Chapter 6
Noncommutative Spaces

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In this chapter we present some of the basic concepts needed to describe noncommutative spaces and their topological and geometrical features. We therefore complement the previous chapters where noncommutative spaces have been described by the commutation relations of their coordinates. The full algebraic description of ordinary (commutative) spaces requires the completion of the algebra of coordinates into a $C^{\star}$-algebra, this encodes the Hausdorff topology of the space. The smooth manifold structure is next encoded in a subalgebra (of "smooth" functions). Relaxing the requirement of commutativity of the algebra opens the way to the definition of noncommutative spaces, which in some cases can be a deformation of an ordinary space. A powerful method to study these noncommutative algebras is to represent them as operators on a Hilbert space. We discuss the noncommutative space generated by two noncommuting variables with a constant commutator. This is the space of the noncommutative field theories described in this book, as well as the elementary phase space of quantum mechanics. The Weyl map from operators to functions is introduced in order to produce a $\star$-product description of this noncommutative space.

6.1 Commutative geometry (and topology)

In Hilbert's foundations of geometry [1] the concepts of points, lines, and planes are considered intuitive and no attempt is made to define them. These "undefined" points are nevertheless the basis of any topological space, differentiable manifold, bundle, and so on, all geometrical concepts built on spaces made of points. This gave the impression that geometrical notions cannot survive without points. Quantum mechanics forced a change of this attitude. While in classical mechanics the state of a system can be described by a point in a phase space, Heisenberg's uncertainty principle makes the concept impossible in quantum mechanics. This led von Neumann [2] to speak of *pointless* geometry.

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In the following we introduce the basic mathematics of noncommutative geometry at an unspecialized level, that of a high-energy physics student for example. Sometimes we sacrifice rigor and refer to some of the classic reference books [3–6] for details and proofs. An extended and more rigorous treatment of the topics of this chapter will also appear in the forthcoming book [7].

In the present section we discuss ordinary topology and geometry from a point of view which enables its generalization to a noncommutative setting. The main tool is the transcription of the usual geometrical concepts in terms of algebras of operators. The starting point is a series of theorems due to Gel'fand and Naimark (for a review see for example [4, 5]). They established a complete equivalence between *Hausdorff topological spaces* and *commutative C^*-algebras*. From a physicist point of view one can look at this activity as describing the topology (and geometrical properties) of a space not seeing it as a set of points, but as the set of fields defined on it. In this sense the tools of noncommutative geometry resemble the methods of modern theoretical physics.

6.1.1 Topology and algebras

A topological space M is a set on which a *topology* is defined: a collection of open subsets obeying certain conditions, this enables the concept of convergence of succession of points $x_n \in M$ to a limit point $x = \lim_n x_n$. Together with the concept of convergence goes the notion of continuous function. A function from a topological space into another topological space is continuous if the inverse image of an open set is open, but as a consequence it maps convergent sequences into convergent sequences:

$$\lim_n f(x_n) = f(x). \tag{6.1}$$

A Hausdorff topology makes the space separable, i.e., given two points it is always possible to find two disjoint open sets each containing one of the two points. The common topological spaces encountered in physics (for example, manifolds) are separable.

Of particular interest in this context is the set of *complex-valued* continuous function. They form a commutative algebra because the sum or product of two continuous functions is still continuous. We will show how it is possible to define the topology of a space from the algebra of continuous functions on it. Moreover, we will show how to construct the topological space starting from the abstract algebra. On one hand every Hausdorff topological space defines naturally a commutative algebra, the algebra of continuous complex-valued functions over it. Remarkably, under certain technical assumptions spelled below, the reverse is also true, i.e., given a commutative algebra $\mathscr{A}$ as an abstract entity, it is always possible to find a topological space whose algebra of continuous functions is $\mathscr{A}$. Therefore, we can establish a complete equivalence between topological spaces and algebras. In the following we will describe these mathematical structures from an "user" point of view, keeping the technicalities at a minimum and refer the literature for proofs and details.

The technical assumptions we have mentioned are resumed in the fact that the algebra $\mathscr{A}$ must be a C^*-algebra. This is, first of all, a vector space with the structure of an associative algebra over the complex numbers $\mathbb{C}$, i.e., a set on which we can define two operations, sum (associative and commutative) and product (associative but not necessarily commutative), and the product of a vector by a complex number, with the following properties:

(1) $\mathscr{A}$ is a vector space over $\mathbb{C}$, i.e., $\alpha a + \beta b \in \mathscr{A}$ for $a, b \in \mathscr{A}$ and $\alpha, \beta \in \mathbb{C}$.
(2) It is distributive over addition with respect to left and right multiplication, i.e., $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$, $\forall\, a, b, c \in \mathscr{A}$.

$\mathscr{A}$ is further required to be a Banach algebra:

(3) It has a norm $\|\cdot\| : \mathscr{A} \to \mathbb{R}$ with the usual properties

a) $\|a\| \geq 0$, $\quad \|a\| = 0 \iff a = 0$
b) $\|\alpha a\| = |\alpha|\, \|a\|$
c) $\|a+b\| \leq \|a\| + \|b\|$
d) $\|ab\| \leq \|a\|\, \|b\|$

The Banach algebra $\mathscr{A}$ is called a $*$-algebra if, in addition to the properties above, it has a hermitian conjugation operation $*$ (analogous to the complex conjugation defined for $\mathbb{C}$) with the properties

(4) $(a^*)^* = a$
(5) $(ab)^* = b^* a^*$
(6) $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$
(7) $\|a^*\| = \|a\|$
(8) $\|a^* a\| = \|a\|^2$

for any $a, b \in \mathscr{A}$ and $\alpha, \beta \in \mathbb{C}$, where $\bar{\alpha}$ denotes the usual complex conjugate of $\alpha \in \mathbb{C}$. Finally,

(9) It is complete with respect to the norm.

C^*-algebras play a very important role in mathematics because as we will see their study is basically the study of topology. A good introduction to their properties is found in the book [8].

