to be identified with the probability of obtaining $\hbar \ell$ in measuring the OAM in that state [8,78].

The measurement of the OAM has been a hoary problem both in quantum and classical optics: it has received a great deal of interest from several authors and has been rightfully considered challenging. Such interest rests upon the fact that a fast and reliable measurement of the OAM content of a light beam is highly desirable for both fundamental and application issues. No doubt, the problem of measurement plays a vital role in OAM-based classical or quantum information transfer and communications. The interest for OAM in this fields arises from its infinite-dimensional nature, which potentially enables to increase the capacity of communication systems. In sect. 6, the role played by OAM in quantum information and, in particular, in quantum communications will be discussed in details [191]. In the classical framework, owing to their mutual orthogonality optical beams carrying OAM and/or SAM can be multiplexed together via mode-division multiplexing to transmit multiple independent data channels, related to different OAM and/or SAM values, on the same wavelength, over a single path. Free-space information transfer based on the orbital angular momentum of light was first demonstrated by Gibson et al. [192] and recently terabit-scale data transmission via OAM multiplexing has been reported both in free-space [193] and in optical fibers [194].

The traditional methods pertaining to the OAM measurement are aimed at measuring either the topological charge of the field or the power spectrum of the helical modes it contains. Measuring the topological charge usually requires a direct visual inspection of either the interference pattern with a reference wave or the far diffraction pattern of a suitably shaped aperture, such as in the case of a multi-pinhole interferometer [195] or in the case of a simple triangular aperture [196]. In the former case, the topological charge of the beam is obtained by counting the $\ell$ spiral fringes forming the interference pattern produced by superimposing a helically phased beam with a plane wave [197-199]. Such a method gives no information on the OAM probability distribution and cannot be applied to single-photon measurement, since it depends on having many photons in the same state. In the latter case, the method relies on the possibility of recognizing the sign of $\ell$ or even its value from the peculiarities of the far-field diffraction pattern from a suitably designed diffraction grating. The far-field pattern of an LG beam through a single edge consists in a cut intensity profile of the same LG beam rotated towards left or right according to the sign of $\ell$ [200]. When more complex diffraction gratings, such as arrays of pinholes, are envisaged, the topological charge can be obtained [195,196]. Also these methods give no information on the OAM probability distribution and cannot be applied to single-photon measurement.

The measurement of the spectrum of the helical modes and of their relative phases for OAM is the counterpart of the measurement of the Stokes parameters for SAM. The dichotomus nature of SAM spectrum can be simply and efficiently analyzed and several methods are available to reconstruct the whole SAM state. In contrast, the OAM is defined in an infinite-dimensional Hilbert space and a spectrum analysis is quite challenging, since an ideal device should have an infinite number of output channels, each corresponding to a distinct OAM state. Helical mode probabilities within a given optical field can be obtained one by one or clustered according as the adopted devices select a single mode or multiple spatially separated modes (mode sorters). Other methods produce such probabilities by resorting to optical wavefront reconstruction via a spatial resolved second-order field correlation, rather than mode selection or separation. This is the case for work reported in ref. [201], where a multi-pinhole interferometer is used. Typical
single-mode selectors are the traditional spiral phase plates or single-vortex holograms, or the more recent \( q \)-plates. In essence, a beam carrying an OAM of \( h\ell \) illuminating a single-vortex hologram, such as a forked diffraction grating, with charge \( q = \ell \) produces a plane wave Gaussian beam in a given diffraction order. To ensure that the output beam is a pure single-mode one, it is coupled through a single-mode fiber and sent to a photodetector. Adopting high-quality photomultipliers or avalanche photodiode, modes, or complex superpositions of modes, can be measured even at single photon level [28]. The main drawback of such method is the possibility of measuring only one mode at a time, and if a large state space (as in the case of OAM) is to be measured, one must test for each of the modes in turn. The efficiency of such an approach, therefore, can never exceed \( 1/N \), where \( N \) is the maximum number of modes to be detected. In order to measure simultaneously as many probabilities as possible, a device should select as many helical modes as possible and sort each of them into a specific direction, where a single-mode fiber and a detector are located. This behaviour has been achieved through complex holograms, with an efficiency again approximately equal to the reciprocal of the number of distinct modes [202, 203, 192], since the incident energy is split between the outputs. A highly efficient, though very complex, mode sorter has been also reported [204, 205], whose \( 2^n \)-ports implementation is obtained by cascading \( 2^n - 1 \) binary sorters, arranged in \( n \) stages. Each binary element is made up of a Mach-Zender interferometer containing two Dove prisms. An efficient and much handier mode sorter has been developed more recently that is based on custom refractive components and separates optical vortex states by transforming them into inclined plane waves, which are afterwards focused to different positions by a standard lens [206-208]. The degree of information provided by the OAM spectral analysis has the undeniable merit of returning the whole probability distribution and, therefore, all the moments of that observable, ultimately the mean and the variance which are immediately related to mechanical effects on matter.