Example 6.1.
Examples of C^*-algebras are $n \times n$ matrices, bounded operators on an infinite-dimensional Hilbert space, as well as compact operators. The norm is the supremum norm in all these cases. These are noncommutative, examples of commutative algebras are $\mathbb{C}$ itself, or the continuous functions on the plane. Note that several commonly used algebras do not satisfy all of the definitions. For example, the set of upper triangular matrices does not have the hermitian conjugation, trace class operators are not complete, and the Hilbert space of L^2 functions has a norm which does not satisfy item (8) above. □

Every Hausdorff topological space has a natural *commutative* C^*-algebra associated with it: the algebra of continuous complex-valued functions. If the space is compact this algebra contains the unity $\mathbb{I}$ and is called unital. The converse is also true. Every unital commutative C^*-algebra is the C^*-algebra of continuous functions on some compact topological space. Nonunital algebras are similarly associated with noncompact Hausdorff spaces.

6.1.2 Reconstructing the space from the algebra

We now show how the topological space can be reconstructed from the algebra. We first introduce the notion of *state*. A state is a linear functional from a C^*-algebra $\mathscr{A}$ (not necessarily commutative) into complex numbers:

$$\phi : \mathscr{A} \longrightarrow \mathbb{C}, \tag{6.2}$$

with the positivity and normalization requirements

$$\phi(a^* a) \geq 0 \;\; \forall a \in \mathscr{A}, \;\; \|\phi\| = 1 \; . \tag{6.3}$$

In this case the norm is defined as

$$\|\phi\| = \sup_{\|a\| \leq 1} \{\phi(a)\} \; . \tag{6.4}$$

If the algebra is unital $\phi(\mathbb{I}) = 1$.

The space of states is convex, i.e., any linear combination of states of the kind $\cos^2\lambda\phi_1 + \sin^2\lambda\phi_2$ is still a state for any value of λ. Some states cannot be expressed as such convex sum, they form the boundary of the set and are called *pure states*.

Example 6.2.
Consider the case of $n \times n$ complex-valued matrices. A state is given by a matrix (which with an abuse we still call ϕ) with the definition

$$\phi(a) = \mathrm{Tr}\phi a. \tag{6.5}$$

Positivity requires the matrix ϕ to be self-adjoint with positive eigenvalues, and normalization requires it to have unit trace. Since the matrix is self-adjoint it can be diagonalized. There are two possibilities. Either more than one eigenvalues is different from zero, and in this case it is immediate to see that we can write it as the convex sum of two diagonal matrices of trace 1. Alternatively only one eigenvalue is different from zero, and it must be the unity. In this case it is not possible to express ϕ as the convex sum of two matrices of trace 1, since positivity requires diagonal elements to be positive numbers less than 1. So pure states are nothing else but pure density matrices, which correspond to the projectors, these in turn are in a *one-to-one correspondence with the normalized n-dimensional vectors (the rays). This construction can be carried over to the infinite-dimensional case, considering bounded operators on an infinite-dimensional separable Hilbert space.* □

Consider next a commutative algebra and its set of pure states. We can give to this set a *topology* as follows: given a succession of pure states δ_{x_n} we find its limit as

$$\lim_n \delta_{x_n} = \delta_x \Leftrightarrow \lim_n \delta_{x_n}(a) = \delta_x(a) \;\; \forall a \in \mathscr{A} \; . \tag{6.6}$$

We have constructed the topological space associated with the C^*-algebra $\mathscr{A}$. We have therefore a *duality* between topological spaces and C^*-algebras: a topological space determines the C^*-algebra of its continuous complex-valued functions. Conversely any commutative C^*-algebra, using uniquely algebraic techniques, determines a topological space whose algebra of continuous functions is the initial C^*-algebra.

The reconstruction of the topological space from the algebra via the set of pure states is one of various equivalent ways to obtain the space from the algebra. It is worth to briefly comment on some of the alternatives since in the noncommutative case these are not anymore the same and capture different aspects of the noncommutative geometry. For commutative algebras it turns out that the space of pure states is the same as the state of irreducible one-dimensional representations. It is possible to give a topology (called regional topology) [9] directly on the space of representations of an algebra, and in the commutative case this topology is the same as the one described earlier. In this case the space of points is also the same as the space of *maximal ideals* of the algebra. An ideal of an algebra is a subalgebra $\mathscr{I}$ with the property that

$$ab \in \mathscr{I} \; \forall a \in \mathscr{A} , \; \forall b \in \mathscr{I} \; , \tag{6.7}$$

the relevant example of ideal for the algebra of functions on some space is the set of functions vanishing in some closed set. Recall that if a continuous function vanishes on some set of a topological space, it will vanish also on the closure of the set, therefore the structure of ideals *feels* the topology of the underlying space. A maximal ideal is an ideal which is not contained in any other ideal (and is not the whole algebra). Since the ideal of functions vanishing in a given set is contained in the ideal of functions vanishing in any smaller set contained in the first set, it is intuitively obvious that the functions vanishing at a given point are an ideal not contained in any other ideal, hence the one-to-one correspondence between points and maximal ideals. A topology based on the closure of the set of ideals can be given (called hull-kernel topology), thus giving a third (equivalent) manner to reconstruct a space from a C^*-algebra. We have seen three different sets that we can build exclusively form the algebra:

- pure states
- irreducible (one-dimensional) representations
- maximal ideals

On this set we can build, purely algebraically, three topologies, which turn out to be same for commutative algebras.

6.1.3 Geometrical structures

We have mainly dealt so far with the topology of Hausdorff spaces, which we can call, in the spirit of these notes, *commutative spaces*. What about the other geometrical structures? We can transcribe all standard concepts of geometry at the algebraic level, as properties of C^*-algebras and of other operators. This program, started by Connes [3], has been going for some time in the construction of some sort of *dictionary* transcribing the concepts of commutative geometry into concepts connected to C^*-algebras. The aim of this exercise should be evident: once we have translated pointwise geometry into operations at the algebraic level, these are more robust and can still be used at the level of noncommutative C^*-algebras, thus describing a noncommutative geometry. Let us give a few entries of this continuously evolving dictionary.

The presence of a smooth structure, i.e., a manifold structure, is equivalent to considering a subalgebra $\mathscr{A}_\infty \subset \mathscr{A}$ of "smooth" functions. This subalgebra can be given the structure of a Fréchet algebra, which is a locally convex algebra with its topology generated by a sequence of seminorms $\|\cdot\|_k$ which separate points: that is, $\|a\|_k = 0 \forall k \Leftrightarrow a = 0$. The seminorms for this algebra are

$$\|a\|_k = \sup_{x\in M} \{ \, |\partial^\alpha a(x)| \text{ for } |\alpha| \le k \, \} \; . \tag{6.8}$$

A theorem of Serre and Swan establishes an equivalence between bundles and modules. A bundle E over topological space M (called the base) is a triple composed by E (which is also a topological space), M, and a continuous surjective map $\pi : E \to M$ and such that for each $x \in M$ the space $\pi^{-1}(x)$ is homeomorphic to a space F, called the typical fiber. When F is a vector space we have a *vector bundle*. Locally the bundle is trivial, i.e., there is a covering U_i of M such that locally $\pi^{-1}(U_i) = U_i \times F$. A *section* of a bundle is a map $s : E \to M$ such that $\pi \circ s = \mathrm{id_M}$. Examples of bundles abound in physics, often with the further structures, like fiber bundles, which are vector bundles together with the action of a group G on the fiber F. Yang–Mills fields are sections of fiber bundles. It turns out that a vector bundle over a manifold M is completely characterized by its space of smooth sections $\mathscr{E} = \Gamma(E,M)$.

It is possible to substitute the concept of bundle with the one of *projective module*. A left module $\mathscr{E}$ is a vector space over $\mathbb{C}$ on which the algebra acts, that is, for $a,b \in \mathscr{A}$, $\eta, xi \in \mathscr{E}$ we have

$$a\eta \in \mathscr{E} \tag{6.9}$$

and

$$(ab)\eta = a(b\eta) \; , \;\; (a+b)\eta = a\eta + b\eta \; , \;\; a(\eta+\xi) = a\eta + a\xi \; . \tag{6.10}$$

The definition of right module is analogous. We have purposive used the same symbol for the sections of a bundle and for the module, since the latter is a relevant example of the former, where the algebra is the algebra of continuous functions over the base. A module $\mathscr{E}$ is finite if it is generated by a finite number of generators, and it is projective if given any two other modules $\mathscr{M}$ and $\mathscr{N}$, and a homomorphism $\phi : \mathscr{M} \to \mathscr{N}$ connecting them, then any surjective homomorphism $\phi_M : \mathscr{E} \to \mathscr{M}$ can be lifted to a homomorphism such that $\phi_N = \phi \circ \phi_M$. What this means actually is quite simple, it is saying that, heuristically, any finite projective module is made of matrices with elements in the algebra. The Serre–Swan theorem then says that any finite projective module over the algebra of smooth functions is isomorphic to the space of sections of a bundle and that conversely the space of sections of a bundle is isomorphic to a finite projective bundle. This means that it is always possible to write the space of sections $\Gamma(E,M) = p\mathscr{A}^N$ where is a matrix of elements of the algebra $p \in M_N(\mathscr{A})$ with the property that $p^2 = p$.

AU: Please check the words "where is a matrix" in the sentence "This means that it is ...". There seems to be a missing variable.

The transcription in algebraic terms of geometry comprises several more entries. Differential forms are realized as operators with the help of a generalized Dirac operator D, integrals of functions are calculated as traces of the corresponding operators, and the list goes on to comprise several more entries. It is possible to characterize a manifold, given by an algebra $\mathscr{A}_\infty$ with its differential structure, given by the generalized Dirac operator D, exclusively in algebraic terms [10]. The dimensionality is encoded in the growth of the eigenvalues of D, differentiability is given by multiple commutators of the elements of the algebra with D, as well as the domain of D^m acting on the Hilbert space on which $\mathscr{A}_\infty$ is represented. There are other conditions which mirror smoothness. We refer to the cited literature for details of this and the other entries of the dictionary and proceed to the generalization of this commutative geometry to noncommutative spaces.

6.2 Noncommutative spaces

In the previous section we have established the one-to-one correspondence between commutative C^*-algebras and ordinary Hausdorff spaces and we have shown how to reconstruct the points using purely algebraic methods. It now is possible to go beyond commutativity and define a noncommutative space as the object described by a *noncommutative* C^*-algebra. One can now ask if there are still points and a topology to recognize in this novel setting. In general we can still recognize a set of pure states, of representations (possibly of dimension larger than one), and of maximal ideals (now one has to distinguish among left, right, or bilateral ideals). These spaces now do not coincide anymore. Moreover, the algebra of continuous functions on the "points", being commutative, cannot anymore be the starting algebra. The concept of point becomes evanescent, and in some cases one is forced to abandon it altogether. Take for example the set of $n \times n$ complex matrices. It has only one representation (n-dimensional), but not one-dimensional representations. It has n unitarily equivalent pure states and no maximal ideals (apart from the whole algebra). One could be tempted to say that it describes a single point, but there is more structure in this algebra than in its commutative counterpart (complex numbers). The same can be said in the infinite-dimensional case of compact operators. We will see below that the equivalence of these algebras with $\mathbb{C}$ as far as the representations are concerned is captured by Morita equivalence.

6.2.1 The GNS construction

Still it is possible to do geometry, noncommutative geometry. This means that we extract geometric information directly form the algebra. The main technique is to represent the C^*-algebra in a Hilbert space. There is another result due to Gel'fand and Naimark (and Segal) which states that any C^*-algebra can be represented as bounded operators on a Hilbert space, and this of course strikes a chord in the hearth of physicists! The proof is constructive, namely, given a C^*-algebra one has a natural procedure (called GNS construction) to build a Hilbert space on which the algebra acts as bounded operators, with the C^* norm given by the operatorial norm.

The GNS construction is based on the fact that since every algebra has an obvious action on itself, we can consider the algebra itself as the starting vector space for the construction of the Hilbert space. To make this space a Hilbert space we first need an inner product with certain properties, and then we need to complete in the norm given by this product. Note that the Hilbert space norm is not the original norm of the C^*-algebra.