Another measurement method has been very recently proposed that is significantly different from those above mentioned, since it returns the mean and variance of the OAM distribution in any optical field avoiding the full spectral analysis. The method, which is based on a polarization homodyne detection, works both in continuous wave and in photon-counting regime. Through this method an ideal infinite port device can be replaced by a two only port system, returning the mean and the variance of the photon OAM distribution. Such quantities, though generally inadequate for fully reconstructing the distribution, may still provide useful information about the source of the radiation field and about the OAM conservation in radiation-matter interactions. The principle here introduced to perform direct measurements of OAM mean and variance could possibly be adopted to implement free-space communication systems where the restrictions on sender-receiver alignment are weaker than in the systems based on traditional spectrum measurements [209].

6. – Relevance of the orbital angular momentum of light in quantum optics

Photonic entanglement in various degrees of freedom, involving single or multiple photons, is considered a crucial resource for both fundamental physics investigations and quantum communication protocols [210]. On the one hand, entangled states of quantum systems are ideal to perform experiments based on the Einstein, Podolosky, Rosen (EPR) argument [211] or to test, on experimental grounds, Bell’s inequalities [212]. On the other hand, polarization [213-217], path [218], time-bin [219], color [220], and, above all, OAM [28, 192, 221-223] have proved to be valuable quantum information carriers.
The advantages of the OAM spatial modes in the quantum information field are essentially of two kinds: those related to the intrinsically infinite dimension of the OAM Hilbert space, which allows one to robustly encode a large amount of information in single photons [224,225], and those related to the possibility of realizing hyperentanglement [177] or hybrid entanglement in combination with photonic polarization [226,227]. Twelve-dimensional entanglement has been demonstrated with OAM eigenstates [178], and hyperentanglement of OAM and polarization has been used to demonstrate superdense coding, with performances exceeding the classical channel capacity limit [228]. Relevant for its potential use in quantum communications is the recently demonstrated transmission and measurement of the spatial entanglement of photons in two-mode optical fibers [229].

6.1. OAM in fundamental tests of quantum mechanics. – As far as the fundamental issues of OAM-entangled photon states are concerned, it was recognized that, as a consequence of the conservation of energy, momentum and angular momentum, in spontaneous parametric down-conversion, the output lower-frequency beams, although spatially incoherent, are coherent with respect to each other [167]. The output beams are therefore entangled spatial modes [176] and measuring one of the conserved observables for a photon in one of such spatial modes gives information about the same observable for the simultaneously generated photon in the other mode. This leads to nonlocal correlations analogous to those involved in the Einstein, Podolosky, Rosen (EPR) paradox [211]. The context pre-represented in the EPR paradox has been experimentally implemented by using a wide range of variable pairs including, position-momentum [230], time-energy [231] and more recently optical angle-OAM variables [232]. The azimuthal angular position of a photon around the axis of the mode and the OAM it carries are conjugate variables [233] and turn out to be particularly suitable for demonstrating the existence of EPR correlations. As a matter of fact, the first experimental attempts to obtain strong correlations in OAM in a complementary basis come back to 2001, by Mair et al. [28], and 2005, by Oemrawsingh et al. [222], who observed that the strength of the correlation depended on both the magnitude and the relative phase of the modes in OAM-state-superpositions.