First we note that any state ϕ gives a bilinear map with some of the properties of inner product: $\phi(a^*b)$. The problem with this map is that there may be instances in which $\phi(a^*a)$ is zero, even if a is not the null vector. To this end consider the space of null elements defined as,

$$\mathscr{N}_\phi = \{a \in \mathscr{A} \mid \phi(a^*a) = 0\} \ . \tag{6.11}$$

This space turns out to be a left ideal. This can be proven using the relation

$$\phi(a^*b^*ba) \leq \|b\|^2 \phi(a^*a) \ , \tag{6.12}$$

so that $a \in \mathscr{N}_\phi \Rightarrow ba \in \mathscr{N}_\phi \, \forall b \in \mathscr{A}$. This ideal of null states can be eliminated by considering the space of equivalence classes of the elements of $\mathscr{A}$ up to elements of $\mathscr{N}_\phi$. We can then equip this space with the scalar product

$$\langle [a], [b] \rangle_\phi = \phi(a^*b) \ . \tag{6.13}$$

This product is by definition independent from the representative of the equivalence class. It defines a norm, and the Hilbert space is the topological completion of the space of equivalence classes with respect to this norm.

The algebra $\mathscr{A}$ is naturally represented on the Hilbert space by associating to any element $a \in \mathscr{A}$ an operator $\hat{a}$ with action

$$\hat{a}[b] = [ab] \ , \tag{6.14}$$

and again the action does not depend on the representative.

Thus, we have a representation of our algebra on the Hilbert space. The operators corresponding to the elements of $\mathscr{A}$ are bounded, in fact, expressing with

$$\|\hat{a}\|_{\phi} = \sup_{\phi(b^*b)\leq 1} \phi(b^*a^*ab), \tag{6.15}$$

we have the operator norm on the Hilbert space, using (6.12)

$$\|\hat{a}[b]\|^2 = \phi(b^*a^*ab) \leq \|a\|^2 \phi(b^*b), \tag{6.16}$$

and considering the supremum over $\phi(b^*b) \leq 1$ one obtains $\|\hat{a}\|_{\phi} \leq \|a\|$. Therefore, since all operators of a C^*-algebra have finite norms, $\hat{a}$ is a bounded operator on the Hilbert space $\mathscr{H}_{\phi}$ that we have just built. Note that the association of an operator to the element of the algebra depends on the choice of the state ϕ.

Conversely, given an algebra of bounded operators on a Hilbert space, any normalized vector $|\xi\rangle$ defines a state with the expectation value

$$\phi_{\xi}(a) = \langle \xi | \hat{a} | \xi \rangle \ . \tag{6.17}$$

It results that to any state ϕ it corresponds a *vector state*, i.e., there is a vector $\xi_{\phi} \in \mathscr{H}_{\phi}$ such that

$$\left\langle \xi_{\phi} \middle| \hat{a} \middle| \xi_{\phi} \right\rangle = \phi(a). \tag{6.18}$$

The vector ξ_{ϕ} is defined by

$$\xi_{\phi} := [\mathbb{I}] = \mathbb{I} + \mathscr{N}_{\phi} \tag{6.19}$$

and is readily seen to verify (6.18). Furthermore, the set $\{\pi_{\phi}(a)\xi_{\phi} \mid a \in \mathscr{A}\}$ is just the dense set $\mathscr{A}/\mathscr{N}_{\phi}$ of equivalence classes. This fact is encoded in the definition of *cyclic vector*. The vector ξ_{ϕ} is cyclic for the representation $(\mathscr{H}_{\phi}, \pi_{\phi})$. By construction, a cyclic vector is of norm one: $\|\xi_{\phi}\|^2 = \|\phi\| = 1$.

The cyclic representation $(\mathscr{H}_{\phi}, \pi_{\phi}, \xi_{\phi})$ is unique up to unitary equivalence. It can be shown that this representation of the algebra is irreducible if ϕ is a pure [11] state.

Example 6.3.
Let us consider the example of the commutative algebra of continuous functions on the real line vanishing at infinity. Choosing as pure state

$$\delta_{x_0}(a) = a(x_0), \tag{6.20}$$

the null space is given by all functions vanishing at x_0. The inner product is then given by

$$\langle a, b \rangle_{\delta} = a(x_0)^* b(x_0) \ , \tag{6.21}$$

and the Hilbert space turns out to be just $\mathbb{C}$. The algebra acts on this space by multiplication of complex numbers:

$$\hat{a}[b] = a(x_0) b(x_0) \ . \tag{6.22}$$

We should not be surprised of the fact that the Hilbert space is $\mathbb{C}$, the state is pure, and the only irreducible representations of a commutative algebra are one-dimensional.

The situation is different if we choose a non-pure state, for example,

$$\phi(a) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, \mathrm{e}^{-x^2} a(x). \tag{6.23}$$

This time there are no nonzero elements of the algebra such that $\phi(a^* a) = 0$. The Hilbert space therefore contains the continuous functions, the inner product is given by

$$\langle a, b \rangle_\phi = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, \mathrm{e}^{-x^2} a^*(x) b(x). \tag{6.24}$$

The completion of this space gives the space $L^2(\mathbb{R})$ with a gaussian measure. Then the operator representation of the algebra is just given by the pointwise multiplication of functions

$$\hat{a} b(x) = a(x) b(x) \ . \tag{6.25}$$

□

Example 6.4.
Let us give a noncommutative example: the matrix algebra $\mathbb{M}_2(\mathbb{C})$ with the two pure states

$$\phi_1 \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = a_{11} \ , \quad \phi_2 \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = a_{22} \ . \tag{6.26}$$

The corresponding representations are equivalent, being indeed both equivalent to the defining two-dimensional one. The ideals of elements of "vanishing norm" of the states ϕ_1, ϕ_2 are, respectively,

$$\mathcal{N}_1 = \left\{ \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix} \right\} \ , \quad \mathcal{N}_2 = \left\{ \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} \right\} \ . \tag{6.27}$$

The associated Hilbert spaces are then found to be

$$\begin{aligned} \mathcal{H}_1 &= \left\{ \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \end{bmatrix} \right\} \simeq \mathbb{C}^2 = \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} \ , \ \langle X, X' \rangle = x_1^* x_1' + x_2^* x_2' \ , \\ \mathcal{H}_2 &= \left\{ \begin{bmatrix} 0 & y_1 \\ 0 & y_2 \end{bmatrix} \right\} \simeq \mathbb{C}^2 = \left\{ Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\} \ , \ \langle Y, Y' \rangle = y_1^* y_1' + y_2^* y_2' \ . \end{aligned} \tag{6.28}$$

As for the action of any element $A \in \mathbf{M}_2(\mathbb{C})$ on $\mathcal{H}_1$ and $\mathcal{H}_2$, we have

$$\pi_1(A) \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} x_1 + a_{12} x_2 & 0 \\ a_{21} x_1 + a_{22} x_2 & 0 \end{bmatrix} \equiv A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ ,$$

$$\pi_2(A)\begin{bmatrix}0 & y_1\\ 0 & y_2\end{bmatrix} = \begin{bmatrix}0 & a_{11}y_1 + a_{12}y_2\\ 0 & a_{21}y_1 + a_{22}y_2\end{bmatrix} \equiv A\begin{pmatrix}y_1\\ y_2\end{pmatrix} . \tag{6.29}$$

The two cyclic vectors are given by

$$\xi_1 = \begin{pmatrix}1\\0\end{pmatrix} , \quad \xi_2 = \begin{pmatrix}0\\1\end{pmatrix} . \tag{6.30}$$

The equivalence of the two representations is provided by the off-diagonal matrix

$$U = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} , \tag{6.31}$$

which interchanges 1 *and* 2 *:* $U\xi_1 = \xi_2$*. Since* π_1 *and* π_2 *are irreducible representations and since any nonvanishing vector is a cyclic vector if the representation is irreducible, we see that* π_1 *and* π_2 *are unitary equivalents and can therefore be identified.* □

6.2.2 Commutative and noncommutative spaces

Sometimes, even in the presence of a noncommutative algebra, we are still in the presence of an ordinary space. Consider functions from a manifold into $n \times n$ complex-valued matrices. In this case the algebra can obviously still be associated with the original manifold, and we cannot really talk of a noncommutative geometry. Note that in this case, since algebra of $n \times n$ matrices has only one representation, we have one representation for each point of the original manifold, as in the commutative case. There are more pure states, as in (6.26), but they are unitarily equivalent. It is like we had points, but with an inner structure. This is sometimes refereed to as an "almost commutative geometry".

AU: Please check the change of "uitatily" to "unitarily" in the sentence "There are more pure . . ."

This characteristic is captured by the concept of (strong) Morita equivalence [12]. Two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ are Morita equivalent if there exists a complex vector space $\mathscr{E}$ which is a left module for $\mathscr{A}$ and a right one for $\mathscr{B}$. In $\mathscr{E}$ two inner products,[1] are defined with values in the two algebras, such that the representations are continuous and bounded, and with the property

$$\langle \eta, \xi \rangle_{\mathscr{A}}\, \chi = \eta\, \langle \xi, \chi \rangle_{\mathscr{B}} \quad \forall\, \eta, \xi, \chi \in \mathscr{E}. \tag{6.32}$$

The important property of Morita equivalent algebras is that they have the same space of (classes of unitary inequivalent) representations with the same topology. In particular all algebras, Morita equivalent to commutative algebras, are algebras of function from some Hausdorff topological space which can be uniquely reconstructed. Morita equivalent algebras also have the same (algebraic)

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[1] There are other requirements of continuity and density for the definition. The two inner products are sesquilinear forms with the usual properties. For details see [5].

K-theory. Hence in some sense two Morita equivalent algebras are algebras of functions on the same "space".

6.2.3 Deformations of spaces

There are several noncommutative spaces that have been studied: deformations (with one or more continuous parameters) of commutative algebras of functions on a topological space, or algebras of matrix-valued functions on commutative spaces. There are then truly noncommutative structures that are not linked to a "classical" (commutative) manifold. In some instances these NC algebras can be associated with non-Hausdorff spaces, the typical example being that of torus foliations for irrational theta. Similarly examples of spaces with a nonseparating topology in which a finite set of points can keep track of the homotopy of the original space are described in [13].

The standard example of a genuine noncommutative geometry is the *noncommutative torus* [3, 5] which we now briefly describe. In Fourier transform one can write functions on the torus (characterized by $x_i \in [0,1]$) as

$$f(x) = \sum f_{mn} U_1^n U_2^m, \tag{6.33}$$

with $U_i = e^{2\pi i x_i}$ and obviously $U_1 U_2 = U_2 U_1$. In this setting continuous functions are the ones with coefficients such that $\lim_{n_i \to \pm\infty} f_{n_1 n_2} \to 0$ faster than n_i^{-2}. From this C^*-algebra it is possible to reconstruct the torus as a topological space as shown in the previous section. If one now generalizes the commutation relation of the U's to the case

$$U_1 U_2 = e^{i2\pi\theta} U_2 U_1, \tag{6.34}$$

the algebra generated by (6.33) is a noncommutative algebra; it describes a deformation of the torus called noncommutative torus. When θ is irrational there is no ordinary space underlying it, in this case we are in the presence of a truly noncommutative space. The name noncommutative torus is given to various completions, with different norms, of the algebra (6.33) with the relation (6.34), corresponding to functions continuous, differentiable, analytic, etc. They all correspond to the various classes of functions of a "manifold" whose coordinates obey the commutation relation $[x_1, x_2] = i\theta$. It should however be kept in mind that this is just an heuristic view, as it is impossible to talk of a topological space in this case. We do not have the points of the space in this case!