6.2. OAM in quantum information science. – The building blocks of any hypothetical quantum computer are gates allowing the value of one variable to switch that of another. This behaviour is typical, for instance, of a controlled-NOT gate (CNOT), such as that proposed by L. Deng et al. [234], consisting in a SAM-CNOT gate and an OAM-CNOT gate for single-photon two-qubit quantum logic. The qubit is a two-dimensional quantum state and represents the standard unit of quantum information. Higher-dimensional quantum states, called qudits, can also be used to encode information and work out more complex protocols. OAM allows the encoding of qudits by exploiting any two-dimensional subspace spanned by two distinct OAM values. Due to the strict analogy with the SAM space, a special role is held by the subspaces spanned by opposite OAM values \( m = \pm \ell \), denoted as \( o \ell \). Besides, OAM provides an obvious possibility for qudit optical implementation, by exploiting a larger subspace. Moreover, OAM can be readily combined with other degrees of freedom of the photon in order to further expand the Hilbert space or to realize the so-called hyper-entanglement, where different degrees of freedom of two particles are simultaneously entangled [177,228]. The combination of OAM with SAM degrees of freedom has proved to be very fruitful for both implementing logical operations and for creating a frame-invariant encoding of quantum information [235,236]. A non-negligible contribution to the development of methods and schemes based on SAM-to-OAM conver-
sion or on SAM and OAM combination, also leading to hybrid entanglement \cite{226,227}, originated from the introduction of devices such as the above-mentioned \( q \)-plates \cite{26} or \( q \)-plate-like azimuthal-mode-converters reported in ref. \cite{114}.

In order to describe the action of a \( q \)-plate on a single photon quantum state, let us introduce the ket notation \(|P,m\rangle = |P\rangle_\pi |m\rangle_o\) for the single photon states, where \( P \) stands for the polarization state (e.g., \( P = L, R, H, V \) for left- and right-circular polarizations and horizontal and vertical linear polarizations) and \( m \) is the OAM value in units of \( \hbar \).

When the OAM-subspace adopted is \( o_\ell \), the radial part of the spatial mode, depending on \(|\ell|\) can be factorized out and, for all practical purposes, ignored. The action of an optimally tuned \( q \)-plate, represented by the linear operator \( \hat{Q}P \), is defined as follows (here the subscript \( \pi \) denotes the SAM space and the subscript \( o \) the OAM one):

\[
\hat{Q}P |L\rangle_\pi |m\rangle_o = |R\rangle_\pi |m + 2q\rangle_o,
\]

\[
\hat{Q}P |R\rangle_\pi |m\rangle_o = |L\rangle_\pi |m - 2q\rangle_o.
\]

When applied to an input linearly polarized light (e.g., horizontal) having \( m = 0 \), we obtain the following output state:

\[
\hat{Q}P |H\rangle_\pi |0\rangle_o = \frac{1}{2} (|L\rangle_\pi |2q\rangle_o + |R\rangle_\pi |2q\rangle_o).
\]

The right-hand-side expression can be interpreted as an entangled state of the SAM (or polarization) and OAM degrees of freedom of the same photon. These predictions have been tested experimentally on heralded single photon states obtained by spontaneous parametric down-conversion (SPDC) \cite{89}.

As already stated above, the main task of a \( q \)-plate consists in converting the SAM degree of freedom of a light beam into the OAM degree of freedom, and vice versa. In the framework of quantum information this corresponds to the possibility of using a \( q \)-plate or a \( q \)-plate-like converter for realizing optical devices that can transfer the quantum information initially stored in the polarization degree of freedom of the photon into the OAM degree of freedom, or vice versa. This action can be strictly regarded as the physical implementation of an isomorphism between the SAM space and OAM \( o_2q \) subspace. In essence such devices enable the generation of an arbitrary qubit photon state in the OAM space merely by transformation of its isomorphic counterpart previously prepared in the SAM space. Any OAM state can also be conveniently analyzed by simply transferring it back to the SAM space. On the grounds of this technology, several experiments have been recently performed. The optical schemes reported in refs. \cite{89,116,237} implement the following transformations (in both directions):

\[
|\psi\rangle_\pi |0\rangle_o \Leftrightarrow |H\rangle_\pi |\psi\rangle_o,
\]