Noncommutative tori are very different mathematical structures in the cases of θ rational or irrational. In the first case, $\theta = p/q$, p, q integers, the noncommutative torus is Morita equivalent to the algebra of functions on the ordinary torus [14], they are in fact isomorphic to the algebra of $q \times q$ matrices on a torus. In the irrational case the algebra does not describe any Hausdorff topological space. It can be seen that they describe the space of orbits of the points of a circle under the action of rotation of an angle $2\pi\theta$. As is known every orbit is dense, and therefore in the neighborhood of any point there is the whole space. If one considers then the circle quotiented by these rotations the Hausdorff topology would give a single point. Likewise if we consider functions which are constant on any given orbit, we obtain only functions constant on the circle. In noncommutative geometry there is a well-defined procedure, called the crossed product, which, starting from the action of functions on a manifold and the action of a group on it, gives the algebra on the quotient space. The application of this procedure to the case of the circle with the action of discrete irrational rotations gives the algebra of the noncommutative torus. Hence the noncommutative algebra captures more structure. In general noncommutative tori with parameters θ and θ' are Morita equivalent iff the parameters are connected by a $SL(2,Z)$ transformation:

$$\theta' = \frac{a\theta + b}{c\theta + d} \ , \ ad - bc = 1, \tag{6.35}$$

with a, b, c, d integers.

Finally, a relevant example of noncommutative spaces is that of quantum groups and Hopf algebras, discussed in the next chapter.

6.3 The noncommutative geometry of canonical commutation relations

The original example of a noncommutative space is quantum phase space; this is a well-established concept from the early days of quantum mechanics, with $\hbar$ a dimensionful quantity, with the dimensions of the area of the phase space of a one-dimensional particle. It is a "small" parameter, in the sense that in the limit in which it goes to zero, classical mechanics should emerge. In the usual view, for example in the courses of the standard physics curriculum, quantum and classical mechanics, however, are two different theories, using different mathematical tools, and the passage from one to the other (the classical limit) is not an immediate and unambiguous procedure. In reality there is a procedure, *deformation quantization*, which connects the two. In this case quantum mechanics is seen as a deformation of the classical theory, and the two theories are both seen as a theory of states on the $*$-algebra of observables. The crucial difference between the two theories is that in the quantum case the algebra is noncommutative.

The geometry underlying Hamiltonian classical mechanics is a Poisson (symplectic) geometry. The space of position and momenta, the *phase space*, is equipped with a Poisson bracket, and time evolution is generated by a Hamiltonian vector field. The set of functions on phase space is the set of observables of the theory: position, momentum, angular momentum, energy, temperature, etc. Under the conditions described in the first sections of this chapter it is possible to reconstruct the phase space from these observables. It is important that we can shift the emphasis from the points of phase space to the observables. The points are then the (pure) states of the algebra of observables.

Quantum mechanics forces the loss of the classical phase space; positions and momenta are substituted by *noncommuting* self-adjoint operators. We like to say that quantization is the rendering of a classical phase space a noncommutative geometry. In this section we will discuss the quantum phase space of a one-dimensional particle. This is an important and relevant example per se, but if we change the notation and send the pair p,x into the pair x_1,x_2, and $\hbar \to \theta$, we are then considering the standard, canonical, noncommutative geometry discussed in most of this book.

The barest minimum for a manifold to be seen as a phase space is the presence of the *Poisson bracket*, a bilinear map among $C^\infty(M)$ functions on M

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M) \ , \tag{6.36}$$

with the properties of being antisymmetric, satisfying the Jacobi identity, and the Leibniz rule

$$\{f, gh\} = g\{f,h\} + \{f,g\}h \ . \tag{6.37}$$

A Poisson bracket is defined by a Poisson bivector $\Lambda \in \Gamma(M, \wedge^2 TM)$, which satisfies the (Jacobi) property

$$\Lambda^{il}\partial_l\Lambda^{jk} + \Lambda^{jl}\partial_l\Lambda^{ki} + \Lambda^{kl}\partial_l\Lambda^{ij} = 0, \tag{6.38}$$

where $\partial_i := \partial/\partial u^i$ and the u's are the local coordinates of M.

We consider the case of a two-dimensional phase space $M = \mathbb{R}^2$ with global coordinates (x,p) and Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} \ . \tag{6.39}$$

A state of the physical system is a point of the phase space, or more generally a probability distribution. The terminology is the same as in Sect. 6.1, and indeed the pure states are the points, while the non-pure states are probability distributions, in which the system is in a probabilistic superposition of states.[2] Classical observables are (real) functions on M, and the C^*-algebra they generate carries all topological information of the phase space. Some observables are the infinitesimal generators of a physically relevant transformation, the infinitesimal variation being given by the Poisson bracket, for example, time evolution is generated by the Hamiltonian function

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\} \ , \tag{6.40}$$

rotations are generated by the angular momentum, etc.

[2] Note that the ensuing uncertainty of measurement is not the one inherent to measurement in quantum mechanics, but it only reflects the possibility that the state of a system is not completely known.

All this has to change drastically upon quantization and the presence of an uncertainty principle. Observables are not defined anymore as functions, but as operators on a Hilbert space, and they form a noncommutative algebra. Contact with classical mechanics is via the correspondence principle, which associates to each classical observable f an element $\hat{f}$ of a noncommutative algebra, with the basic requirement that the Poisson bracket is replaced (to leading order) by the commutator:

$$\{f,g\} \longmapsto -\frac{\mathrm{i}}{\hbar}[\hat{f},\hat{g}] \, . \tag{6.41}$$

This brings about the usual commutation relation between position and momentum:

$$[\hat{x},\hat{p}] = \mathrm{i}\hbar \, . \tag{6.42}$$

As is known $\hat{x}$ and $\hat{p}$ are unbounded operators, but it is possible to exponentiate them to obtain unitary operators and use them to build a C^*-algebra. This can then be represented as an algebra of bounded operators on some Hilbert space. The most common representation is on $L^2(\mathbb{R}_x)$, the square integrable functions of position, but one could use functions of momentum. Another commonly used representation is in terms of the eigenstates of an operator with discrete spectrum, say the Hamiltonian of the harmonic oscillator. In this case the basis of the Hilbert space is countable, and the operators can be seen as infinite matrices. We will see later on in Example 6.6 how the GNS construction applies to this case.

On square integrable functions of x the operators $\hat{x}$ and $\hat{p}$ are represented as

$$\hat{x}\psi(x) = x\psi(x), \;\; \hat{p}\psi(x) = -\mathrm{i}\hbar\partial_x\psi(x). \tag{6.43}$$

The association of an operator to other functions of x and p is, however, ambiguous, and moreover it is preferable to deal with bounded operators. Weyl [15] has given a well-defined map from functions into operators, this procedure was implicitly used in Appendix 1.9. We first define the operator (sometimes called the quantizer [5] in this context)

$$\hat{W}(\eta,\xi) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\xi\cdot\hat{p}+\eta\cdot\hat{x})}. \tag{6.44}$$

The correspondence is then defined as

$$f(p,x) \longmapsto \hat{\Omega}(f)(\hat{p},\hat{x}) = \int d\xi d\eta \tilde{f}(\xi,\eta)\hat{W}(\xi,\eta) \, , \tag{6.45}$$

where

$$\tilde{f}(\xi,\eta) = \int \frac{dxdp}{2\pi} f(p,x)\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\eta x+\xi p)} \tag{6.46}$$

is the Fourier transform of f. If we were to forget the hat on p and x in (6.44), the expression (6.45) would look just like the expression which Fourier transforms back $\tilde{f}$ to the original function. Because of the operatorial nature of $\hat{W}$, instead it associates an operator to functions, with the property that real functions are mapped into hermitian operators. The inverse of the Weyl map is called the *Wigner map* [16]:

$$\Omega^{-1}(\hat{F})(p,x) = \int \frac{d\eta d\xi}{(2\pi)^2\hbar}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\eta x+\xi p)} \mathrm{Tr}\hat{F}\,\hat{W}(\xi,\eta). \tag{6.47}$$

The Weyl map gives a precise prescription associating an operator to any function for which a Fourier transform can be defined. It has the characteristic of mapping real functions into hermitian operators and is a vector space isomorphism between L^2 functions on phase space and Hilbert–Schmidt operators [17–19].

The correspondence between functions and operators implicitly defines a new noncommutative product [20, 21] among functions on phase spaces defined as follows:

$$f \star g = \Omega^{-1}(\hat{\Omega}(f)\hat{\Omega}(g)) \; . \tag{6.48}$$

This product called the Grönewold–Moyal, or simply Moyal, or $\star$-product, is associative but noncommutative and it reproduces the standard quantum mechanical commutation relation:

$$x \star p - p \star x = \mathrm{i}\hbar. \tag{6.49}$$

There are several integral expressions (see for example [22, 23]) for the $\star$-product, with a fairly large domain of definition. In the context of this book it is useful to see the $\star$-product as a *twisted* convolution of Fourier transforms. Given two functions f and g the Fourier transform of their product is

$$\widetilde{(f \star g)}(\xi,\eta) = \int \frac{d\xi' d\eta'}{2\pi}\, \mathrm{e}^{\,\mathrm{i}\hbar(\xi\eta'-\xi'\eta)}\, \tilde{f}(\xi',\eta')\tilde{g}(\xi-\xi',\eta-\eta') \; . \tag{6.50}$$

Without the exponential this expression would just give the commutative convolution product among Fourier components. The exponential breaks the symmetry between f and g and gives noncommutativity.

Another very common form of the product is the differential expansion of the product (6.50) given by

$$(f \star g)(u) := f(u) \exp\left(\frac{\mathrm{i}\hbar}{2} \overleftarrow{\partial_i} \Lambda^{ij} \overrightarrow{\partial_j}\right) g(u) \; , \tag{6.51}$$

where the notation $\overleftarrow{\partial_i}$ (resp. $\overrightarrow{\partial_i}$) means that the partial derivative acts on the left (resp. right). This expression is an asymptotic expansion of the integral one [24], obtained by expanding the exponential in (6.50). The product can be seen as acting with the twist operator

$$\mathscr{F} = \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}(\partial_x \otimes \partial_p - \partial_p \otimes \partial_x)} \tag{6.52}$$

on the tensor product of the two functions, before evaluating them on the same point. In this sense, as is discussed at length in this book, the $\star$-product is a twisted product.

Expressions (6.51) and (6.50) have different domains of definition, but they are both well defined if both function are Schwarzian functions, and in this case their product is still Schwarzian. The star product (both in the differential and integral forms) is also well defined on polynomials, which however do not belong to the C^*-

algebra, and in fact they are not mapped into bounded operators. It is nevertheless useful to consider them, which is what we do when we talk of the commutation relations (6.42). If one is not interested in the presence of the norm, then one can define the algebra of formal series in the generators x and p. This is basically the construction described in Sect. 1.2.

The asymptotic form (6.50) is convenient because it enables to write immediately the $\star$-product of two functions as a series expansion in the small parameter $\hbar$. The first term of the expansion is the ordinary commutative product. In this sense this product is a *deformation* [25] of the usual pointwise product. The second term in the expansion is proportional to the Poisson bracket:

$$f \star g = fg + \mathrm{i}\hbar\{f,g\} + O(\hbar^2)\ . \tag{6.53}$$

Considering less trivial phase spaces, starting from the work of [26, 27] a whole theory of deformed products with the property that to first order in $\hbar$ they reproduce the Poisson bracket has been developed, under the name of $\star$-quantization or (formal) deformation quantization. This culminated in the work of Kontsevich [28] who proved that it is always possible, given a manifold with a Poisson bracket, to construct a $\star$-product that quantizes the Poisson structure. That is, such that the product is associative and whose commutator, to first order in the deformation parameter, is proportional to the Poisson bracket.

Consider the Heisenberg equation of motion for observables which do not depend explicitly on time:

$$\frac{d\hat{f}}{dt} = \mathrm{i}\,\frac{[\hat{f},\hat{H}]}{\hbar} \tag{6.54}$$

and the classical analogous in terms of the Poisson bracket

$$\frac{df}{dt} = \{f,H\}, \tag{6.55}$$

where f and H are observable and the Hamiltonian for classical system, respectively, and $\hat{f},\hat{H}$ the operators obtained with the Weyl correspondence. In terms of a deformed classical mechanics we can define

$$\frac{df}{dt} = \frac{1}{\mathrm{i}\hbar}(f \star H - H \star f) = \{f,H\} + O(\hbar^2). \tag{6.56}$$

Here we can see that the two evolutions coincide in the limit $\hbar \to 0$. In this sense classical mechanics can be seen as the classical limit of quantum mechanics. The $\star$-commutator is called the *Moyal bracket* [21] and plays the role of a quantum mechanics Poisson bracket.