where \(|\psi\rangle\) here stands for an arbitrary qubit state and \(|H\rangle_\pi \) has been chosen as the “blank” state of polarization and \(|0\rangle_o \) as the blank state of OAM, other choices being obviously possible. These quantum information transfer devices have been experimentally verified in the simpler probabilistic schemes within the heralded single photon regime. Adopting an electrically tunable \( q \)-plate \cite{87} and a Sagnac interferometer with a Dove prism \cite{117}(fig. 26), the deterministic transfer of a generic qubit initially encoded in the orbital angular momentum of a single-photon to its polarization was also experimentally
Fig. 26. – Experimental setup for the implementation of the deterministic quantum transferrer $\pi \rightarrow o_2$. The input photon, coming from the left, is prepared by a probabilistic transferrer ($\pi \rightarrow o_2$) (first two wave plates, QP1 and PBS) into an arbitrary $o_2$ state with polarization $H$. After this generation stage, the PSI and the QP2 realize the deterministic transferrer ($o_2 \rightarrow \pi$). The outgoing polarization state is analyzed in the last part of the setup (wave plates, PBS, detectors D1 and D2). C is a phase compensation stage to correct all the unwanted phase shifts introduced by the setup. All QPs are electrically tuned. (Reprinted with permission from ref. [119]. Copyright 2012, Optical Society of America.)

implemented [119]. The transfer of quantum information was completely reversible and was shown to exhibit high fidelity and low losses. In ref. [89], the $q$-plate-based optical scheme reported in fig. 27 was adopted to show that two-photon quantum correlations such as those resulting from coalescence interference can be directly transferred into the OAM degree of freedom. A biphoton state, i.e. a single optical mode having exactly two photons, having polarization correlations was initially generated by SPDC, and the SAM-OAM transfer device was then used to generate the final state with OAM correlations. This experiment was indeed the first evidence of quantum correlation transfer from SAM to OAM degree of freedom, resolving any doubt possibly arising from the adoption of the heralded mode and its resemblance with classical optics experiments. This result, namely the successful operation of SAM-OAM information transfer tools, paved the way to the first experimental demonstration of Hong-Ou-Mandel (HOM) coalescence [238] of OAM-carrying photons in a beam-splitter [30]. HOM coalescence plays a key role in quantum information, since it underlies many other quantum information protocols, such as quantum teleportation, quantum cloning, etc. Its experimental viability through a SAM-to-OAM transfer device represents for the latter a robust test. And, indeed, in the same paper the implementation of optimal quantum cloning of OAM-encoded photonic qubits was also demonstrated [116]. Transfer devices have also recently enabled the preparation of spin-orbit hybrid-entangled photons [226, 227]. Two opposite approaches have been demonstrated. By starting with a polarization (SAM) entangled pair generated by SPDC, one can transfer the quantum information of only a single photon of the pair from SAM to OAM to obtain the hybrid entanglement [226]. Conversely, by starting with an OAM entangled pair (also generated by SPDC, as first demonstrated in [28]), it is possible to reach the hybrid entanglement by an OAM-to-SAM quantum transfer [227].
Fig. 27. – Schematic of the biphoton OAM coalescence setup. The SPDC source generates pairs of correlated photons having $H$ and $V$ polarizations. The $q$-plate converts this correlation in the OAM space. The correlations can then be tested by the vanishing coincidence measurements on opposite OAM states, as filtered using a fork hologram. A quartz plate can be used to introduce a delay between the two input photons, thus destroying the quantum correlations. (Reprinted with permission from ref. [89]. Copyright 2009, American Physical Society.)

In these works, the entanglement has been confirmed by testing the violation of a Bell’s inequality [226, 227, 239]. The nonseparability (or single-particle entanglement) of SAM and OAM degrees of freedom has also been investigated, using $q$-plates or interferometric layouts, both in a classical intense beam regime [227, 240] and in a heralded single photon one [89, 227, 239, 241]. The remote preparation of single-particle hybrid-entangled states has also recently been demonstrated (not using $q$-plates), by exploiting an SAM-OAM hyper-entangled photon pair source [242]. A proposal for hybrid entanglement multiphoton manipulations exploiting the $q$-plate has also recently been put forward [243].