Example 6.5.
The algebra of functions on the (p,x) plane with the $\star$-product is isomorphic to the algebra of operators generated by $\hat{p}$ and $\hat{x}$. For further illustration in this example, we see how the algebra with the $\star$-product as well can be seen as a (infinite) matrix algebra.

Consider first the function[3]*:*

$$\varphi_0 = 2\,\mathrm{e}^{-\frac{p^2+x^2}{2}} \ . \tag{6.57}$$

This function is a projector, that is,

$$\varphi_0 \star \varphi_0 = \varphi_0 \ . \tag{6.58}$$

It is in fact the first of a whole class of projectors, as it is the function obtained applying the Wigner map to the projection operator corresponding to the ground state of the harmonic oscillator

$$\varphi_0 = \Omega^{-1}(|0\rangle\,\langle 0|). \tag{6.59}$$

Consider then the functions

$$a = \frac{1}{\sqrt{2}}(x + \mathrm{i}\,p) \quad \bar{a} = \frac{1}{\sqrt{2}}(x - \mathrm{i}\,p) \ . \tag{6.60}$$

These two operators are easily recognized as the functions corresponding (with the Wigner map) to the usual creation and annihilation operators. They have the property that for a generic function f

$$a \star f = af + \frac{\partial f}{\partial \bar{a}} \quad f \star a = af - \frac{\partial f}{\partial \bar{a}}, \tag{6.61}$$

and analogous relations involving $\bar{a}$.

Define now the functions [22]

$$\varphi_{nm} = \frac{1}{\sqrt{2^{n+m} m! n!}} \bar{a}^n \star \varphi_0 \star a^m \ . \tag{6.62}$$

These are the functions corresponding via the Wigner map to the operators $|m\rangle\,\langle n|$ and have the property

$$\varphi_{mn} \star \varphi_{kl} = \delta_{nk} \varphi_{ml} \ , \tag{6.63}$$

which is easily proven using (6.61) and (6.58).

The φ_{mn} are a basis for the functions of p and q, or alternatively of a and $\bar{a}$:

$$f = \sum_{m,n=0}^{\infty} f_{mn} \varphi_{mn}, \tag{6.64}$$

relation (6.63) ensures that

$$(f \star g)_{mn} = \sum_{p=0}^{\infty} f_{mp} g_{pn}. \tag{6.65}$$

---

[3] For simplicity set $\hbar = 1$.

In this sense the deformed algebra can be seen as multiplication of (infinite) matrices. □

Example 6.6.
Example 6.3 can be immediately generalized to arbitrary size matrices and even to infinite matrices (operators on $\ell^2(\mathbb{Z})$). In fact using the matrix basis described in Example 6.5 for functions $f = \sum_{mn} f_{mn}\varphi_{mn}$ the same construction can be applied using the state

$$\phi(f) = f_{00} = \int dp\,dx\ \varphi_0 \star f \star \varphi_0\ . \tag{6.66}$$

The ideal $\mathcal{N}_\phi$ is given by functions with $f_{0m} = 0$ and we can identify the Hilbert space with functions of the kind

$$\psi = \sum_n \psi_n \varphi_{n0}\ . \tag{6.67}$$

Upon recalling that $\varphi_{n0} = \frac{1}{\sqrt{2^n n!}}\bar{a}^n \star \varphi_0$ one recognizes the usual countable basis of the Hilbert space $L^2(\mathbb{R})$ composed of Hermite polynomials multiplied by a gaussian function. □

Example 6.7.
The noncommutative torus is a compact version of the algebra described in this section. It can be seen as a deformation of the algebra of functions on the torus in the sense of Moyal. Given a function on the torus with Fourier expansion

$$f(x) = \sum_{n_1,n_2=-\infty}^{\infty} f_{n1n_2}\,\mathrm{e}^{in_1x_1}\,\mathrm{e}^{in_2x_2}, \tag{6.68}$$

we associate to it the operator

$$\hat{f} = \sum_{n_1,n_2=-\infty}^{\infty} f_{n_1n_2}\hat{U}_1^{n_1}\hat{U}_2^{n_2}, \tag{6.69}$$

where the operators U_i act on the Hilbert space of infinite sequences of complex numbers $c = \{c_n\}$ as

$$\left(\hat{U}_1 c\right)_n = \mathrm{e}^{in\theta} c_n\ \ ;\ \ \left(\hat{U}_2 c\right)_n = c_{n+1}. \tag{6.70}$$

It is not difficult to see that the $\hat{U}$'s satisfy the relation (6.34) and the $\star$-product defined as in (6.48) can also be expressed as

$$(f\star g)(x) = e^{i\varepsilon^{ij}\theta\partial_{\xi_i}\partial_{\eta_j}} f(\xi)g(\eta)\Big|_{\xi=\eta=x}. \tag{6.71}$$

□

6.4 Final remarks

Noncommutative geometry started from the need to describe quantum mechanics, and it has led to see it as a deformation of classical mechanics. The freedom from the need to describe spaces as sets of points opened a whole new quantum world, needed on physical grounds to describe atomic physics. This deformation was the main stimulus for large body of mathematical literature, which not only helped to clarify and develop quantum mechanics, but also led to the construction of several other "noncommutative geometries", together with their symmetries. The catalog of noncommutative spaces is already large, and still growing, and noncommutative geometry has proven to be an useful tool also to understand standard, commutative geometries.

Historically quantum mechanics started from a "cutoff", imposed by Planck to avoid the ultraviolet divergences in the calculation of the black body spectrum. It is natural to think that the tools of noncommutative geometry may help the solution of the other ultraviolet divergences that we are encountering in the search for a theory that unifies quantum mechanics and gravity. Hence the study of field theories on noncommutative spaces, which is the main object of this book.

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AU: Please provide year for the reference [6,7]

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