Fig. 28. – Experimental apparatus for implementing the $1 \rightarrow 2$ optimal quantum cloning of polarization-OAM photon ququarts. (Reprinted with permission from ref. [31]. Copyright 2010, American Physical Society.)
Quantum states with dimension higher than two can be implemented by either encoding different qubits in different photons (multi-particle protocols) [244-247] or encoding as many qubits as possible in a single photon by exploiting different degrees of freedom (multi-degree-of-freedom approach). The latter approach is not ultimately scalable, but it is very advantageous for implementing intermediate numbers of qubits with the present technology. Single photon ququarts, obtained by encoding a qubit in SAM and another qubit in an OAM subspace, were prepared and measured through a q-plate based apparatus [32]. The optimal quantum cloning 1 \rightarrow 2 of ququarts was also achieved by exploiting the four-dimensional HOM effect [31], using the setup illustrated in fig. 28. In ref. [248], two novel quantum-state manipulation processes have been introduced and
Fig. 30. – Misalignment-immune single-photon qubits. Alice and Bob need to carefully control the relative orientation between their horizontal (H) and vertical (V) axes to faithfully implement free-space quantum communications. Unknown misalignments around the propagation axis manifest as rotations of the transmitted qubits by unknown angles $\theta$ in the $H$-$V$ plane. (a) Qubits can be equivalently encoded in both polarization and transverse modes: $H/V$ denote horizontal/vertical linear polarizations, $L/R$ left/right circular polarizations, $h/v$ horizontal/vertical first-order Hermite-Gauss modes, and $l/r$ left- and right-handed first-order Laguerre-Gauss modes. The $L/R$ polarizations are eigenstates with eigenvalues $\pm \hbar$ of the SAM, whereas the $l/r$ modes are the equivalent eigenstates of the OAM. (b) By combining SAM and OAM eigenstates of opposite handedness, two null-eigenvalue eigenstates of the total angular momentum arise. Both these hybrid states are invariant under rotations around the propagation axis, and can therefore encode misalignment-immune logical qubit states, called $0_L$ and $1_L$. (Reprinted with permission from ref. [250]. Copyright 2012, Nature Publishing Group.)
invariant states of single photons and a complete toolbox for the efficient encoding and decoding of quantum information in such photonic qubits was developed. The core of the toolbox is a $q$-plate having $q = 1/2$, mapping polarization-encoded qubits into qubits encoded in hybrid polarization-OAM states of the same photon that are invariant under arbitrary rotations around the propagation direction, and vice versa. The peculiarity of the SAM and $\ell = 1$ OAM eigenstates together is that, as they are defined with respect to the same reference frame, they suffer exactly the same transformation under coordinate rotation. Therefore, they satisfy the collective rotation requirement exactly, constituting an ideal pair to carry rotationally invariant hybrid qubits (see fig. 30).

The generation of NOON-like photonic states of $m$ quanta of angular momentum [251], with $m$ as high as 100, has been very recently demonstrated and applied to implement a setup operating as a “photonic gear”, that converts a mechanical rotation of angle $\theta$ into an amplified rotation of the optical polarization by the angle $m\theta$. Exploiting this effect, the feasibility of angular measurements has been demonstrated with a precision comparable to that of the optimal quantum strategy with $m$ photons, but robust to photon losses. The high “gear ratio” $m$ translates into a similar sensitivity enhancement of optical noncontact angular measurements, boosting the current state of the art by almost two orders of magnitude. This result was demonstrated both in classical and quantum regimes.

7. – Summarizing remarks and future issues of OAM

As mentioned from the very outset, much more could be said on the subject and a number of outstanding papers and authoritative reference textbooks should be looked up for further insight. In this review, we have outlined the issues and challenges in OAM research selecting a path aimed at emphasizing the importance of learning how to “harness” the shape of light: the full control of the local amplitude and phase of light makes the OAM degree of freedom an extraordinarily flexible and powerful tool for the most diverse applications, ranging from fundamental quantum tests to the realization of photonic polarization gears for ultra-sensitive angular measurements, to the manipulation of micron-sized particles trapped by optical tweezers and so on. From all this, we can see at least an invitation to look at light through new eyes. Old and new challenges have to be faced, such as—just to name one—the spin-orbit interaction beyond the paraxial approximations. The research in the spin-orbit effects of light is drawing an increasingly amount of attention especially for its potential practical implications. Miniaturization of optical devices for integrated optics and more specifically in nano-optics, photonics, plasmonics, and singular optics all require the management of highly nonparaxial beams and therefore of the spin-orbit interaction of light, due to the subwavelength characteristic scales involved.

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